A Language of Form
The Two-Dimensional Isometric System

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The Department of Design at The University of Kansas, under the direction of Professor Richard Branham, has been investigating a new approach to teaching basic design. A more precise understanding of the relationships possible in two and three dimensions has been our goal. The use of classical symmetry theory offers a very precise study of relationships found in many natural and man-made forms and has served as the basis of our search.

Symmetry theory is a powerful intellectual tool which can be used to clarify the nature of relationships found in objects, processes, and ideas. Man typically is aware of and utilizes only a small portion of the relationships possible within a system. He uses only those relationships which are relevent to his questions. Symmetry theory brings to light many of the relationships possible within a system.

A system can be anything we choose to consider. Symmetry theory studies the possible relationships in systems. A system can have any number of possible relationships, and those relationships are limited only by the imagination of the observer; however certain relationships have proven to be more beneficial than others.

The physical scientist deals with the structural relationships present in particles, atoms, crystal, the earth, the solar system, and our galaxy. He has observed (most often indirectly) and confirmed that certain spatial relationships appear to operate in our world. These relationships reflect the order (although never perfectly realized) thought to be present in time and space. Certain of these relationships are included under the concept of *Symmetry*.

Few designers have observed or used the concept of a system of possible relationships operating in their activities, a concept which could lead to more solutions to their problems. Many prefer to reiterate the intuitive and subjective experiences of the past, rather than to identify the framework of a problem and the invariants and transformations possible therein. Symmetry theory can be useful in arranging information and thus facilitating its manipulation in a variety of situations.

Form makers (artists, craftsmen, and architects) have been aware of symmetry relationships for centuries. The band patterns, found on Greek vases and in Greek architecture can be classified through the use of symmetry theory. The Egyptians and Persians have also left evidence that they were aware of the symmetry of two-dimensional space-filling patterns. It is interesting to note that it was the form makers who first discovered and used the possible regular ways of creating patterns, but that it was not until the late nineteenth century that mathematicians confirmed what had been practiced two thousand years before.

Many take as a beginning of symmetry theory a pamphlet by the astronomer Johannes Kepler who in 1611 wrote "A New Year's present; on hexagonal snow." Kepler's uneasy speculation about the possibility of 'the regular geometric arrangement of minute and equal brick-like units' was thought by himself to be of "nothing." Crystallographers have described the characteristic angles between the faces of crystals. This was first done by Stensen in 1669. This led by many routes to the laws of symmetry, the indexing of faces, and the classification of crystals in 32 classes by Hessel in 1830 and also by Gadolin in 1867. The question of how many pure three-dimensional translation lattices exist was solved by Auguste Bravis in 1850. The fourteen lattices Bravis discovered still carry his name. Fedorov and Schoenflies in the 1890's answered the question of how many symmetry operations are possible on a system containing a three-dimensional lattice. However, not until the application of the x-ray in the second decade of the century was it possible to confirm that the symmetry many men had speculated about was indeed present in three dimensions!

Many designers today will think, like Kepler, that this system has no value, nothing to offer them in their creative work—there is no need of considering the structure of the work they pursue. It is very evident that the persons responsible for patterns used on wall coverings, wrapping paper, ribbon, lines of paper goods, and similar products today, are often unaware of the full range of the theoretical relationships which are possible. They manipulate symbols on surfaces, trapped by the constraints of production tools and traditions. These become insurmountable barriers to new creative acts. They are not aware that a system can be manipulated by the use of symmetry to create and maintain a new level of relationships between many apparently divergent products.

Symmetry is the study of relationships. This paper presents the simplest and most easily understood symmetrical relationships. A two-dimensional form is called symmetrical if certain operations leave the form unchanged after the operation has been performed. An understanding of these basic symmetry concepts makes clear the two-dimensional patterns and can lead to the consideration of the nature of other relationships.

The concepts presented in this paper are not new, they have been presented before and are well documented. I have tried to communicate these concepts, to a new audience, an audience unfamiliar with theory. This has lead me to reorder the use of terminology and to develop another notation system. If the concept of symmetry is as powerful as the theorist would have us believe, and I believe it is, it must be communicated and made understandable to a larger audience.

Visible Language, XIII 4. Author's address: Royal College of Art, House 12, Evelyn Gardens, London SW7.

a

This paper will enumerate the sets of symmetry relationships which are possible in a very small, clearly defined segment of space. (Others are possible but cannot be discussed in this short paper.) Acquaintance with symmetry can lead to a better understanding of the nature of relationships and to new ways of proceeding to search for new relationships. This basic understanding of the theory of the restraints on the symmetry transformations is not widely understood. In what follows, the possible relationships in a symmetrical system will be illustrated by the familiar letterform, a lower-case Helvetica 'a'.

I chose to use this character for several reasons: it has no mirror planes, such as the characters E, U, v, and w; it has no rotation axes, such as the characters N, s, and z; it has no combination of mirror planes and rotation axes such as the characters H, I, O, 1, and x; and its enantiomorphic form is not another character, e.g., such as in the pairs b and d, or p and q. Also, it is a very aesthetically pleasing letter form.

To introduce the idea of forms in relationship we consider a set of examples. Two forms can be called equal if both share a common description. Two letter forms can be considered equal when they belong to the same language. They can, and often have, other relationships. Of the possible relationships between two letter forms, character form (a, b, c), family of type (serif, sanserif), face (Times Roman, Univers, Helvetica), case (upper, lower), weight (bold, medium, light), and size (40 points, 20 points, 10 points) exist. The study of symmetry deals with forms and the relationships which can be observed between them.

Two letter forms are really not equal if they occupy different positions in space. Two forms separated by time or space are essentially unequal. However, it is possible to set up a standard or a criterion against which one can measure relative equality. Symmetry deals with this relative equality of forms.

