A Physical Model of the Nonlinear Sitar String

Sadjad SIDDIQ

Institute for Music Studies, University of Vienna, Universitätscampus AAKH
Spitalgasse 2-4, Hof 9, 1090 Vienna, Austria; e-mail: sadjad.siddiq@gmx.at

The acoustic properties of the sitar string are studied with the aid of a physical model. The nonlinearity of the string movement caused by the bridge acting as an obstacle to the vibrating string is of special interest. Comparison of the model's audio output to recordings of the instrument shows interesting similarities. The effects dispersion and bridge have on the sound of the instrument are demonstrated in the model.

Keywords: sitar, physical modelling, nonlinear string vibration, dispersion.

1. Introduction

The sitar is a stringed instrument from the South Asia known for its very distinctive timbre which is quite different from the sound of most Western plucked instruments. Here a physical model of the sitar's string is designed to verify common theses about the instrument's sound formation as they are found in the literature. Sympathetic strings and the resonating body, although beyond doubt important for its sound because of their filtering effect and because of interactions between them and the vibrating string, are neglected. The physical model could also be the first step in developing a sitar synthesizer, since with all its sympathetic strings the sitar is rather difficult to synthesize with other techniques such as sampling.

In Sec. 2 the sound of the sitar is analyzed and important theses concerning the string movement are summarized. In Sec. 3 the physical model is presented. In Sec. 4 the model is analyzed by comparing its output to recordings of the sitar. It is investigated if the theses presented earlier can be applied to the modelled string as well. Sample sounds of the model and the model itself can be downloaded from www.talaash.at/sitar.

A part of this analysis was presented at the ViennaTalk 2010 (SIDDIQ, 2010).

2. Characteristics of the sitar

2.1. Sound analysis

The sitar sound analyzed was recorded in an anechoic room. All strings except the plucked string

(c with $f_0 \approx 131$ Hz) where damped. Comparison to Western plucked string instruments, such as the guitar, shows four major differences:

1. Descending formants are one striking feature of the sitar sound which can not only be easily heard, but is also very obvious in spectrograms (see Fig. 1).

2. The envelopes of the partials of the sitar sound are very complex (see Fig. 2) in comparison to most Western plucked instruments. Since energy is fed to the system only when the string is plucked, the fluctuations in the partials' envelopes must be caused by an energy flow between partials or between the string and the body.
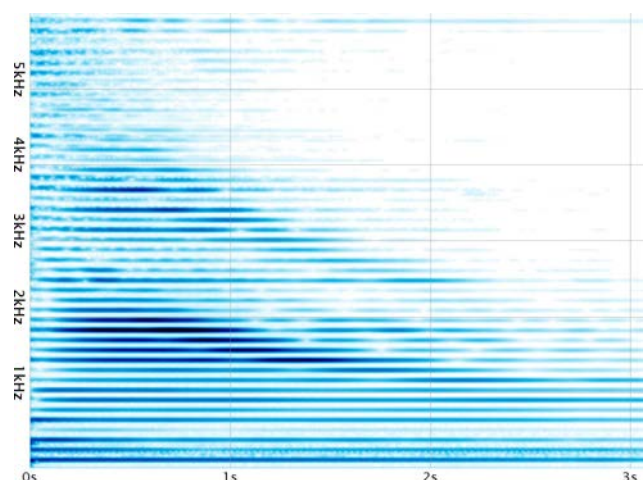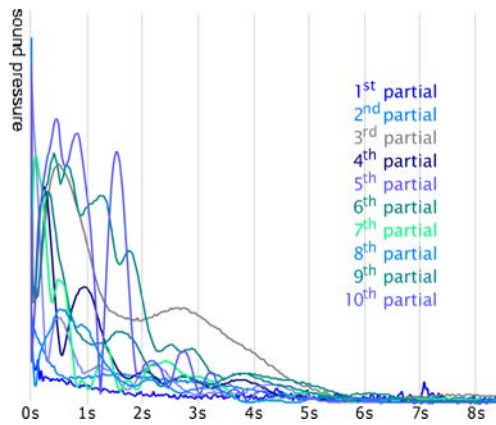

Fig. 1. Spectrogram of the sitar.

Fig. 2. Envelopes of sitar partials.

Fig. 4. "Flat" bridges of the sitar.

