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SCATTERING OF ELASTIC SURFACE WAVES ON A LOCALIZED MECHANICAL LOAD OF THE SURFACE

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This paper presents a model of scattering of elastic surface waves (ESW) on a single reflective elements of dot structures loading the surface of a hexagonal structure with a sixfold rotation axis, normal to the plane of propagation. Qualitative results obtained with the use of TIERSTEN type boundary conditions [7] and reduced Green's functions presented by MARADUDIN and DOBRZYŃSKI [14] are given for a rectangular dot. Resultant components of the scattering amplitude were achieved with the utilization of an approximation of the stationary phase and approximation of far distances.

Przedstawiono model rozpraszania sprężystych fal powierzchniowych SFP na pojedynczych elementach odbiciowych struktur kropkowych, obciążających powierzchnię ośrodka heksagonalnego z sześciokrotną osią obrotu w kierunku normalnym do płaszczyzny propagacji. Podano jakościowe rezultaty dla kropki prostokątnej wykorzystując warunki brzegowe typu TIERSTENA [7] oraz zredukowane funkcje Greena podane przez MARADUDINA i DOBRZYŃSKIEGO [14]. Wynikowe składowe amplitudy rozpraszania uzyskano wykorzystując przybliżenie stacjonarnej fazy oraz przybliżenie dalekich odległości.

1. Introduction

This paper is aimed at the study of the effect of scattering of an elastic surface wave (ESW) on small mutually insulated loading centres considered as elements of reflection lines (so-called dots). Here small means that the greatest linear dimension of the centre is comparable to the length of incident wave. Such a dot is physically made by applying a thin metallic film with required shape, by picking a shallow groove, or by diffusion of another metal into the substrate, etc.

First devices applying the reflection of surface waves from dot arrays were constructed in 1976–1977 in the USA [1–4]. In the first approximation these arrays can be considered as discontinuous stripe structures, where the number of dots in each line determines the reflecting power, i.e. causes amplitude weighting.

In order to increase the efficiency of devices applying structures mentioned above, theoreticians now investigate the influence of the shape of a single reflector (dot) on the system's response. Besides the classical paper [5] which applies the method of partial waves to describe scattering on dots with circular symmetry profiles paper [6] applies Born's approximation for constructing a theory of scattering on dots with arbitrary profile, on anisotropic substrate. Paper [6] also contains a list of publications concerning mentioned above structures.

This paper contains a model of the scattering. It was built with the use of the Green's function method, familiar from the mechanics of continuous media. It can be applied to dots of arbitrary profile, located on the surface of a medium with hexagonal structure or (after algebraic modifications) an isotropic medium (see [14, 15]).

2. Formulation of the problem

Let us consider an elastic half-space loaded with a thin layer of material with constant thickness h and a given shape. "Loaded" means that the velocity of an acoustic transverse wave in the layer of material is lower than that of a transverse wave in the substrate [7, 8]. Figure 1 shows the geometry of the problem.

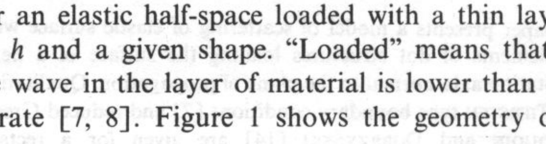

FIG. 1. A rectangular dot located on the surface of an elastic half-space on which a plane surface wave with wave vector k_R is scattered: a, b, h – dimensions of the dot

In further considerations we apply the summation convention Differentiation with respect to time is denoted with a dot above the symbol, and differentiation with respect to the coordinates is donoted with a comma preceding the index. The equation of motion for an elastic half space can be written as follows

$$\varrho \ddot{u}_i = T_{ij,j} + f_i, \tag{2.1}$$

where

$$T_{ij} = C_{ijkl}(\mathbf{x})\eta_{kl}(\mathbf{x}), \tag{2.2}$$

$$C_{ijkl}(\mathbf{x}) = \theta(x_3)C_{ijkl}, \tag{2.3}$$

$$\eta_{kl}(\mathbf{x}) = \frac{1}{2}(u_{k,l} + u_{l,k}), \tag{2.4}$$

and

$$\theta(x_3) = \begin{cases} 1, & x_3 \geqslant 0 \\ 0, & x_3 < 0 \end{cases}.$$

Denotation T_{ij} – stress tensor; η_{kl} – strain tensor; u_i – displacement vector, ϱ – density of the material of the half space, $C_{ijkl}(\mathbf{X})$ – elasticity tensor, dependent on coordinates, C_{ijkl} – classical elasticity tensor, independent of coordinates, f_i – density of external forces acting on the half-space. Inserting (2.2)–(2.4) into (2.1) and taking advantage of the symmetry of C_{ijkl} with respect to k, l, we have

$$\varrho \ddot{u}_i = \delta(x_3) C_{i3kl} u_{k,l} + \theta(x_3) C_{ijkl} u_{k,lj} + f_i. \tag{2.5}$$