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a	≠	Z	Character
a	≠	A	Family
a	≠	A	Case
a	≠	a	Face
a	≠	**a**	Weight
a	≠	a	Size
a	=	a	

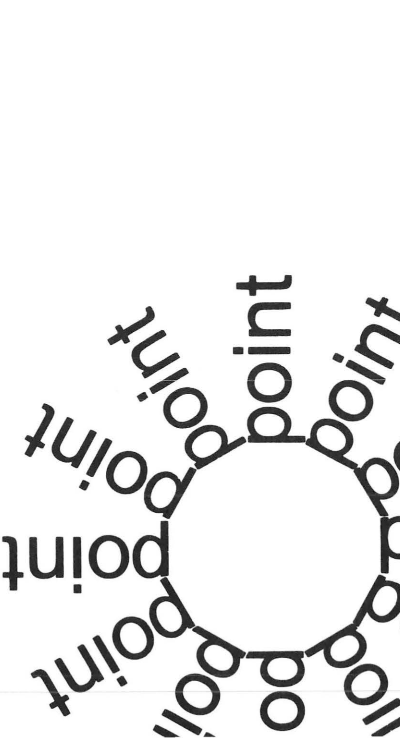
The symmetry systems considered in this paper will be easier to understand if the limits of our consideration are defined. This will allow the concepts and groups of forms to be placed in context. The definitions are quite specific and leave no latitude for interpretation. We define them specifically for two reasons: to identify the limits of consideration and to provide a context within which many forms can be related to each other. Several concepts are defined in the next section so you will be familiar with the context in which they are used in this paper.

Singularity

A point is singular if there are no other equivalent points in the form under consideration. For example, the point at the center of a circle is singular, for there are no other points in the circle which are its equivalent. When a number of points are considered, they will have a specific relationship. For example, on a meter stick there are a great number of pairs of points which are at an equal distance from each other. Because each point of a pair bears a similar relationship to the other, no one such point on the stick can be called singular. Another example is a chess board, where a number of red squares (points) are in the same relationship to the neighboring black squares (points); hence, the squares are equal to one another except for their color. A form has *symmetry* if there is a set of operations—the *Symmetry Group* —which, when applied to the object, leave it apparently unchanged. The concept of singularity can be applied to points and axes of symmetry. A point or axis is singular if only one such point or axis can be located in the set of symmetry operations in the symmetry group of the form.

A point

A point is dimensionless; it defines a position and is the most basic element in geometric constructions. When a point is placed on a plane, it defines a position on that plane. The term "point" in the name of a group indicates that the relationships occur around a singular point. Such a group is called a "point group."



One-sidedness

Consideration is limited to one side of a plane. An ideal plane does not exist in the real world. A piece of paper, often given as an example of a plane, has two distinct sides; a plane in the mathematical sense, however, has absolutely no third dimension. Attention is directed to one side of the plane only. There are groups of forms which explore the possible relationships occurring around the two surfaces of a plane in space, but those relationships are beyond the scope of this short paper. When only one side of a plane is considered, the plane is called one-sided. The term two-dimensional indicates that the relationships occur on one side of the plane only.

A line

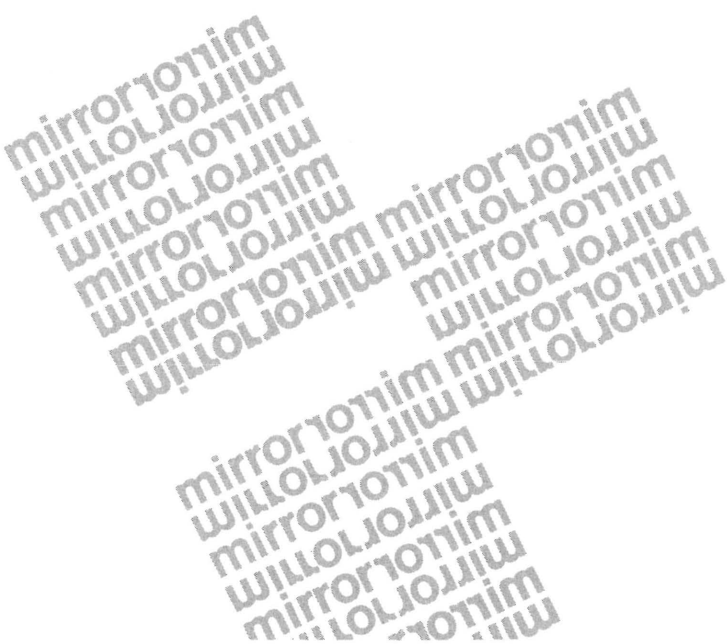
A line can be understood as being composed of a number of points in a certain relationship. No longer can the points which make up the line be called singular, for many points can be identified on the line. When a line is placed on a plane, it defines a direction on that plane. An axis is another name given to a line on a plane. The term mono-axial in the name of a group indicates that the forms occur in relationship to one axis. For example, a certain class of architectural frieze possesses a mono-axial symmetry group. The term bi-axial in the name of a group indicates that the forms occur in relationship to two axis. For example, a wall covering can possess a bi-axial symmetry group.

An isometric system

The group of symmtery operations presented in this paper are isometric. The variable of length is held constant. If a form is a repeated pattern, all lengths in the form are preserved. This means in theory that the forms in the object considered could be repeated infinitely. Other systems can be considered which do not hold the variable of length constant but which hold other variables constant. We are not considering those systems in this paper. The next section of this paper discusses the possible transformations or symmetry operations which preserve the variable of length.

The transformations

A transformation creates a one-to-one correspondence between an initial state and a subsequent state of a system. The transformation is expressed by a rule. An isometry transformation preserves length and creates a congruence among all parts on a plane. It creates a relationship, expressed as a process, which leaves the whole form unchanged but changes the parts. The symmetry operations which are possible on one side of a plane are defined as follows.



