ISSN: 2688-8645

Improvement of Sufficient Conditions for Pancyclic Graphs

Xingxing Wang*, Mengjie Li, Wenxue Shi, Yue Wu, Yuting Zhao

Chuzhou University, Chuzhou, Anhui 239000, China

*2779511915@qq.com

Abstract

The study of cyclic graphs has always been a hot topic in the field of graph theory and has received widespread attention from graph theory practitioners. If an n-order graph G exactly contains cycles of all lengths from 3 to n, then the graph is called a pan cycle graph. This article proves that, after excluding some special cases, when the number of edges in graph G is greater than or equal to $C_{n-3}^{2}+12$, graph G must be a pan cyclic graph.

Keywords

Pan Circle Diagram; Number of Edges; Spectrum.

1. Introduction

This article only studies undirected simple graphs. Let graph G be a simple graph, where $V(G)=\{v_1,v_2,\cdots,v_n\}$ represents the vertex set of n-order graph G and $E(G)$ represents the edge set of graph G. For convenience, this article uses $|V(G)|$ and $e(G)$ to represent the number of vertices and edges of graph G, respectively. $d_G(v)$ represents the degree of point v in graph G, while $\delta(G)$ and $\Delta(G)$ represent the minimum and maximum degrees of graph G, respectively.

If a cycle (path) contains all vertices of graph G, it is called the Hamiltonian cycle (path) of graph G; If a graph G contains at least one Hamiltonian cycle, it is called a Hamiltonian graph; If there is a Hamiltonian path between any two points in graph G, it is called a Hamiltonian connected graph. For two completely disjoint graphs G_1 and G_2, we use $G_1 \cup G_2$ to represent their disjoint union and $G_1 \vee G_2$ to represent their joint graph. Define $G-\{v\}$ to remove point v and all its associated edges in graph G.

The adjacency matrix of graph G is defined as $A(G)=(a_{ij})_{n\times n}$. When point v_i is adjacent to point v_j, denote $a_{ij}=1$, otherwise it is 0. The degree diagonal matrix of graph G is defined as $D(G)=(d_G(v_1),d_G(v_2),\cdots,d_G(v_n))$. The unsigned Laplacian matrix of graph G is defined as $Q(G)=A(G)+D(G)$. Furthermore, let $\lambda(G)$ and $q(G)$ represent the adjacency spectral radius and unsigned Laplacian spectral radius of graph G respectively. For more relevant symbols and terminology, readers can refer to references [1-2].

In 1971, Bondy [1] gave the definition of a pan cyclic graph and provided some sufficient conditions for determining pan cyclic graphs. Obviously, bipartite graphs do not contain odd cycles, therefore, Mitchem et al. [3] further defined bipartite even pan cyclic graphs. Since then, the issue of pan cyclic graphs has gradually attracted the attention of graph theorists. In recent years, Yu Guidong et al. [4] have optimized some previous conclusions through more detailed research and provided stronger criteria for determining pan cycle graphs. Almost simultaneously, Li Rao [5] characterized some conclusions on bipartite bipartite pan cyclic graphs, optimizing some existing results. Inspired by this, this article strengthens the characterization of graph structure, providing the edge number and spectral conditions for pan cyclic graphs, which to some extent optimizes the results of Yu Guidong et al.

2. Related Lemmas

Lemma 2.1: Assuming that graph G is a n-order graph with a degree sequence $d_1 \le d_2 \le \cdots \le d_n$. If there is a positive integer s that allows $d_s \le s$ to derive $d_{n-s} \ge n-s$, then graph G is either a pan cyclic graph or a bipartite graph [6].

Lemma 2.2: Assuming that graph G is an n-order graph, $e(G)=m$ [9]. If the number of edges in graph G satisfies:

$e(G) \ge C_{n-1}^2 + 3$,

so graph G is a Hamiltonian connected graph.

Lemma 2.3: Assuming that graph G is a $n(n \ge 5)$-order simply connected graph of $\delta \ge 2$ [4], if the number of edges in graph G satisfies:

$e(G) \ge C_{n-2}^2 + 3$,

so G is either a pan cyclic graph, a bipartite graph, or belongs to the graph set NP_1.

