TRUNCATING CLASSICAL SOLUTIONS OF BENDING OF SQUARE WOOD-BASE PLATES

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ABSTRACT

Classical solutions for bending of orthotropic plates consist of infinite series and flexural rigidities. The computations of solutions would not be feasible if the computer were not available. The objective of this study was to truncate classical solutions into simplified solutions. Simplified solutions consist of only flexural rigidities with coefficients. Because of the elimination of infinite series in classical solutions, the simplified solutions are easier to solve and can be calculated without using a computer. The simplified solutions give practically the same results as those of the classical solutions. Limited experimental verification of these solutions was made using southern pine plywood and composite sandwich panels (particleboard with veneer faces).

Keywords: Wood-base plates, bending.

INTRODUCTION

The classical solutions for bending of orthotropic plates presented in the engineering books (Lekhnitskii 1968; Timoshenko and Woinowsky-Krieger 1959) are usually expressed in terms of flexural rigidities coupled with infinite series. Some of the infinite series are rapidly convergent so that the approximate solutions can be quickly obtained. But some other series are slowly convergent, thus the calculations become tedious and require the use of a computer. The purpose of this study is to truncate the classical solutions into simplified solutions for bending of square wood-base plates. The truncating process is to replace the infinite series with coefficients.

FUNDAMENTAL THEORY

The center deflection for an isotropic rectangular thin plate, simply supported on four edges and uniformly loaded, is given by Timoshenko (1959) as follows:

$$W_{max} = \frac{16q}{\Pi^6 D} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{(m+n)/2-1}}{mn(m^2/a^2 + n^2/b^2)^2} \qquad (1)$$

where

W_{max} = maximum deflection of plate
q = uniformly distributed load
D = flexural rigidity = $Eh^3/12(1 - \nu^2)$
E = Young's modulus
h = thickness of plate
ν = Poisson's ratio
m, n = odd integers
a, b = dimensions of plate in X-axis and Y-axis

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TABLE 1. *Coefficient K_1 calculated from Eq. (11) for simply supported square plate with concentrated load at center.*

Type of panel[a]	K_1				
	1 term[b]	4 terms	25 terms	100 terms	400 terms
PLW-3/8 in., 3-ply	0.04106	0.04837	0.05083	0.05120	0.05130
PLW-1/2 in., 4-ply	0.04106	0.04573	0.04728	0.04752	0.04758
PLW-5/8 in., 5-ply	0.04106	0.04501	0.04634	0.04655	0.04660
SDW-5/8 in., 3-ply	0.04106	0.04686	0.04882	0.04912	0.04920
SDW-7/8 in., 5-ply	0.04106	0.04447	0.04562	0.04580	0.04584
Average	0.04106	0.04609	0.04778	0.04804	0.04810
PTB-3/8 in.	0.04106	0.04500	0.04634	0.04654	0.04659

[a] The first 3 letters designate plywood (PLW), sandwich (SDW), and particleboard (PTB). The following numbers designate the thicknesses of panels in inches. Sandwich panels were made with 3/8-in. urea particleboard core and 1/8-in. veneers.
[b] Number of terms used for computation.

If the plate is square (a = b), then Eq. (1) becomes:

$$W_{max} = \frac{16qa^4}{\Pi^6 D} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{(m+n)/2-1}}{mn(m^2 + n^2)^2} \tag{2}$$

This is a rapidly converging series. By taking only the first four terms of the series, the approximation is obtained as follows:

$$W_{max} = 0.00406qa^4/D \tag{3}$$

Although the computation of the series was made with 2,000 terms with a computer, the coefficient still remained the same. Thus, the accuracy of Eq. (3) is verified and it will give satisfactory results for isotropic plates.