A mirror plane—m

The word symmetry generally calls to mind bilateral symmetry. This idea is very old and very familiar, since bilateral forms can be found in many places in our world. Most mammals (including man), fish, birds, and insects exhibit this type of symmetry in their exterior form. Many of the objects that humans make for their own use (chairs, tables, cars, and airplanes) have one half identical to the mirror image of the other half. Often this symmetrical relationship is the result of functional characteristics, but sometimes it appears, apparently, for aesthetic reasons alone. Bilateral symmetry locates a mirror plane between two halves of a form so that the two halves are related as an object is related to its reflection in a mirror.

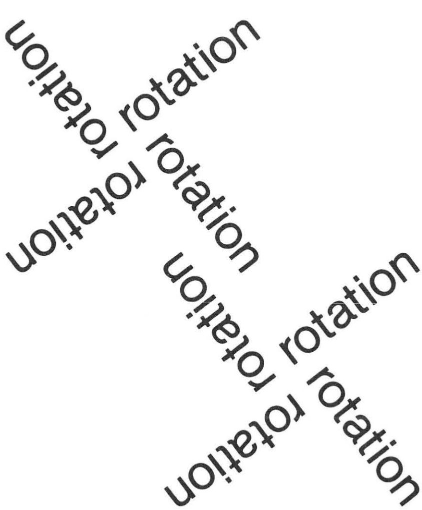
When the symmetry operation of reflection through a mirror plane occurs, the parts of the form superimpose. Similar points on the form can be connected through the mirror plane. Observe that in the form illustrated, corresponding points are located at an equal distance on either side of the mirror plane, and that the connecting lines are at right angles to the mirror plane. Reflection through a mirror plane produces forms which have a mirror plane relationship. A form can have one or many mirror planes in various relationships.

A rotation axis—r

General understanding of symmetry is usually limited to the symmetry operation of a mirror plane. Several other concepts, the first of which is rotational symmetry, need introduction. Not only do rotation axes occur in the natural world, but the principle of rotation has significant application in the inventions of man. The wheel, which is one of man's most useful and labor-saving inventions, is most easily understood as a rotation of a form around an axis perpendicular to its plane. For example, the wheels and gears used in bicycles, automobiles, and tractors rotate around axes. Likewise, the petals on a daisy are arranged in a rotational manner around a stem; and the sections of a grapefruit or the pieces of a pie all are related by a rotation axis.

When the symmetry operation around a rotation axis is repeated, the parts of the form coincide several times. The smallest angle of rotation which causes the form to coincide with itself is called the *elementary angle* of rotation. The distance from the rotation axis to equivalent parts and the angular relationships at the rotation axis are the same for all parts of the object. Repeated rotation of an element through an elementary angle around a rotation axis produces a form which has a rotation axis of symmetry.

The number of times the object superimposes on itself in a complete (360°) rotation is called the *order* of the rotation axis or the *folds* of rotation. A two-dimensional axial object may have from one to an infinite number of identical forms, depending on the order of the axis. A form has any number of one-fold axes, such that if the form is rotated through 360°, it comes back into coincidence with itself. This "one-fold" rotation axis must be included for theoretical reasons. A form that has axes of only one-fold rotation is called an identity. Some forms, on the other hand, have infinite folds of rotation that is, they come into coincidence no matter how small or large the elementary angle of rotation; a perfect circle. Between these extremes, a large number of forms exist which have a specific number of folds of rotation. Illustration of rotation axes is here limited to one-, two-, three-, four-, and six-fold rotation axes, for no other axes of rotation are possible when a whole system is considered which repeats itself, after being moved along lines in a plane.



A translation axis—p

Translation axes occur in the natural world, but the principal of translation has significant application in the inventions of man. Ropes, cables, chains, and wires are examples of repetition by translation occurring in forms made by man. Forms which have translation axes repeat themselves in a given length. They are often used to transmit effort or power over a distance, to hold other forms in relationship, to move information from one place to another, or to decorate a surface. A translation axis may occur along with other symmetry relationships.

When the symmetry operation along a translation axis occurs, the parts of the form coincide several times. The translation does not change the relative positions of the parts of the form but only their positions relative to the other equivalent parts along a straight line. The shortest distance causing an element to superimpose with itself is called the *elementary translation* or *period*. The elementary translation is equal between all neighboring parts of the form, and it may be repeated many times in either direction along the translation axis. Any line parallel to the translation axis is equivalent to the translation axis. The translation axis can be taken to be any line which lies parallel to this form.

Translation axes may occur in one direction or two non-parallel directions on the surface of a plane. When two directions of translation are allowed, four two-dimensional lattice systems can be defined. The two-dimensional lattice systems are defined as having two axes of translation which are not parallel and as having an angular relationship between the two axes of translation. The four lattice systems are called oblique, rectangular, square, and hexagonal. A fifth translation system exists which has a relationship to the rectangular lattice. It is call the centered rectangular or rhombic translation lattice. The measurable qualities of the four two-dimensional lattice systems are discussed in the following section.

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The two-dimensional, oblique lattice system has two non-parallel translation periods which are not equal, and the angle between the two translation periods is not equal to 90°.

The two-dimensional, rectangular lattice system has two non-parallel translation periods which are not equal, and the angle between the two translation periods is equal to 90°.

The two-dimensional, square lattice system has two non-parallel translation periods which are equal, and the angle between the two translation periods is equal to 90°.

The two-dimensional, hexagonal lattice system has two non-parallel translation periods which are equal, and the angle between the two translation periods is equal to 120° or 60°. The 60° angle identifies an equilateral triangle which divides the hexagon into six parts.

The two-dimensional, centered rectangular lattice system is related to the rectangular lattice. It is possible to translate to a point in the center as well as to the four usual points associated with the rectangular lattice unit. This is a unique translation which is possible in the rectangular lattice. If this were to occur in the other lattice systems, a unique relationship is not created but only a lattice unit which has translation lengths different from the original length. This relationship is possible in the context of the two-dimensional bi-axial forms. The centered rectangular lattice is often called a rhombic lattice.