3. Equation of the scattering

In order to determine the equation governing the scattering of elastic waves on the described before type of disturbance, we will apply the method given in [9]. Let us consider two fields, u_i and v_i in an undisturbed medium, which satisfy the following equations of motion in the region R (elastic half-space, in our case)

$$\varrho \ddot{u}_i = \delta(x_3) C_{i3kl} u_{k,l} + \theta(x_3) C_{ijkl} u_{k,lj} + f_i. \tag{3.1}$$

$$\varrho \ddot{v}_i = \delta(x_3) C_{i3kl} v_{k,l} + \theta(x_3) C_{ijkl} v_{k,lj} + g_i. \tag{3.2}$$

where f_i and g_i as before are non-homogeneities in these differential equations.

If

$$\begin{aligned} u_i(\mathbf{x}, t) &= u_i(\mathbf{x}) \exp(-i\omega t), \\ f_i(\mathbf{x}, t) &= f_i(\mathbf{x}) \exp(-i\omega t), \\ v_i(\mathbf{x}, t) &= v_i(\mathbf{x}) \exp(-i\omega t), \\ g_i(\mathbf{x}, t) &= g_i(\mathbf{x}) \exp(-i\omega t), \end{aligned} \tag{3.3}$$

then equations (3.1) and (3.2) are reduced to the following form

$$\delta(x_3) C_{i3kl} u_{k,l} + \theta(x_3) C_{ijkl} u_{k,lj} + \varrho \omega^2 u_i + f_i = 0, \tag{3.4}$$

$$\delta(x_3) C_{i3kl} v_{k,l} + \theta(x_3) C_{ijkl} v_{k,lj} + \varrho \omega^2 v_i + g_i = 0. \tag{3.5}$$

From a combination of these equations we have

$$v_i f_i - u_i g_i + \delta(x_3) C_{i3kl}(v_i u_{k,l} - u_i v_{k,l}) + \theta(x_3) C_{ijkl}(v_i u_{k,lj} - u_i v_{k,lj}) = 0. \tag{3.6}$$

and further

$$v_i f_i - u_i g_i + \delta(x_3) C_{i3kl}(v_i u_{k,l} - u_i v_{k,l}) + \theta(x_3) C_{ijkl}(v_i u_{k,l} - u_i v_{k,l})_{,j} + \\ - \theta(x_3) C_{ijkl}(v_{i,j} u_{k,l} - u_{i,j} v_{k,l}) = 0. \tag{3.7}$$

The last term in this equation is equal to zero, because of the symmetry of C_{ijkl} with respect to pairs of indices ij, kl.

Substituting now

$$g_i = \delta_{im}\delta(\mathbf{x}-\mathbf{x}'),$$
$$v_i = G'_{im}(\mathbf{x}, \mathbf{x}'), \tag{3.8}$$

where G'_{im} is a Green's function of equation (3.2) with the condition $(3.8)_1$, we reach

$$f_i G'_{im} - u_i\delta_{im}\delta(\mathbf{x}-\mathbf{x}') + \delta(x_3)C_{i3kl}(G'_{im}u_{k,l} - u_i G'_{km,l}) + \\ + \theta(x_3)C_{ijkl}(G'_{im}u_{k,l} - u_i G'_{km,l})_{,j} = 0. \tag{3.9}$$

After integrating the above expression in terms of volume and taking advantage of the Gauss theorem in region R we have:

$$u_m(\mathbf{x}') = \int_R \delta(x_3)C_{i3kl}(G'_{im}u_{k,l} - u_i G'_{km,l})dV + C_{ijkl}\int_S (G'_{im}u_{k,l} - u_i G'_{km,l})n_j dS + \int_R dVf_i G'_{im}, \tag{3.10}$$

where n_j is the external normal to the region R limited by the surface S (Fig. 2). In accordance with reasoning in paper [9], the first two integrals depend on boundary conditions only and can be substituted with a field of displacements $u^0_m(\mathbf{x}')$. This field satisfies the following homogeneous equation:

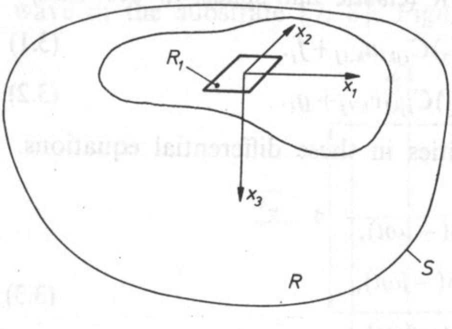

FIG. 2. Coordinates system with the region R, of interest to us, limited by surface S partially contained in the plane of the surface

$$\delta(x'_3)C_{i3kl}u^0_{k,l} + \theta(x'_3)C_{ijkl}u^0_{k,lj} + \varrho\omega^2 u^0_i = 0. \tag{3.11}$$