If the isotropic square plate is simply supported on four edges and with a concentrated load (P) at the center, the center deflection (Timoshenko and Woinowsky-Krieger 1959) is:

$$W_{max} = \frac{4Pa^2}{\Pi^4 D} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} 1/(m^2 + n^2)^2 \tag{4}$$

By taking the first 36 terms of the series, one obtains the approximate solution:

$$W_{max} = 0.0116Pa^2/D \tag{5}$$

Computation of the series was made with 2,000 terms using a computer, and the coefficient still remained the same.

In the case of orthotropic plates, it is assumed that the material has three mutually perpendicular planes with respect to its elastic properties. When a plate is simply supported on all edges and uniformly loaded, the solution can be obtained by the same theory (Timoshenko and Woinowsky-Krieger 1959) as follows:

$$W_{max} = \frac{16q}{\Pi^6} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{(m+n)/2-1}}{mn(m^4D_x/a^4 + 2m^2n^2H/a^2b^2 + n^4Dy/b^4)} \tag{6}$$

where

$$D_x = E_x h^3/12(1 - \nu_{xy}\nu_{yx})$$

TABLE 2. *Coefficient K_2 calculated from Eq. (12) for simple supported square plate with uniform load.*

Type of panel[a]	K_2				
	1 term[b]	4 terms	25 terms	100 terms	400 terms
PLW-3/8 in.	0.01664	0.01575	0.01582	0.01582	0.01582
PLW-1/2 in.	0.01664	0.01610	0.01614	0.01614	0.01614
PLW-5/8 in.	0.01664	0.01620	0.01623	0.01623	0.01623
SDW-5/8 in.	0.01664	0.01595	0.01601	0.01600	0.01600
SDW-7/8 in.	0.01664	0.01627	0.01630	0.01630	0.01630
Average	0.01664	0.01605	0.01610	0.01610	0.01610
PTB-3/8 in.	0.01664	0.01620	0.01623	0.01623	0.01623

[a] Same as in Table 1.
[b] Same as in Table 1.

$D_y = E_y h^3/12(1 - \nu_{xy}\nu_{yx})$
E_x = Young's modulus in X-axis
E_y = Young's modulus in Y-axis
ν_{xy}, ν_{yx} = Poisson's ratios
$H = D_1 + 2D_{xy}$
$D_1 = E_x\nu_{yx}h^3/12(1 - \nu_{xy}\nu_{yx}) = E_y\nu_{xy}h^3/12(1 - \nu_{xy}\nu_{yx})$
$D_{xy} = G_{xy}h^3/12$

In the case of an isotropic plate $D_x = D_y = H = D$, and Eq. (6) coincides with Eq. (1). In the case of a square plate (a = b), Eq. (6) can be simplified as:

$$W_{max} = \frac{16qa^4}{\Pi^6}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{(m+n)/2-1}}{mn(m^4D_x + 2m^2n^2H + n^4D_y)} \tag{7}$$

For a square orthotropic plate with all edges simply supported and with a concentrated load applied at the center of plate, the following solution for the maximum deflection is given (Lekhnitskii 1968):

$$W_{max} = \frac{4Pa^2}{\Pi^4}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m^4D_x + 2m^2n^2H + n^4D_y} \tag{8}$$

This solution is analogous to Eq. (4) for an isotropic plate. These solutions (Eqs. [7] and [8]) for orthotropic plates consisted of infinite series coupled with flexural rigidities (D_x, D_y, and H); hence the computations of solutions are very time-consuming.

SIMPLIFYING ASSUMPTIONS

Analogizing from Eqs. (3) and (5), solutions may be assumed by using only the flexural rigidities of orthotropic plates. (a) For concentrated load (P) at center:

$$W_{max} = K_1\frac{Pa^2}{D_x + 2H + D_y} \tag{9}$$

(b) For uniform load (q):