3. On the ideal string, when plucked at one fifth of its length, the fifth, tenth, fifteenth etc. modes remain silent, because the string is plucked at one of their nodes (FLETCHER, ROSSING, 1998). This rule, known as the Young-Helmholtz law, is not valid for the sitar (RAMAN, 1921), as Fig. 3 shows.
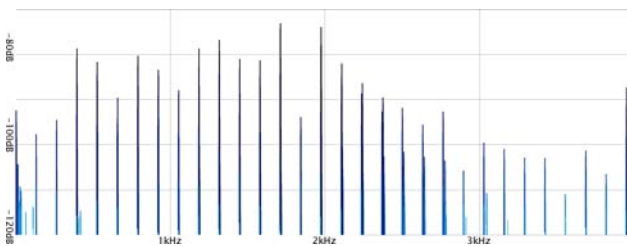

Fig. 3. Spectrum of the sitar string plucked at one fifth of its length.

2.2. The sound formation of the sitar

2.2.1. The effect of the flat bridges

RAMAN (1921) identified the flat bridges[1] (see Fig. 4) of the sitar as the main reason for its distinctive sound (RAMAN, 1921). The bridge disturbs a free vibration of the string which is resting on it varying the length of the vibrating part periodicaly. Laws which assume that the string has a constant length are thus not applicable.

2.2.2. The importance of dispersion

Although commonly neglected in the literature describing string vibration, dispersion plays a key role in the effects described above, as noted by BERTRAND (1992) and VALETTE, CUESTA (1993). If are looks at

[1]It should be noted that the bridges of the sitar are not really flat. Their surface is rather a curved plane. The adjective "flat" was chosen as a default prevalent term to emphazise that a string segment of a certain length rests on that curved surface of the bridge, or collides with it when vibrating.

one wavefront travelling towards the bridge the importance of dispersion becomes clear: when the lower frequency waves of the wavefront, which usually let the string vibrate with a higher amplitude, cause the string to collide with the bridge, a part of the higher frequency waves will already have passed the point of impact and thus do not die away when the string and bridge collide (VALETTE, CUESTA, 1993). Furthermore, dispersion also alters the shape of the vibrating string (LIEBER, 1992), so that the contact of the string and bridge is affected.

3. Physical modelling

3.1. Simplifications of the string vibration

In the proposed model the string is broken down into a finite number of points, of which only vertical displacement in a plane perpendicular to the bridge is considered. Longitudinal waves, torsional waves and transversal waves in a plane parallel to the bridge are neglected, although especially the latter might be of great importance for the sitar string. Such simplifications are, however, acceptable, because the coupling between transversal waves perpendicular to the bridge and the body is more effective than the coupling between longitudinal waves or transversal waves parallel to the bridge and the body (FLETCHER, ROSSING, 1998). Additionally, the main interaction between the bridge and string takes place in the modelled plane.

3.2. Difficulties in modelling the sitar string

As the bridges damping effect on the strings movement increases with the amplitude of string vibration, the amplitudes of the input and output signals are not related in a linear way. Therefore, the string cannot be described as a linear system.

Dispersion is not a trivial problem either, and most works about vibrating strings with constraints tend to neglect it. BURRIDGE, KAPPRAFF and MOR-

SHEDI (1982), as well as VYASARAYANI, BIRKETT and MCPHEE (2009), describe the movement of the sitar string with sets of equations, while TAGUTI (2007) describes the vibration of a biwa or shamisen string (with similar vibrational constraints as caused by the sitar's bridge) with finite differences. In all these works dispersion is neglected, which causes a huge difference that we shall see below.

VALETTE and CUESTA (1993), who consider dispersion and vibrational constraints (for the tanpura, which is slightly different), mention a simulation of the string vibration but do not tell the reader how they do it.

3.3. Design of the model

Waveguides are a fast and efficient way to model the string vibration and will be used in the sitar string model. However the string's interaction with the bridge is difficult to model in waveguides since they are based on the wave equation of the ideal string, and nonlinearities cannot be inserted into the model directly. The part of the string colliding with the bridge is therefore modelled using finite differences and coupled to the model as described in the work by KRISHNASWAMY and SMITH (2003).