The boundary condition for S usually has the following form in the problem of scattering

$$u^0_i(\mathbf{x}, t) = u^0_i \exp[i(\mathbf{kx}) - \omega t)], \tag{3.12}$$

i.e. $u^0_i(\mathbf{x}, t)$ is an incident wave with frequency ω and wave vector $\mathbf{k}$. In general the amplitude u^0_i can be a function of coordinate x_3. Finally, we can write:

$$u_m = u^0_m + \int_R dVf_i G'_{im} \quad \text{for } \mathbf{x}' \text{ in } R, \tag{3.13}$$

where Green's function G'_{im} satisfies the equation

$$\left(\delta_{ji}\varrho\omega^2+\delta(x_3)C_{j3ik}\frac{\partial}{\partial x_k}+\theta(x_3)C_{jlik}\frac{\partial^2}{\partial x_k\partial x_l}\right)G'_{im}(\mathbf{x},\mathbf{x}';\omega)=\delta_{jm}\delta(\mathbf{x}-\mathbf{x}'). \quad (3.14)$$

4. Thin layer deposited on the surface of an elastic half-space as a disturbance

In a case of thin isotropic films ($h \ll \lambda$) loading a substrate, the film can be substituted with an equivalent stress pattern [10]:

$$T_{31}(x_3=0)=\left\{-h\left(k^2\frac{c'^2_{11}-c'^2_{12}}{c'_{11}}-\varrho'\omega^2\right)u_1^{(0)}\right\}\exp(ikx_1), \quad (4.1)$$

$$T_{32}(x_3=0)=\left\{-h\left(k^2\frac{c'_{11}-c'_{12}}{2}-\varrho'\omega^2\right)u_2^{(0)}\right\}\exp(ikx_1), \quad (4.2)$$

$$T_{33}(x_3=0)=\{h\varrho'\omega^2u_3^{(0)}\}\exp(ikx_1), \quad (4.3)$$

In general $T_{3i}(x_3=0)=T_i\exp(ikx_1)$. In above expressions, h is the film's thickness, k denotes the wavevector length, ω – frequency of wave propagating in the direction of x_1 in the elastic half-space, ϱ' – mass density of the film, c'_{11} and c'_{12} reduced components of the elasticity tensor in Voigt's notation [11] of the thin isotropic layer.

In the case of a thin film, the disturbing force in equation (3.13) is equal to

$$f_i=T_{ik}n_k, \quad (4.4)$$

where n_k is the external normal $n_k=[0, 0, -1]$ with the dimension of the surface. As the disturbance occurs in plane $x'_3=0$, so

$$f_i(\mathbf{x}')=\delta(x'_3)\,T_{ik}(\mathbf{x}')n_k\theta(R_1), \quad (4.5)$$

where

$$\theta(R_1)=\begin{cases}1 & (x'_1, x'_2)\in R_1,\\ 0 & (x'_1, x'_2)\notin R_1\end{cases}$$

and R_1 – disturbance region in plane x'_1, x'_2. Including (4.5) in equation (3.13), we get

$$u_m(\mathbf{x})=u_m^{(0)}(\mathbf{x})+\int_R f_i(\mathbf{x}')G_{im}(\mathbf{x},\mathbf{x}')\,d\mathbf{x}'=$$

$$=u_m^{(0)}(\mathbf{x})+\int_R\delta(\mathbf{x}'_3)\,T_{ik}(\mathbf{x}')n_k\theta(R_1)G_{im}(\mathbf{x},\mathbf{x}')\,d\mathbf{x}'.$$

And finally

$$u_m(\mathbf{x})=u_m^{(0)}(\mathbf{x})+\int_{R_1}-T_{i3}(x'_1, x'_2, x'_3=0)\,G_{im}(\mathbf{x}'_{\parallel}-\mathbf{x}_{\parallel}, x'_3=0)\,dx'_1\,dx'_2.$$

The minus sign in the above expression is the result of the convention accepted in equation (4.4), that the so-called "compressive" stress has a negative sign. In order to shorten the notation, index $\|$ was introduced in expression (4.6). This sign denotes the projection of a corresponding vector (e.g. **x**) on the plane of the half-space's surface.

5. Amplitude of ESW scattering on a single dot

The method of determining the Green's function G_{im}, satisfying equation (S. 0), in quadratures is presented in the Supplement. Comparing (3.14) and (S. 0) we see that the sought function G'_{im} is equal to

$$G'_{im} = G_{im}\frac{1}{\varrho}. \tag{5.0}$$

Let us rewrite the equation (S. 1) in the following form

$$G_{im}(\mathbf{x}, \mathbf{x}'; \omega) = \int_0^\infty \int_0^{2\pi} \exp(ikr\cos\gamma)\, d_{im}(k, \theta, \omega|x_3 x'_3)\, d\gamma\, k dk, \tag{5.1}$$

where $\mathbf{r} = \mathbf{x}_\| - \mathbf{x}'_\|$, $\nu = \theta - \Phi$ — angle between vectors $\mathbf{k}$ and $\mathbf{r}$, $r = r\cos\Phi$, $r_2 = r\sin\Phi$, $k_1 = k\cos\theta$, $k_2 = k\sin\theta$.