3. Main Results

Theorem 3.1: Assumes that graph G is a $n(n \ge 14)$-order non bipartite simple graph of $\delta \ge 2$. If the number of edges in graph G satisfies:

$e(G) \ge C_{n-3}^2 + 12$

Then G is a pan cyclic graph, unless $G \subseteq NP$.

Proof: (proof by contradiction) Assume that graph G is not a pan cyclic graph and $d_1 \le d_2 \le \cdots \le d_n$ is the degree sequence of graph G. According to Lemma 2.1, it can be inferred that: there exists a positive integer s such that $d_s \le s$ and $d_{n-s} \le n-s-1$. Therefore,

$$2e(G) = \sum_{i=1}^{n} d_i = \sum_{i=1}^{s} d_i + \sum_{i=s+1}^{n-s} d_i + \sum_{i=n-s+1}^{n} d_i \le s^2 + (n-2s)(n-s-1) + s(n-1)$$

If $f(x) = 3s^2 - (2n-1)s + 6n - 36$, then it can be obtained from the above equation:

$2e(G) \le n^2 - 7n + 36 + f(s)$.

Therefore, there must be $f(s) \ge 0$. Obviously, $f(s)$ is a convex function about s. When $n \ge 14$, $f(3) = -6 < 0$ and $f(\frac{n-1}{2}) = -\frac{1}{4}(n-11)(n-13) < 0$, So, when $3 \le s \le \frac{n-1}{2} < \frac{n}{2}$, $f(s) < 0$, contradiction! Therefore, there must be $s = 2$.

At this point, the degree sequence of graph G satisfies the following conditions:

$d_1 = d_2 = 2, d_3 \le \cdots \le d_{n-2} \le n-3, d_{n-1} \le d_n \le n-1$.

If graph   contains 3 or more vertices of degree 2, then:

$e(G) \le C_{n-3}^2 + 6 < C_{n-3}^2 + 12$, contradicts!

Combining degree sequences, graph   has and only has 2 2-degree vertices, which can be denoted as v_1, v_2. There are two scenarios for discussion:

Scenario 1: v_1 and v_2 are not adjacent.

If the adjacent sets of v_1 and v_2 are completely identical, then graph G is a subgraph of $k_2 \vee (k_{n-4} + 2k_1)$, that is, $G \subseteq NP$.

If the neighboring sets of v_1 and v_1 are not completely consistent, let $G' = G - v_1$. Obviously, G' is a connected graph, $|V(G')| = n-1$ and $e(G') = e(G) - 2 = C_{n-3}^2 + 10 > C_{(n-1)-2}^2 + 3$. According to Lemma 2.5, G' is a pan cyclic graph.

This means that the graph G contains all the circles with lengths from 3 to G. The proof graph G also contains a Hamiltonian cycle, therefore the graph G is a pan cycle graph, thus deriving the contradiction.

Without loss of generality, assuming $C \equiv (y_1, y_2, \cdots, y_{n-1})$ is a Hamiltonian cycle in the clockwise direction of graph G', points y_1 and y_2 are the two adjacent points of v_1, and points y_1^+ and y_2^+ are the next points of y_1 and y_2 on cycle C, respectively. Obviously, point sets $\{v_1, y_1^+, y_2^+\}$ and $\{v_1, v_2\}$ are independent sets, otherwise graph G contains a Hamiltonian cycle, which is contradictory.

Assume $d_G(y_1^+) + d_G(y_2^+) \geq n$, since graph G' only has $n-1$ vertices, $v_2 \neq y_2^+$, otherwise $d_G(y_1^+) + d_G(y_2^+) = d_G(y_1^+) + d_G(v_2) \leq (n-3) + 2 = n-1$, which is contradictory.

Therefore, there must be a pair of consecutive vertices y_i and y_{i+1} on circle C, where y_i and y_1^+ are adjacent points and y_{i+1} and y_2^+ are adjacent points. For convenience, $\vec{C}$ and $\overleftarrow{C}$ are used here to represent the clockwise and counterclockwise directions of circle C. At this point, graph G contains a Hamiltonian cycle with $v_1 y_1 \overleftarrow{C} y_{i+1} y_2^+ \vec{C} y_i y_1^+ \vec{C} y_2 v_1$, which is contradictory. Therefore, $d_G(y_1^+) + d_G(y_2^+) \leq n-1$.