$$W_{max} = K_2\frac{qa^4}{D_x + 2H + D_y} \tag{10}$$

TABLE 3. *Factors affecting the values of coefficient K_1 and K_2.*

E_x/E_y ratio	E_x	E_y (1,000 psi)	G_{xy}	h (in.)	ν_{xy}	ν_{yx}	K_1[a]	K_2[a]
Effect of E_x/E_y ratio								
1	1,150	1,150	100	0.860	0.300	0.300	0.04556	0.01632
2	1,600	800	100	0.860	0.409	0.205	0.04585	0.01630
3	1,700	567	100	0.600	0.380	0.127	0.04625	0.01626
4	1,800	450	100	0.600	0.368	0.092	0.04672	0.01622
5	1,900	380	100	0.480	0.456	0.091	0.04727	0.01617
10	1,900	190	100	0.600	0.473	0.047	0.04955	0.01597
15	2,100	140	100	0.350	0.310	0.021	0.05148	0.01580
20	2,200	110	100	0.300	0.400	0.020	0.05313	0.01567
Effect of G_{xy}								
5	1,900	380	80	0.480	0.456	0.091	0.04721	0.01617
5	1,900	380	100	0.480	0.456	0.091	0.04727	0.01617
5	1,900	380	120	0.480	0.456	0.091	0.04734	0.01616
5	1,900	380	140	0.480	0.456	0.091	0.04740	0.01616
5	1,900	380	160	0.480	0.456	0.091	0.04746	0.01615
5	1,900	380	180	0.480	0.456	0.091	0.04752	0.01615
Effect of Poisson's ratio								
5	1,900	380	100	0.480	0.6	0.12	0.04737	0.01616
5	1,900	380	100	0.480	0.5	0.10	0.04730	0.01617
5	1,900	380	100	0.480	0.4	0.08	0.04724	0.01617
5	1,900	380	100	0.480	0.3	0.06	0.04717	0.01618
5	1,900	380	100	0.480	0.2	0.04	0.04710	0.01618
Effect of actual values of E_x and E_y (at the same E_x/E_y ratio)								
5	2,200	440	100	0.480	0.456	0.091	0.04723	0.01617
5	2,000	400	100	0.480	0.456	0.091	0.04726	0.01617
5	1,800	360	100	0.480	0.456	0.091	0.04729	0.01617
5	1,600	320	100	0.480	0.456	0.091	0.04733	0.01616
5	1,400	280	100	0.480	0.456	0.091	0.04739	0.01616
5	1,200	240	100	0.480	0.456	0.091	0.04746	0.01615
5	1,000	200	100	0.480	0.456	0.091	0.04756	0.01615

[a] K_1 and K_2 were calculated according to Eqs. (11) and (12) with 400 terms of the series.

From Eqs. (8) and (9), one obtains

$$K_1 = \frac{4(D_x + 2H + D_y)}{\Pi^4} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^4D_x + 2m^2n^2H + n^4D_y} \quad (11)$$

From Eqs. (7) and (10), one obtains

$$K_2 = \frac{16(D_x + 2H + D_y)}{\Pi^6} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{(m+n)/2-1}}{mn(m^4D_x + 2m^2n^2H + n^4D_y)} \quad (12)$$

By using the orthotropic elastic constants of plywood and sandwich determined in a previous report (Lee and Biblis 1977), the coefficients K_1 and K_2 are computed with a computer and listed in Tables 1 and 2. The average value of K_1 calculated with 400 terms is 0.0481. Thus, the approximate solution for a simply supported square wood-base plate with a concentrated load at the center is:

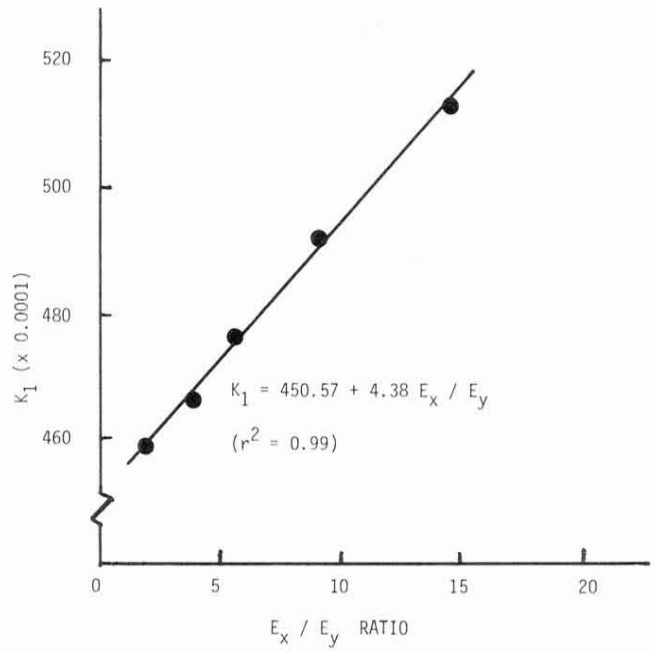

FIG. 1. Coefficient K_1 at various E_x/E_y ratios of square wood-base orthotropic plates in bending.

$$W_{max} = 0.04810\frac{Pa^2}{D_x + 2H + D_y} \tag{13}$$

For a wood-base orthotropic plate with uniformly distributed load, the average value of K_2 is 0.0161. Thus, the approximate solution is:

$$W_{max} = 0.01610\frac{qa^4}{D_x + 2H + D_y} \tag{14}$$

The coefficients K_1 and K_2 calculated for ⅜-in. isotropic particleboard are listed in the last rows of Tables 1 and 2. They are used here to check whether the simplified solutions for orthotropic plates would agree with the solutions for isotropic plates. In this case, the solutions applied to isotropic plates of ⅜-in. particleboard are:

$$W_{max} = 0.04659\frac{Pa^2}{D_x + 2H + D_y} \tag{15}$$

$$W_{max} = 0.01623\frac{qa^4}{D_x + 2H + D_y} \tag{16}$$

Since for an isotropic plate $D_x = D_y = H = D$, Eqs. (15) and (16) agree very well with Eqs. (5) and (3), respectively. This indicates that the approximate solutions for orthotropic plates are applicable to isotropic plates as well.

Calculated values of K_1 and K_2 (Tables 1 and 2) varied among the different constructions of panels. A further investigation was made to determine the effect of E_x/E_y ratio, G_{xy}, Poisson's ratios, and actual values of E_x and E_y (at the same

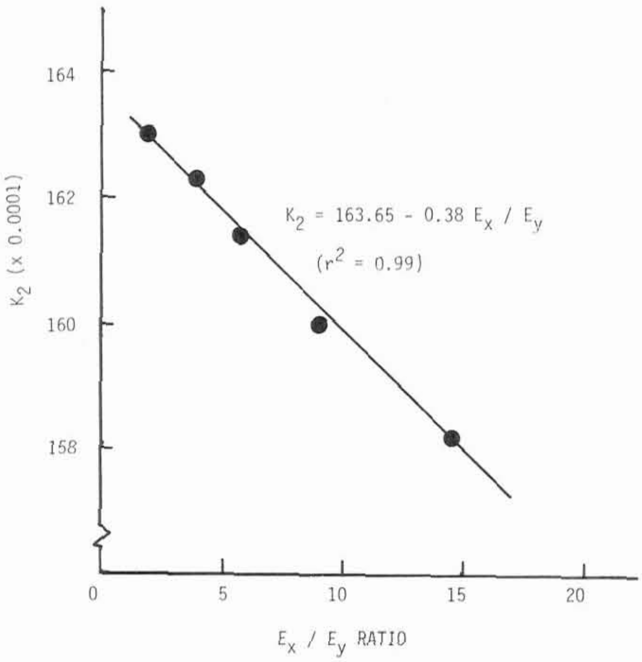

FIG. 2. Coefficient K_2 at various E_x/E_y ratios of square wood-base orthotropic plates in bending.