We will take advantage of the approximation of the stationary phase (see e.g. [12], page 46) in order to integrate with respect to angle ν. This result in

$$G_{im}(\mathbf{x}, \mathbf{x}'; \omega) = \frac{1}{(2\Pi)^2}(2\Pi)^{1/2} \int \frac{\exp[i(kr + \Pi/4)]}{(2kr)^{1/2}}\, d_{im}(k, \Phi, \omega|x_3 x'_3)\, kdk. \tag{5.2}$$

Tensor d_{im} can be easily found if we write equation (S. 7) in the matrix form

$$\tilde{d} = \tilde{S}^{-1}\tilde{g}\tilde{S}$$

hence

$$\tilde{d} = \begin{bmatrix} \cos^2\Phi g_{11} + \sin^2\Phi g_{22}; & \cos\Phi\sin\Phi(g_{11} - g_{22}); & \cos\Phi\, g_{13} \\ \cos\Phi\sin\Phi(g_{11} - g_{22}); & \sin^2\Phi g_{11} + \cos^2\Phi g_{22}; & \sin\Phi\, g_{13} \\ \cos\Phi\, g_{31}; & \sin\Phi\, g_{31}; & g_{33} \end{bmatrix}. \tag{5.3}$$

Including (5.0) and (5.2) in (4.6), and applying the theory of residua (for an analogic approach — see [13]) we obtain resulting components of displacement vector for the scattering wave

$$u_m(\mathbf{x}) = \frac{\exp(i\Pi/4)}{i2(2\Pi)^{1/2}} \sum_S \operatorname{Res}\left\{ \int T_{i3}(x'_1, x'_2, x'_3 = 0) \frac{\exp(ik_s|\mathbf{x}_\| - \mathbf{x}'_\||)}{(k_S|\mathbf{x}_\| - \mathbf{x}'_\||)^{1/2}} \times \right.$$

$$\left. \times \frac{1}{\varrho} d_{im}(k_s, \Phi, \omega|x_3 x'_3 = 0)\, k_s\, d\bar{x}'_\| \right\}, \tag{5.4}$$

where k_s denotes a pole associated with every possible type of wave occuring in the crystal and it is a solution to equation $D(k_s) = 0$ for surface waves or $\alpha_i(k_S) = 0$ for bulk waves (see "Supplement" expressions (S. 12) and (S. 13)).

Coefficients accompanying $\exp(ik_S x_{\|})/(x_{\|})^{1/2}$ are the amplitudes of scattering for individual modes.

In order to isolate them we will proceed as in the quantuum mechanics theory of scattering and apply the following approximation in the expression in the exponent

$$k_s|\mathbf{x}_{\|} - \mathbf{x}'_{\|}| \approx k_s x_{\|} - \mathbf{k}_s \mathbf{x}'_{\|} \tag{5.5}$$

$(\mathbf{k}_s \| \mathbf{x}_{\|})$ when at the same time $\mathbf{x}'_{\|}$ neglected in the denominator in expression (5.4). Angle Φ will now be the angle between vector $\mathbf{x}_{\|}$ and the Ox_1 axis. Our considerations result in an expression for scattering of ESW on a localized stress pattern

$$u_m(\mathbf{x}) = \sum_s A^s_m(\Phi, \omega) \frac{\exp(ik_s x_{\|})}{(x_{\|})^{1/2}} \tag{5.6}$$

with amplitudes of scattering

$$A^s_m(\Phi, \omega) = \frac{\exp(i\pi/4)}{i2(2\Pi)^{1/2}} \operatorname*{Res}_{k_s} \{d_{im}(k_s, \Phi, \omega | x_3 x'_3 = 0)\, k_s \times$$

$$\times \int T_{i3}(\mathbf{x}'_{\|}, x'_3 = 0) \exp(ik_s x'_{\|})\, d^2 \bar{x}'_{\|}\}. \tag{5.7}$$

The asymptotic expression (5.6)–(5.7) is a solution to the problem of scattering of ESW on localized thin films loading a crystalline half-space of one of the types listed at the beginning of the "Supplement". The complicated form of relation $D(k_S) = 0$ requires numerical calculations. For particular case of isotropic substrate only (see [15]), an analytical solution can be found [16]. Qualitative relationships can bè relatively easily obtained for a case when an elastic surface wave with vector k_R incides onto a rectangular dot (Fig. 1) with dimensions a and b. In such a case the components of the amplitude of scattering for a surface wave associated with vector k_R can be determined. The integral in expression (5.7) assumes the following value:

$$ab\, Sa[(1+\cos\Phi)\, k_R a/2]\, Sa\,[\sin\Phi\, k_R\, b/2], \tag{5.8}$$

where $Sa(x) = \sin(x)/x$.