A glide plane—g

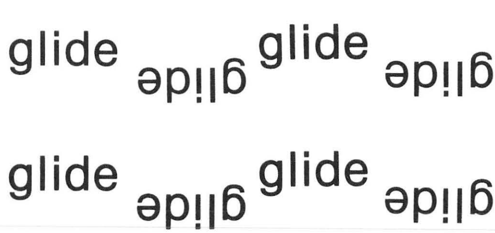
The compound symmetry operation of a glide plane is possible in two-dimensional forms. The glide plane operation can be understood as the sequential application of the reflection and translation operations. The glide plane operation makes a form coincide with itself after reflection through a mirror plane which lies parallel to the translation axis and translation through one half of the elementary translation in the form. If the mirror plane and the elementary translation remain constant, it does not matter which symmetry operation is performed first—the results are the same. If considered separately, the reflection and translation do not make the form coincide with itself; it is the compound operation which makes the form coincide. When performed twice, the glide plane operation creates the elementary translation in the form.

Orientation to the system

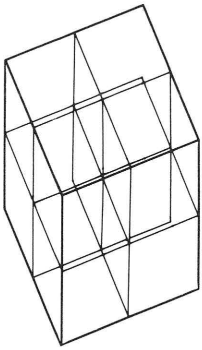
As a means of orientation, three symmetrical groups are presented in this paper. A two unit by two unit square can be used to represent the three groups and their relationship to one another.

The three symmetry groups will be related by a *general* orientation of the axes in the system. The axes are tools used to understand the relationships in each group and among the three groups. The axes give consistency to certain relationships and are a standard for the notation systems. There are exceptions to the *general* orientation of the axes when three-, four-, and six-fold rotation occurs. The axes identified relate to the forms under consideration, not to the world at large or the usual understanding of axes.

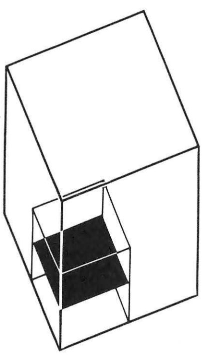
Relationships in the *general* orientation refer to the a-axis pointing off the two unit by two unit square, the b-axis pointing back on the square, and the c-axis pointing to the right in the square. There are specific exceptions to the general orientation of the b- and c-axes, which change with specific rotational transformations.
tions.

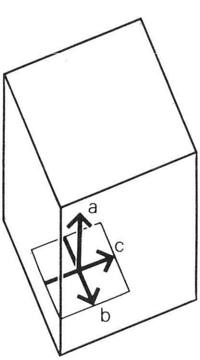


The notation system



Written and graphic notation systems are necessary for the forms which are members of the symmetry groups. The written and graphic notation systems parallel each other. The notation systems are based on a coordinate axis system, which places three axes in specific relationships to one another. The general orientation of axes serves as a standard for all relationships notated. There are exceptions to the general orientation when forms which have three-, four-, and six-fold rotation are considered.





The written notation system

The written notation system uses letters and numbers placed in sequence to denote the symmetry operations found in periodic forms. The written notation has four positions. The first position denotes the symmetry operations of translation, the second denotes the symmetry operations related to the a-axis, the third denotes the symmetry operations related to the b-axis, and the fourth denotes symmetry operations related to the c-axis.

The symmetry operations of a form are symbolized in relationship to the axes with which they are identified. The written symbols in the second, third, and fourth positions denote the symmetry operations related to the coordinate axes. The number 1 is used in the written notation to maintain the relative positions of the written symbols.

In the written notation system the presence of a translation axis is denoted with a letter and subscript in the first position. The primitive translations are denoted with the letter p. The centered rectangular translation net is denoted with the letter c. The subscript used in conjunction with the letter denotes the number of translation axes present; zero denotes no translation, one denotes one axis of translation, two denotes two axes of translation.

The presence of a mirror plane is denoted with the letter m. Mirror planes are noted perpendicular to axes. A mirror plane which is denoted in the b-axis position lies perpendicular to the b-axis.

The presence and order of a rotation axis in a form is denoted with a whole number. Rotation axes are noted parallel to axes. A rotation axis which is denoted in the a-axis position lies parallel to the a-axis, etc.

The presence of a glide plane is denoted with the letter g and a subscript. Glide planes are noted perpendicular to axes. The subscript notes the translation direction present in the glide plane. A glide plane which is denoted g_c in the b-axis position lies perpendicular to the b-axis and has a translation component parallel to the c-axis.

The graphic notation system uses symbols placed in relationship to denote the symmetry operations found in forms. The graphic notation has four positions. The first position denotes the lattice unit found in the form, the second denotes the symmetry operations in relationship to the a-axis, the third denotes the symmetry operations in relationship to the b-axis, and the fourth denotes the symmetry operations in relationship to the c-axis.

The symmetry operations in a form are symbolized in relationship to the axes with which they are identified. The graphic symbols in the second, third, and fourth positions denote the symmetry operations in relationship to the coordinate axes. An unfilled space is used in the graphic notation to maintain the relative positions of the graphic symbols.

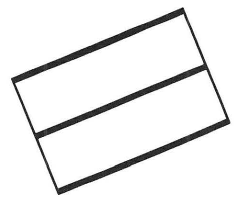
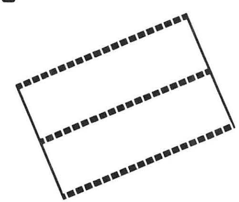
In the graphic notation system the translation unit is denoted in the first position. The translation unit is determined by the symmetry operations in the form. The three coordinate axes are shown in relationship to the translation unit in the first position.

The location of a mirror plane is denoted with a line. Mirror planes are noted perpendicular to axes. A mirror plane which is denoted in the b-axis position lies perpendicular to the b-axis.

The location and order of the rotation axis is denoted with a symbol having the same number of sides as the order of the rotation axis. Rotation axes are noted parallel to axes. A rotation axis which is denoted in the a-axis lies parallel to the a-axis.