E_x/E_y ratio) on K_1 and K_2. The elastic constants E_x, E_y, G_{xy}, h, ν_{xy}, and ν_{yx} were chosen in the ranges corresponding to those of panels tested in the previous study (Lee and Biblis 1977). The results are presented in Table 3.

First, a series of E_x/E_y ratios from 1 to 20 were chosen to compute K_1 and K_2. Coefficient K_1 increases from 0.04556 to 0.05313 and K_2 decreases from 0.01632 to 0.01567 as E_x/E_y ratio increases from 1 to 20. The effect of G_{xy}, Poisson's ratios, and actual values of E_x and E_y (at the same E_x/E_y ratio) on coefficients K_1 and K_2 are also presented in Table 3. Although K_1 and K_2 are affected by these three factors, the magnitude of changes in K_1 and K_2 is relatively small and insignificant compared to those affected by the E_x/E_y ratio.

The relationships of K_1 and K_2 values versus E_x/E_y ratios of three types of plywood and two types of sandwich are shown in Figs. 1 and 2, respectively. Linear relation between K_1, K_2, and E_x/E_y ratio was observed to have a highly significant coefficient of determination (0.99).

EXPERIMENTAL VERIFICATION

One panel (4 by 8 ft) of each construction was made at the same time with the same quality as those boards tested in the previous study (Lee and Biblis 1977). Three 20-in.-square plates were cut from each of three southern pine plywood constructions and two composite sandwich constructions. All plates were cut with face grain oriented parallel to two opposite edges of the specimen. In addition, three 20-in.-square particleboard plates (3/8-in.-thick) were tested. All plates were conditioned to reach equilibrium moisture content at 65% relative humidity and 72 F temperature prior to testing.

Plates were simply supported on four edges with 1/2-in.-diameter steel bars.

TABLE 4. *Deflection of plate bending test with all edges simply supported and with a = b = 19.5 in.*

Specimen no.	W_{max} (in.) at p = 100 pounds			W_{max} (in.) at q = 1 psi	
	(classical)	(simplified)	(exp.)	(classical)	(simplified)
	Col. (1)	Col. (2)	Col. (3)	Col. (4)	Col. (5)
PTB-⅜ in.-1	0.2199	0.2199	0.2411	0.2914	0.2914
PTB-⅜ in.-2	0.2199	0.2199	0.2129	0.2914	0.2914
PTB-⅜ in.-3	0.2182	0.2182	0.2224	0.2891	0.2891
Average	0.2193	0.2193	0.2255		
PLW-⅜ in.-1	0.1960	0.1968	0.2071	0.2299	0.2299
PLW-⅜ in.-2	0.1960	0.1968	0.1911	0.2299	0.2299
PLW-⅜ in.-3	0.1977	0.1985	0.1766	0.2318	0.2318
Average	0.1966	0.1973	0.1916		
PLW-½ in.-1	0.0727	0.0726	0.0792	0.0937	0.0938
PLW-½ in.-2	0.0727	0.0726	0.0754	0.0937	0.0938
PLW-½ in.-3	0.0727	0.0726	0.0809	0.0937	0.0938
Average	0.0727	0.0726	0.0785		
PLW-⅝ in.-1	0.0353	0.0354	0.0383	0.0467	0.0467
PLW-⅝ in.-2	0.0348	0.0348	0.0395	0.0460	0.0460
PLW-⅝ in.-3	0.0351	0.0352	0.0361	0.0465	0.0465
Average	0.0351	0.0351	0.0380		
SDW-⅝ in.-1	0.0415	0.0414	0.0437	0.0513	0.0514
SDW-⅝ in.-2	0.0411	0.0410	0.0482	0.0508	0.0509
SDW-⅝ in.-3	0.0407	0.0406	0.0435	0.0503	0.0504
Average	0.0411	0.0410	0.0451		
SDW-⅞ in.-1	0.0099	0.0099	0.0106	0.0134	0.0134
SDW-⅞ in.-2	0.0100	0.0100	0.0117	0.0135	0.0135
SDW-⅞ in.-3	0.0098	0.0098	0.0122	0.0133	0.0133
Average	0.0099	0.0099	0.0115		

Column (1) was calculated from Eq. (8) with 400 terms.
Column (2) was calculated from Eq. (9) with K_1 from Fig. 1.
Column (3) was experimental result.
Column (4) was calculated from Eq. (7) with 400 terms.
Column (5) was calculated from Eq. (10) with K_2 from Fig. 2.