Components of the amplitude of scattering are expressed by

$$A^R_1(\Phi, k_R) = A(\Phi, k_R)\{\cos^2\Phi\, G_{11} T_1 + \cos\Phi \sin\Phi\, G_{11} T_2 + \cos\Phi\, G_{31} T_3\},$$

$$A^R_2(\Phi, k_R) = A(\Phi, k_R)\{\cos\Phi \sin\Phi\, G_{11} T_1 + \sin^2\Phi\, G_{11} T_2 + \sin\Phi\, G_{31} T_3\},$$

$$A^R_3(\Phi, k_R) = A(\Phi, k_R)\{\cos\Phi\, G_{13} T_1 + \sin\Phi\, G_{13} T_2 + G_{33} T_3\},$$

where

$$A(\Phi, k_R) = \frac{ab\exp(i\Pi/4)}{i2(2\Pi)^{1/2}} k_R Sa[(1+\cos\Phi) k_R a/2]\, Sa[\sin\Phi k_R b/2],$$

$$G_{im}(k_R) = \frac{1}{\varrho} \operatorname*{Res}_{k=k_R} [g_{im}(k) - g^P_{im}(k)].$$

Quantities g_{im} and G^P_{im} are explained in the "Supplement"

Supplement

The Green's function for a half-infinite hexagonal crystal

In order to understand given above considerations it is necessary to present in short at least, results of Dobrzyński's and Maradudin's paper [14]. It presents a method of finding Green's function G_{im} $(\mathbf{x}, \mathbf{x}'; \omega)$ for a half-infinite hexagonal crystal in quadratures.

This function satisfies the following equation

$$\sum_l \left(\delta_{ji}\omega^2 + \frac{1}{\varrho}\delta(x_3) \sum_k C_{j3ik} \frac{\partial}{\partial x_k} + \frac{1}{\varrho} \sum_{mk} C_{jlik} \frac{\partial^2}{\partial x_l \partial x_k} \right) G_{im}(\mathbf{x}, \mathbf{x}', \omega) = \delta_{jm}(\mathbf{x}-\mathbf{x}'). \quad \text{(S.0)}$$

As previously, latin indices accept values from the set (1, 2, 3), which denote directions in the cartesian coordinates system.

a) Bivariate Fourier distribution G_{im} $(\mathbf{x}, \mathbf{x}'; \omega)$. Green's function G_{im} $(\mathbf{x}, \mathbf{x}'; \omega)$ for an elastic medium with arbitrary symmetry which occupies half-space $x_3 > 0$ can be distributed according to the Fourier distribution as follows:

$$G_{im}(\mathbf{x}, \mathbf{x}'; \omega) = \int \frac{d^2 k}{(2\Pi)^2} \exp[i\mathbf{k}(\mathbf{x}_{\parallel} - \mathbf{x}'_{\parallel})]\, d_{im}(\mathbf{k}\omega : x_3 x'_3), \quad \text{(S.1)}$$

where $\mathbf{x}_{\parallel}$ and $\mathbf{k}$ are vectors with components $(x_1, x_2, 0)$ and $(k_1, k_2, 0)$, respectively. If equation (5.1) together with representation

$$\delta(\mathbf{x}-\mathbf{x}') = \delta(x_3 - x'_3) \int \frac{d^2 k}{(2\Pi)^2} \exp[ik(\mathbf{x}_{\parallel} - \mathbf{x}'_{\parallel})] \quad \text{(S.2)}$$

are included in equation (S.0) and the resulting equation is specified for a case of a hexagonal crystal structure with a sixfold rotation axis in direction x_3, i.e. structure included in one of crystal classes 6, $\bar{6}$, 6/m, 6 mm, $\bar{6}$m2, 62,6/mm in the Herman-Maugin's notation, then the equation fulfilled by Fourier's coefficients $d_{im}(\mathbf{k}\omega;\, x_3 x'_3)$ assumes the following form

$$\sum_j L_{ij}\ (\mathbf{k}\omega\colon x_3)\, d_{jm}(\mathbf{k}\omega\colon x_3 x'_3) = \delta_{im}(x_3 - x'_3). \quad \text{(S.3)}$$

b) Transformation of the set of equations (S.3) Elements of the matrix of the differential operator $L_{ij}(\mathbf{k}\omega\colon x_3)$ have very complicated form [14]. It can be simplified, if we take advantage of the isotroppy of the hexagonal structure in the plane perpendicular to the sixtuple axis of rotation (plane $x_3 = 0$, in this case). We will transform the set of equations (S.3) with respect to matrix $\tilde{S}(k)$, noted as:

$$\tilde{S}(\bar{k}) = \begin{pmatrix} \hat{k}_1, & \hat{k}_2, & 0 \\ -\hat{k}_2, & \hat{k}_1, & 0 \\ 0, & 0, & 1 \end{pmatrix},$$