The location of a glide plane is denoted with a broken line. Glide planes are noted perpendicular to axes. A glide plane which is denoted in the b-axis position lies perpendicular to the b-axis.

The notation systems for the two-dimensional bi-axial forms

	Printed symbol	Graphic symbol	Operator
Translation axes	p_2 c_2	Various — Dependent upon symmetry operations present	Movement along two axes
A mirror plane	m		Reflection plane
A rotation axis	1 2 3 4 6	None ⬮ ▲ ■ ⬢	One-fold (360°) axis Two-fold (180°) axis Three-fold (120°) axis Four-fold (90°) axis Six-fold (60°) axis
A glide plane	g_b		Reflection and a one-half translation length in the b-axis
	g_c		Reflection and a one-half translation length in the c-axis

Group 1
The two-dimensional centric forms

Having defined the system, the symmetry operations and an orientation to the system, we can now define the three groups of forms—the two-dimensional centric forms, the two-dimensional mono-axial forms, and the two-dimensional bi-axial forms—considered in this paper, and present the classes in each. It is important to remember that the first group contains a singular point, the second group a singular axis, and the third group two axes.

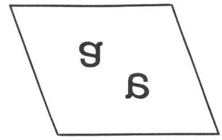
A two-dimensional centric form is a figure which is organized around a singular point on a plane. (The term two-dimensional indicates that the relationships occur on one side of the plane.) The term centric indicates that the relationships occur around a singular point or center. A two-dimensional centric form is a figure which possesses a singular point on a one-sided plane. The space considered in this group of forms, and the transformations which are possible, are limited by this definition.

Two series of forms can be identified by using the symmetry operations of mirror planes and rotation axes in the two-dimensional centric forms. A series is a subset composed of a number of classes of forms which have common transformations. The first series, forms with a single rotation axis—r, is composed of classes which have rotation axes as a common transformation. The second series, forms with a single rotation axis and a parallel mirror plane—r || m, is composed of classes which have as a common transformation a mirror plane parallel to the rotation axes.

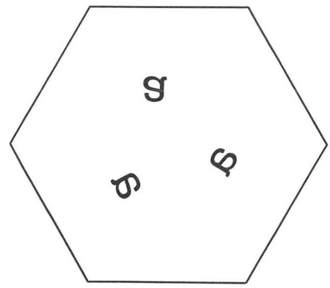
Finite objects on a one-sided plane contain at least one singular point. If more than one singular point exists in such a finite object, it possesses no rotational symmetry other than one-fold axes.

When we consider the limits of the system of which the two-dimensional finite forms are a part, there are only ten classes in the group. Other finite forms are possible, but they lie outside the system. We illustrate the ten two-dimensional finite forms and their notation systems.

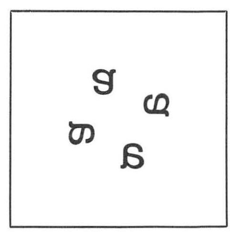
Forms with a single rotation axis—r



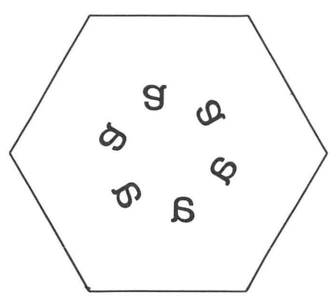
p_o 2 1 1



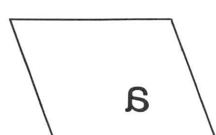
p_o 3 1 1



p_o 4 1 1

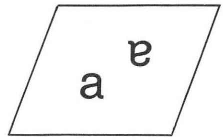


p_o 6 1 1



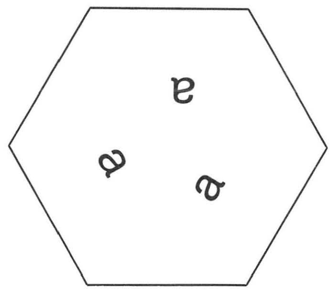
p_o 1 1 1

Enantiomorphism

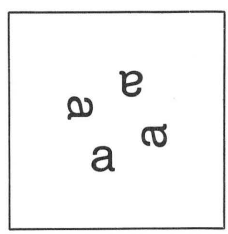


p_o 2 1 1

Enantiomorphism is a concept which becomes important when dealing with forms containing only axes of rotation. The transformations around a rotation axis allow rotation in a left-handed or a right-handed direction. Two forms which are constructed of similar parts and which have the same order of rotation are enantiomorphic if they exist as left- and right-handed forms. The two forms have the same shape, size and order of rotation, but they do not coincide when placed on top of each other. An enantiomorphic relationship exists between a form with a single rotation axis and the mirror image of that form.



p_o 3 1 1



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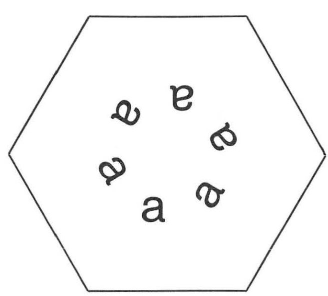
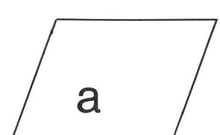
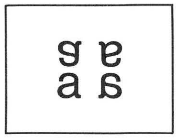


p_o 6 1 1

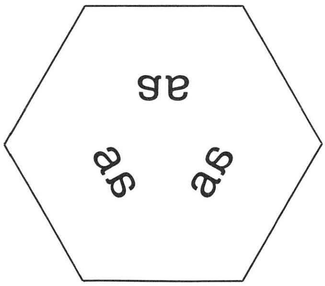


p_o 1 1 1

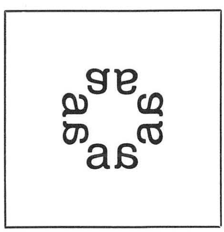
Forms with a single rotation axis and a parallel mirror plane—r || m