Let $G'' = G - \{v_1, y_1^+, y_2^+\}$ and $G''' = G - \{v_1, y_1, y_2, y_1^+, y_2^+\}$. Note that $|V(G'')| = n-3$,. So, the number of edges of G''' satisfies: $|V(G''')| = n-5$, $d_{G''}(y_1) \leq n-4$, $d_{G''}(y_2) \leq n-4$.

Therefore, according to Lemma 2.4, it can be concluded that G'' is a Hamiltonian connected graph.

We can always find a pair of vertices z_1 and z_2 in G that are adjacent to y_1^+ and y_2^+, respectively. If not, then the degree of z_1 and z_2 in G is at most 3. Therefore, $e(G) \leq C_{n-5}^2 + 2 + 6 + 2(n-5) < C_{n-3}^2 + 12$, which is contradictory. Assuming P is a Hamiltonian path from z_1 to z_2 in G'', there exists a Hamiltonian cycle in Figure G as: $v_1 y_1 y_1^+ z_1 P z_2 y_2^+ y_2 v_1$.Therefore, Figure G is a pan cyclic graph, contradictory.

Scenario 2: v_1 is adjacent to v_2.

If the neighboring sets of v_1 and v_2 are completely identical, then graph G is a subgraph of $k_1 \vee (k_{n-3} + k_2)$, that is, $G \subseteq NP$.

If the neighboring sets of v_1 and v_2 are not completely consistent, then $H = G - \{v_1, v_2\}$ is taken, indicating that $|V(H)| = n-2$, $e(H) = e(G) - 3 = C_{n-3}^2 + 9$. Similar to scenario 1, it can be inferred from Lemma 2.4 and Lemma 2.5 that a graph is both a pan cyclic graph and a Hamiltonian connected graph. Therefore, graph G is also a pan cyclic graph, which is contradictory.

In summary, if the number of edges of graph G satisfies the condition, then G is either a pan cyclic graph or $G \subseteq NP$, the proof is complete.

It can be seen that when $n \geq 14$, the bound of Theorem 3.1 is better than Lemma 2.5, which can be seen to some extent as an optimization of Lemma 2.5. Moreover, due to the sufficient number of vertices in the condition of Theorem 3.1, the conclusion of this theorem needs to be more complete.

Acknowledgments

This work is supported by Chuzhou College Student Innovation and Entrepreneurship Training Program Project, "*Improvement of Sufficient Conditions for Pancyclic Graphs*" (Project Number: S202210377171).

References

[1] J.-A. Bondy. Pancyclic graphs I[J]. J. Combin.Theory Ser. B, 1971, 11:80-84.
[2] J.-A. Bondy, U.S.R. Murty, Graph Theory, GTM 244, Springer, New York, 2008.
[3] E. Mitchem, J. Schmeichel, Bipartite graphs with cycles of all even length, J. Graph Theory, 1982, 6 :429-439.
[4] G.-D. Yu, T. Yu, et al. Spectral Sufficient Conditions on Pancyclic Graphs[J]. Complexity, 2021, 1:1-8.
[5] R Li. Spectral Conditions for the Bipancyclic Bipartite Graphs[J]. 2021, 10.48550/ arXiv. 2106.13741.
[6] D. Bauer, E. Schmeichel. Hamiltonian degree conditions which imply a graph is pancyclic[J]. Journal of Combinatorial Theory, 1990, 48(1):111-116.
[7] H Yuan. A bound on the spectral radius of graphs[J]. Linear Algebra Appl, 1988,108:135-139.
[8] L.H. Feng, G.H. Yu, On three conjectures involving the signless Laplacian spectral radius of graphs, Publ. De L Institut Math. 2009, 85: 35-38.
[9] O. Ore, Hamiltonian connected graphs, J Math Pures Appl. 1963, 42 :21-27.