$$\tilde{S}^{-1}(\bar{k}) = \begin{pmatrix} \hat{k}_1, & -\hat{k}_2, & 0 \\ \hat{k}_2, & \hat{k}_1, & 0 \\ 0, & 0, & 1 \end{pmatrix}, \tag{S.4}$$

where $\hat{k}_1 = k_1/k$ and $\hat{k}_2 = k_2/k$. This real, orthogonal matrix is a matrix which rotates vector $\mathbf{k}$ into vector $(k, 0, 0)$. Equation (S.3) has the following form after the transformation:

$$\sum_j \mathscr{L}_{ij}(k\omega\colon x_3)\, g_{jm}(k\omega\colon x_3 x_3') = \delta_{im}\delta(x_3 - x_3'), \tag{S.5}$$

where

$$\tilde{\mathscr{L}}(k\omega\colon x_3) = \tilde{S}(\mathbf{k})\, \tilde{L}(\mathbf{k}\omega\colon x_3)\, \tilde{S}^{-1}(\mathbf{k}) \tag{S.6}$$

and

$$d_{im}(\mathbf{k}\omega\colon x_3 x_3') = \sum_{jk} S_{ji}(\mathbf{k})\, S_{km}(\mathbf{k})\, g_{jk}(\mathbf{k}\omega\colon x_3 x_3'). \tag{S.7}$$

The above transformation eliminates certain elements of the matrix of the differential operator, and makes other elements dependent on the modulus of vector $\mathbf{k}$, only.

c) Solution of the set of equations (S.5) The solution of the set of equations (S.5) is reduced to a solution of the following set of equations noted in matrix form

$$\begin{bmatrix} \omega^2 - \frac{c_{11}}{\varrho}k^2 + \frac{c_{44}}{\varrho}\frac{d^2}{dx_3^2} & 0 & \frac{i}{\varrho}(c_{13}+c_{44})\,k\frac{d}{dx_3} \\ 0 & \omega^2 - \frac{c_{11}-c_{12}}{2\varrho}k^2 + \frac{c_{44}}{\varrho}\frac{d^2}{dx_3^2} & 0 \\ \frac{i}{\varrho}(c_{13}+c_{44})\,k\frac{d}{dx_3} & 0 & \omega^2 - \frac{c_{44}}{\varrho}k^2 + \frac{c_{33}}{\varrho}\frac{d^2}{dx_3^2} \end{bmatrix} \times$$

$$\times \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} = \delta(x - x') \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{S.8}$$

Elements of matrix $\tilde{g}$ have to satisfy noted below boundary conditions on plane $x_3 = 0$.

$$\frac{c_{44}}{\varrho}\frac{d}{dx_3}g_{1i}+i\frac{c_{44}}{\varrho}kg_{3i}=0,$$

$$\frac{c_{44}}{\varrho}\frac{d}{dx_3}g_{2i}=0, \tag{S.9}$$

$$i\frac{c_{13}}{\varrho}kg_{1i}+\frac{c_{33}}{\varrho}\frac{d}{dx_3}g_{3i}=0,$$

where $i = 1, 2, 3$. We will omit rather complicated calculations and write the final form of the matrix of reduced Green's functions $g_{jk}(k\omega\colon x_3x_3')$ for a hexagonal system described above [14]

$$\tilde{g}=\begin{pmatrix} g_{11} & 0 & g_{13} \\ 0 & g_{22} & 0 \\ g_{31} & 0 & g_{33} \end{pmatrix}. \tag{S.10}$$

Elements not equal to zero assume values given below. Quantities g_{11} and g_{31}

$$g_{11}(k\omega\colon x_3x_3') = [D(k\omega)]^{-1}[A_{11}(k\omega)\exp\{-\alpha_1(x_3+x_3')\}+$$
$$+A_{12}(k\omega)\exp\{-\alpha_1x_3-\alpha_2x_3'\}+A_{21}(k\omega)\exp\{-\alpha_2x_3-\alpha_1x_3'\}+$$
$$+A_{22}(k\omega)\exp\{-\alpha_2(x_3+x_3')]+g_{11}^p(k\omega\colon x_3x_3'), \tag{S.11 a}$$

$$g_{31}(k\omega\colon x_3x_3') = -i\frac{c_{44}}{(c_{13}+c_{44})\alpha_1k}\left(\alpha_1^2-\frac{c_{11}}{c_{44}}k^2+\frac{\varrho\omega^2}{c_{44}}\right)[D(k\omega)]^{-1}\times$$
$$\times\left(A_{11}(k\omega)\exp\{-\alpha_1(x_3+x_3')\}+A_{12}(k\omega)\exp\{-\alpha_1x_3-\alpha_2x_3'\}\right)-$$
$$-i\frac{c_{44}}{(c_{13}+c_{44})\alpha_2k}\left(\alpha_2^2-\frac{c_{11}}{c_{44}}k^2+\frac{\varrho\omega^2}{c_{44}}\right)[D(k\omega)]^{-1}\times$$
$$\times\left(A_{21}(k\omega)\exp\{-\alpha_2x_3-\alpha_1x_3'\}+A_{22}(k\omega)\exp\{-\alpha_2(x_3+x_3')\}\right)+$$
$$+g_{31}(k\omega\colon x_3x_3'). \tag{S.11 b}$$