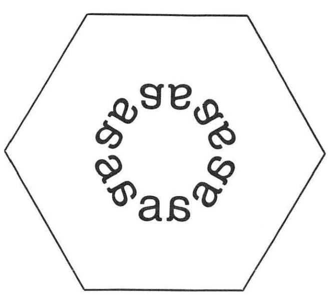
p_o 2 m m



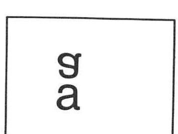
p_o 3 m 1



p_o 4 m m



p_o 6 m m



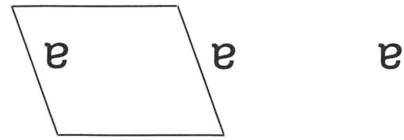
p_o 1 m 1

Generalizations

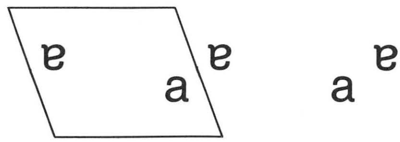
The following generalizations can be made about the forms with a single rotation axis and a parallel mirror plane: the interaction of an *even* order rotation axis and a mirror plane produces two sets of different mirror planes intersecting in the single axis; and the interaction of an *odd* order rotation axis and a mirror plane produces one set of identical mirror planes interesecting in the single axis. These generalizations can be applied to rotation axes of any order, and are denoted in the written and graphic notation of the forms.

Group 2
The two-dimensional mono-axial forms

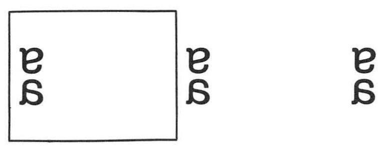
The two-dimensional mono-axial form is a figure which is organized about a singular axis on a plane. The term mono-axial indicates that the relationships occur about a singular axis. A two-dimensional mono-axial form is a figure which occurs about a singular axis, on a one-sided plane. The space considered in this group of forms, and the transformations which are possible, are limited by this definition. There are seven two-dimensional mono-axial forms.



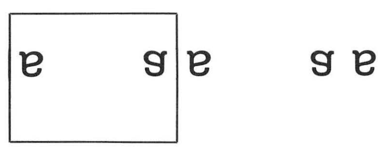
p_1 1 1 1



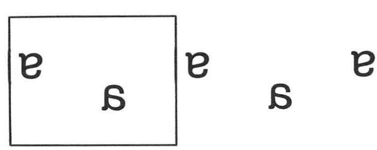
p_1 2 1 1



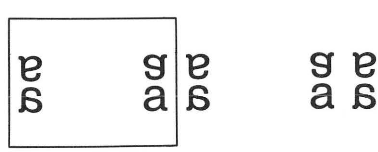
p_1 1 m 1



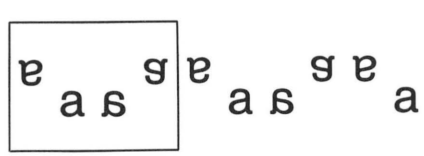
p_1 1 1 m



p_1 1 g_c 1



p_1 2 m m



p_1 2 g_c m

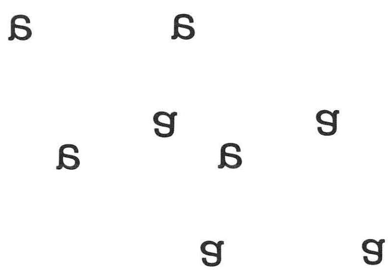
Group 3

The two-dimensional bi-axial forms

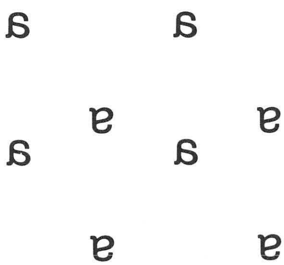
A two-dimensional bi-axial form is a figure which is organized about two axes on a plane. A two-dimensional bi-axial form is a figure which occurs about two axes on a one-sided plane. The space considered in this group of forms, and the transformations which are possible, are limited by this definition. There are seventeen two-dimensional bi-axial forms.