Parallel bars were spaced 19.5 in. and supported full length by four columns. A concentrated load was applied at the center of the plate through a 2.25-in.-diameter disk. Static load was increased in 20-pound increments up to 100 pounds by using the Instron Testing Machine calibration weights.

Deflections were measured at the center of the plate directly beneath the load by a dial gauge with 0.0001-in. precision and 0.5-in. range. Deflection measurement was taken as soon as each increment load was applied. Afterwards, the plate was turned over to test the opposite surface in the same manner.

RESULTS AND DISCUSSION

The classical solutions for bending deflections of particleboard, plywood, and sandwich plates, calculated with a computer based on Eqs. (7) and (8), are listed in Columns (1) and (4) of Table 4. These calculations involved computations of infinite series coupled with flexural rigidities. The flexural rigidities were computed using the elastic constants determined in a previous study (Lee and Biblis 1977).

The simplified solutions, calculated based on Eqs. (9) and (10) with the coefficients K_1 and K_2 shown in Figs. 1 and 2, are listed in Columns (2) and (5). The simplified solutions give results very close to the classical solutions. In fact, the differences are less than 1% in all cases in this study. Therefore, the proposed equations (Eqs. [9] and [10]) eliminate the tedious computations and give practically the same results. Even the overall approximate solutions (Eqs. [13] and [14]) are within a 6% error for wood-base orthotropic plates.

The experimental results of the plate bending test are listed in Column (3) of Table 4. The experimental results of ⅜-in. particleboard and ⅜-in. plywood gave the best fit with less than 3% difference from classical solutions. However, experimental results of sandwich board were up to 16% higher than that of classical solutions. The difference may be attributed to either material variation or loading-supporting method. The primary cause was believed to be material variation, since the panels tested in this study were assumed to be the same quality as those tested in the previous study (Lee and Biblis 1977), and the elastic constants determined in that study were used for computations of classical and simplified solutions.

The loading-supporting method may have some effect on the experimental results because plates were not all perfectly flat and small gaps existed between plate and supporting steel bars. Also, the 100-pound concentrated load produced more stress in thinner plates than in thicker plates. Thus, the deflections obtained for thicker plates represent only the initial portion of the elastic line. Another study (Superfesky et al. 1977) revealed that there was no significant difference between using a 4-in.- and a 1-in.-diameter loading disk to simulate concentrated load. Therefore, the loading on the disk (diameter = 2.25 in.) used in this study shall be considered as concentrated loading.

CONCLUSIONS

Conventionally classical solutions for orthotropic plate bending consist of infinite series coupled with flexural rigidities. A computer must be used to obtain the solutions. Two simplified solutions (Eqs. [9] and [10]) with the coefficients K_1 and K_2 shown in Figs. 1 and 2 give practically the same results as the classical solutions. The simplified solutions are much easier to solve and can be calculated without using a computer. Even the overall average solutions (Eqs. [13] and [14]) for wood-base plate bending give the results with error less than 6%. These solutions are reasonably easy to calculate and can be applied to any simply supported, square, orthotropic, wood-base plate subjected to concentrated center or uniform loadings.

Although the simplified solutions are affected by some of the elastic constants of the plate, the influences are insignificant (refer to Table 3) except for the E_x/E_y ratios.

The experimental tests of six different types of plates in bending indicate that some results are in close agreement with theory. The experimental results for the sandwich plates, however, were up to 16% higher than classical solutions. This was attributed to the material's variation and loading-supporting condition.

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