In these expressions

$$A_{11}=M_{22}C_{11}-M_{12}C_{21},$$
$$A_{12}=M_{22}C_{12}-M_{12}C_{22},$$
$$A_{21}=-M_{21}C_{11}+M_{11}C_{21},$$
$$A_{22}=-M_{21}C_{12}+M_{11}C_{22}$$

all quantities are functions of k and ω, where

$$M_{1(1,2)}(k\omega) = [(c_{13}+c_{44})\alpha_{1,2}]^{-1}\left[-(c_{13}+c_{44})\alpha_{1,2}^2+c_{44}\left(\alpha_{1,2}^2-\frac{c_{11}}{c_{44}}k^2+\frac{\varrho\omega^2}{c_{44}}\right)\right],$$

$$M_{2(1,2)}(k\omega) = [(c_{13}+c_{44})k]^{-1}\left[(c_{13}+c_{44})k^2+\frac{c_{33}c_{44}}{c_{13}}\left(\alpha_{1,2}^2-\frac{c_{11}}{c_{44}}k^2+\frac{\varrho\omega^2}{c_{44}}\right)\right],$$

$$C_{1(1,2)}(k\omega) = \frac{\varrho}{2c_{44}}\frac{1}{\alpha_{1,2}^2-\alpha_{2,1}^2}\left(\alpha_{1,2}^2+\frac{c_{13}+c_{44}}{c_{33}}k^2-\frac{c_{44}}{c_{33}}k^2+\frac{\varrho\omega^2}{c_{33}}\right),$$

$$C_{2(1,2)}(k\omega) = \frac{\varrho k}{2c_{44}c_{13}}\frac{1}{\alpha_{1,2}(\alpha_{1,2}^2-\alpha_{2,1}^2)}\times\left[c_{13}\left(\alpha_{1,2}^2-\frac{c_{44}}{c_{33}}k^2+\frac{\varrho\omega^2}{c_{33}}\right)-(c_{13}+c_{44})\alpha_{1,2}^2\right].$$

Quantities g_{13} and g_{33}

$$g_{13}(k\omega\colon x_3x_3') = [D(k\omega)]^{-1}[B_{11}(k\omega)\exp\{-\alpha_1(x_3+x_3')\}+$$

$$+B_{12}(k\omega)\exp\{-\alpha_1x_3-\alpha_2x_3'\}+B_{21}(k\omega)\exp\{-\alpha_2x_3-\alpha_1x_3'\}+$$

$$+B_{22}(k\omega)\exp\{-\alpha_2(x_3+x_3')\}]+g_{13}^p(k\omega\colon x_3x_3'), \quad \text{(S.11 c)}$$

$$g_{33}(k\omega\colon x_3x_3') = -i\frac{c_{44}}{(c_{13}+c_{44})k\alpha_1}\left(\alpha_1^2-\frac{c_{11}}{c_{44}}k^2+\frac{\varrho\omega^2}{c_{44}}\right)[D(k\omega)]^{-1}\times$$

$$\times(B_{11}(k\omega)\exp\{-\alpha_1(x_3+x_3')\}+B_{12}(k\omega)\exp\{-\alpha_1x_3-\alpha_2x_3'\}]-$$

$$-i\frac{c_{44}}{(c_{13}+c_{44})k\alpha_2}\left(\alpha_2^2-\frac{c_{11}}{c_{44}}k^2+\frac{\varrho\omega^2}{c_{44}}\right)[D(k\omega)]^{-1}+$$

$$+[B_{21}(k\omega)\exp\{-\alpha_2x_3-\alpha_1x_3'\}+B_{22}(k\omega)\exp\{-\alpha_2(x_3+x_3')\}]+$$

$$+g_{33}^p(k\omega\colon x_3x_3'). \quad \text{(S.11 d)}$$

In these expressions

$$B_{11} = M_{22}C'_{11}-M_{12}C'_{21},$$

$$B_{12} = M_{22}C'_{12}-M_{12}C'_{22},$$

$$B_{21} = -M_{21}C'_{11}+M_{11}C'_{21},$$

$$B_{22} = -M_{21}C'_{12}+M_{11}C'_{22},$$

$$C'_{1(1,2)}(k\omega) = \frac{i\varrho k}{2c_{44}c_{33}}\frac{1}{\alpha_{1,2}(\alpha_{1,2}^2-\alpha_{2,1}^2)}\times\left[c_{44}\left(\alpha_{1,2}^2-\frac{c_{11}}{c_{44}}k^2+\frac{\varrho\omega^2}{c_{44}}\right)-(c_{13}+c_{44})\alpha_{1,2}^2\right].$$

$$C'_{2(1,2)}(k\omega) = \frac{i\varrho}{2c_{13}}\frac{1}{\alpha_{1,2}^2-\alpha_{2,1}^2}\left(\alpha_{1,2}^2+c_{13}\frac{c_{13}+c_{44}}{c_{33}c_{44}}k^2-\frac{c_{11}}{c_{44}}k^2+\frac{\varrho\omega^2}{c_{44}}\right).$$