p_2 1 m 1

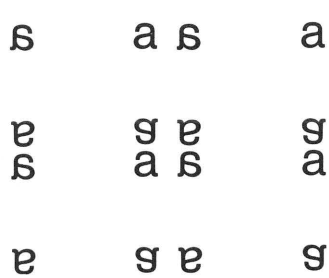
p_2 1 1 1



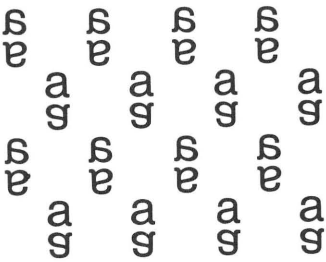
p_2 2 1 1



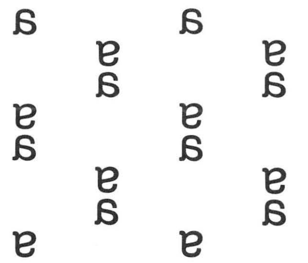
p_2 1 g_c 1



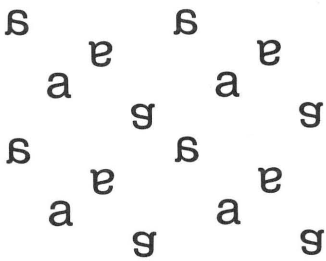
p_2 2 m m



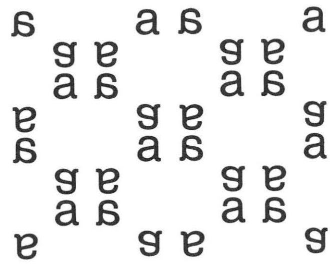
p_2 2 m g_b



c_2 1 m 1

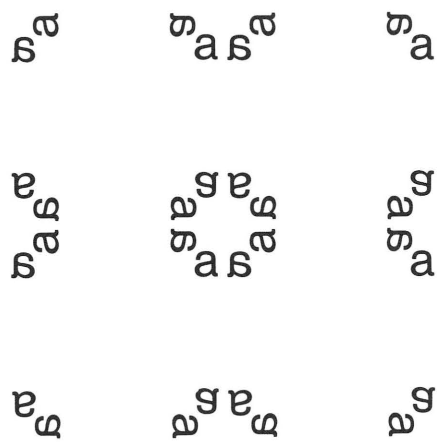


p_2 2 g_c g_b

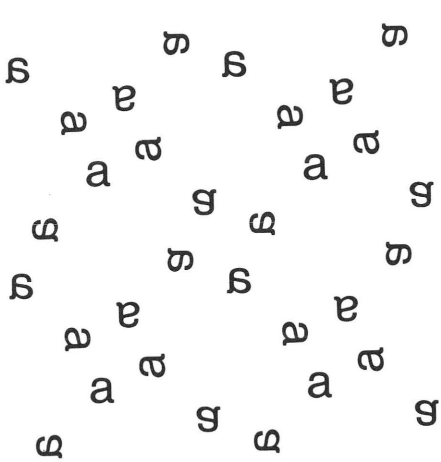


c_2 2 m m

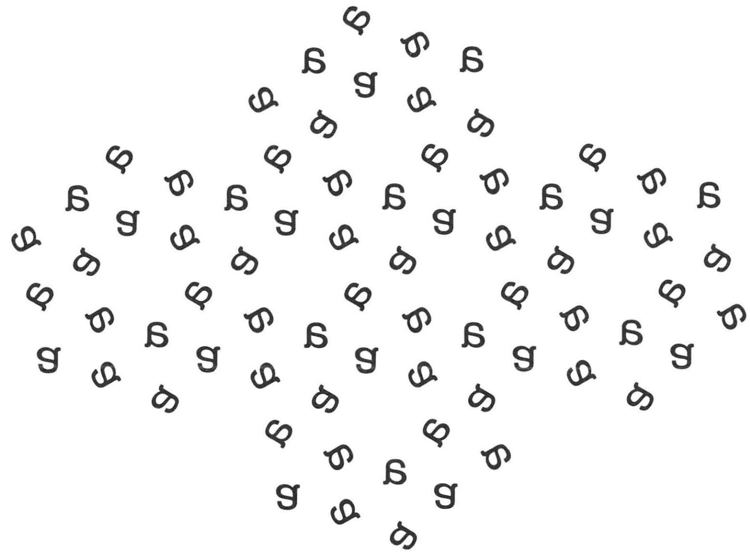
p_2 4 1 1



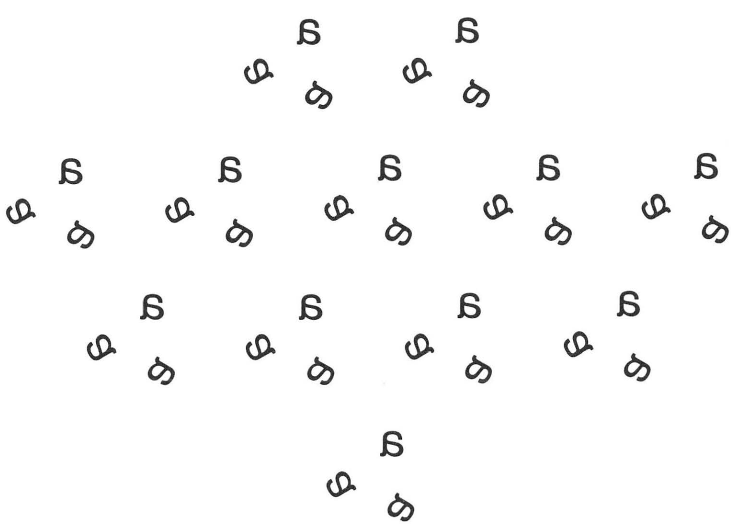
p_2 4 m m



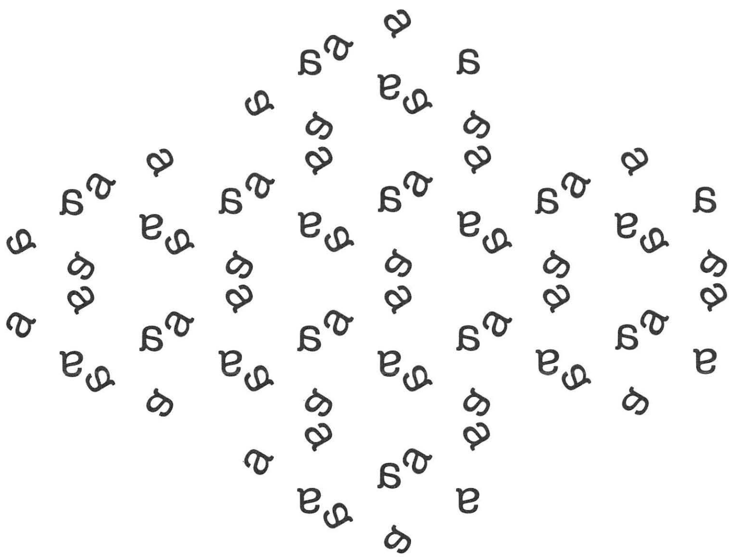
p_2 4 g_b m



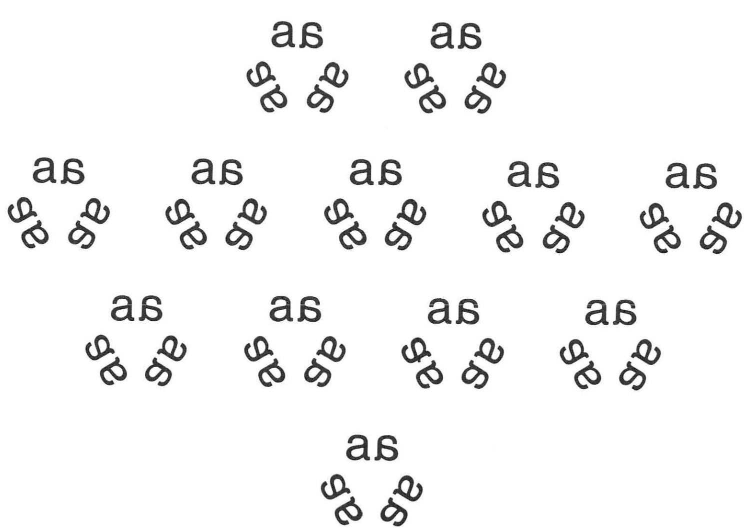
p_2 6 1 1



p_2 3 1 1



p_2 3 m 1

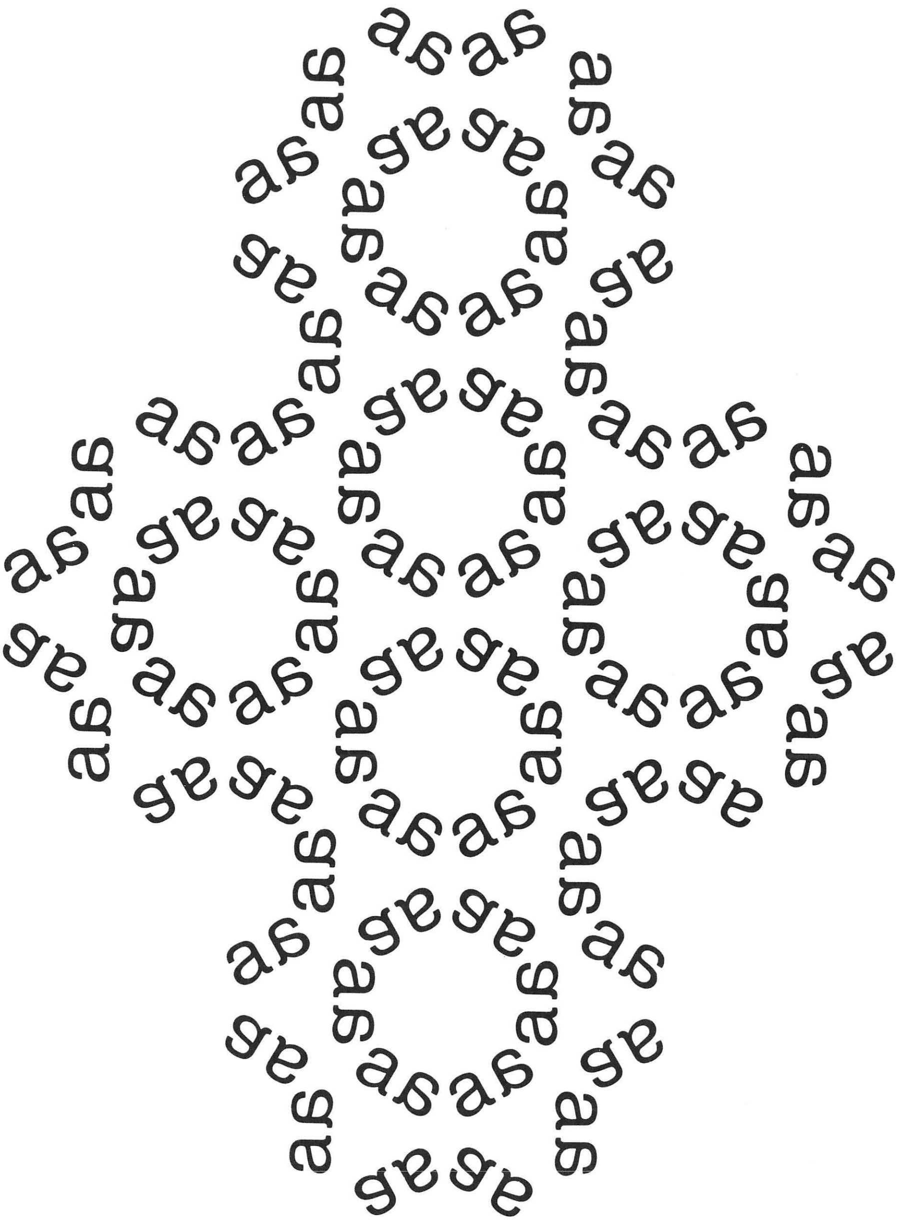


p_2 3 1 m

We have presented the classes which belong to the three groups of the regular two-dimensional forms. The relationships present in these three groups can be applied to an infinite number of forms. Every culture throughout history has made forms which can be understood as belonging to these three groups. Examples range from modern corporate logos to historic wall coverings.

The strength of this system is that it allows consideration of a great number of forms which may not appear to have relationships in common. It looks beyond the surface to the underlying structure of the form and the geometry associated with this time and space. The system considers how the element, an identity, the smallest portion, is manipulated or transformed in relationship. As one becomes familiar with the idea of a system in relationship, transformations are not limited to mirror planes, rotation axes, translation axes, and glide planes but can become any relationship (color, size, placement) which one chooses to consider.

Every problem has restraints, constants, and variables, and it is within these parameters that a number of solutions are possible. Given the restraints of the space considered and the few transformations allowed in the system, the numbers of forms are not great. But when one brings to this system infinite numbers of visual elements (identities), the forms possible become infinite. To know the possible relationships does not limit creativity but gives the information necessary to create. The challenge of discovering visual forms remains, but these forms can reflect the order present in this time and space as man has come to understand it.



p_2 6 mm

References

1
The symmetry of three-dimensional forms is listed in: **Internationl Tables for X-ray Crystallography**, Vol. 1., The International Union of Crystallography, Birmingham, England: The Kynock Press, 1952.