While quantities g_{11}^P, g_{31}^P, g_{13}^P, g_{33}^P

$$g_{31}^P = g_{13}^P = -\frac{i\varrho(c_{13}+c_{44})k}{2c_{33}c_{44}}\frac{1}{\alpha_1^2-\alpha_2^2}[\exp\{-\alpha_1|x_3-x_3'|\}-$$

$$-\exp\{-\alpha_2|x_3-x_3'|\}]\operatorname{sgn}(x_3-x_3')$$

$$g_{11}^P = -\frac{\varrho}{2\alpha_1 c_{44}}\frac{1}{\alpha_1^2-\alpha_2^2}\left(\alpha_1^2-\frac{c_{44}}{c_{33}}k+\frac{\varrho\omega^2}{c_{33}}\right)\exp\{-\alpha_1|x_3-x_3'|\}+ \tag{S.12 a}$$

$$+\frac{\varrho}{2\alpha_2 c_{44}}\frac{1}{\alpha_1^2-\alpha_2^2}\left(\alpha_2^2-\frac{c_{44}}{c_{33}}k+\frac{\varrho\omega^2}{c_{33}}\right)\exp\{-\alpha_2|x_3-x_3'|\}$$

$$g_{33}^P = -\frac{\varrho}{2\alpha_1 c_{33}}\frac{1}{\alpha_1^2-\alpha_2^2}\left(\alpha_1^2-\frac{c_{11}}{c_{44}}k^2+\frac{\varrho\omega^2}{c_{44}}\right)\exp\{-\alpha_1|x_3-x_3'|\}+$$

$$+\frac{\varrho}{2\alpha_2 c_{33}}\frac{1}{\alpha_1^2-\alpha_2^2}\left(\alpha_2^2-\frac{c_{11}}{c_{44}}k^2+\frac{\varrho\omega^2}{c_{44}}\right)\exp\{-\alpha_2|x_3-x_3'|\},$$

where

$$\alpha_1^2 = \frac{1}{2}[x+(x^2-4y^2)^{1/2}] \qquad \alpha_2^2 = \frac{1}{2}[x-(x^2-4y^2)^{1/2}] \tag{S.12 b}$$

with

$$x = (c_{33}c_{44})^{-1}[(c_{44}+c_{11}c_{33})k^2-(c_{13}+c_{44})^2k^2-(c_{33}+c_{44})\varrho\omega^2],$$
$$y^2 = (c_{33}c_{44})^{-1}(c_{44}k^2-\varrho\omega^2)(c_{11}k^2-\varrho\omega^2).$$

Functions α_1, α_2 are defined by equations (S.12) with the following limitations resulting from boundary conditions for $x_3 = +\infty$

$$\operatorname{Re}\alpha_{1,2} > 0 \qquad \operatorname{Im}\alpha_{1,2} < 0.$$

Quantity g_{22}

$$g_{22}(k\omega\colon x_3x_3') = -\frac{\varrho}{2c_{44}\alpha_t}\exp\{-\alpha_t(x_3+x_3')\}+g_{22}^P(k\omega\colon x_3x_3') \tag{S.12 c}$$

$$g_{22}^P = -\frac{\varrho}{2c_{44}\alpha_t}\exp\{-\alpha_t|x_3+x_3'|\},$$

where

$$\alpha_t = \begin{cases} \left(\dfrac{c_{11}-c_{12}}{2c_{44}}k^2-\dfrac{\varrho\omega^2}{c_{44}}\right)^{1/2} & \text{for} \quad \dfrac{1}{2}(c_{11}-c_{12})k^2 > \varrho\omega^2, \\ -i\left(\dfrac{\varrho\omega^2}{c_{44}}-\dfrac{c_{11}-c_{12}}{2c_{44}}k^2\right)^{1/2} & \text{for} \quad \varrho\omega^2 > \dfrac{1}{2}(c_{11}-c_{12})k^2. \end{cases}$$

In the above equations

$$D(k\omega) = M_{11}(k\omega)\,M_{22}(k\omega) - M_{12}(k\omega)\,M_{21}(k\omega). \tag{S.13}$$

Conclusion

The presented model can be used for detailed analysis of scattering of elastic surface waves on a single centre of a dot array, i.e. evaluation of energy radiated in the form of ESW, as well as losses in the form of bulk waves, in hexagonal crystals with a sixfold rotation axis in the direction normal to the surface.

The results can be applied for piezoelectric crystals with good approximation although the electric component of interactions is neglected. Detailed considerations can be found in [6].

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