3.4. Waveguide modelling

In waveguides (SMITH, 1992) the travelling waves on a string are used to model the string vibration. One right travelling and one left travelling wave determine the string's shape according to the simple relation

$$y(x,t) = y_l(x,t) + y_r(x,t), \tag{1}$$

which is based on d'Alembert's solution of the wave equation (SMITH, 1992)

$$y(x,t) = y_r(x - ct) + y_l(x + ct). \tag{2}$$

The pitch of the modelled string is determined by the length of the waveguide L according to the formula (SMITH, 1992)

$$f_0 = \frac{f_S}{2 \cdot L}, \tag{3}$$

where f_S is the sampling rate.

After the initial shapes of the travelling waves are derived from the strings initial displacement according to the relation above (1), they are shifted by one sample into their direction each step. To model the reflection at the fixed ends of the string, samples shifted out of the waveguide are appended to the end of the wave travelling in the opposite direction with an inverted sign (see Fig. 5).

Damping can be modelled by integrating filters into the travelling waves' loops. Using filters with low-pass characteristics can account for a stronger damping of
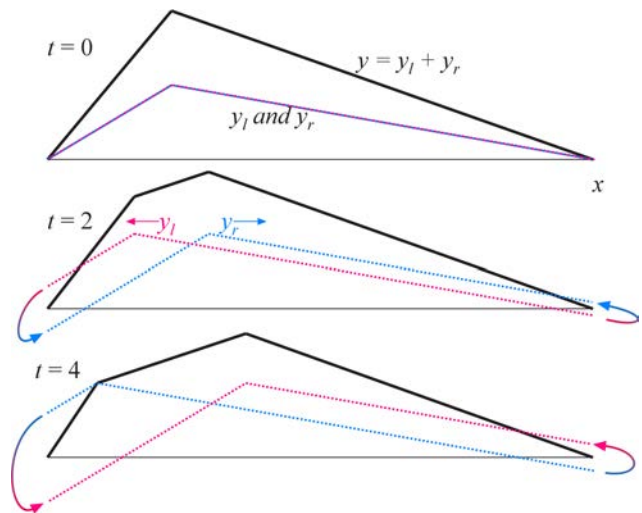

Fig. 5. Two travelling waves determine the shape of the string in a waveguide model.

higher partials, as it can be observed in real strings (SMITH, 1992).

Dispersion means that the propagation speed of waves changes with frequency. The travelling waves' higher frequency components travel faster than the lower frequency components. The effect of dispersion can thus be imitated by inserting all pass filters which cause delay for higher partials into the travelling waves' loop (SMITH, 1992). As proposed by RAUHALA and VÄLIMÄKI (2006), a cascade of several first order all-pass filters is used.

3.5. Finite Difference Modelling (FDM)

The displacement y of a vibrating string with linear density μ and tension T at time t can be calculated with the well known wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \tag{4}$$

where $c^2 = T/\mu$. In FDM (KARJALAINEN, 2002) this equation is solved approximately with finite differences. The first derivation of a function $f(x)$ is approximated as

$$f'(x) \approx \frac{f(x + \Delta x/2) - f(x - \Delta x/2)}{\Delta x}. \tag{5}$$

The second derivation is therefore

$$f''(x) \approx \frac{f'(x + \Delta x/2) - f'(x - \Delta x/2)}{\Delta x} \tag{6}$$

or

$$f''(x) \approx \frac{f(x + \Delta x) - 2f(x) + f(x - \Delta x)}{\Delta x^2}. \tag{7}$$

Neglecting inaccuracy application on the wave equation yields

$$\frac{y(x, t + \Delta t) - 2y(x, t) + y(x, t - \Delta t)}{\Delta t^2}$$

$$= c^2 \frac{y(x + \Delta x, t) - 2y(x, t) + y(x - \Delta x, t)}{\Delta x^2}, \quad (8)$$

where $y(x, t)$ is the displacement of the string at position x and time t.

If we now define $\Delta x = c \cdot \Delta t$, the vertical displacement y at position x of the string in the next time step $t + \Delta t$ can be calculated from the current displacement of the neighbouring points and the previous displacement at x (Karjalainen, 2002):

$$y(x, t + \Delta t) = y(x + \Delta x, t)$$
$$+ y(x - \Delta x, t) - y(x, t - \Delta t). \quad (9)$$

Figure 6 shows this relation schematically. If outermost points are fixed, the movement of a whole string can be calculated.
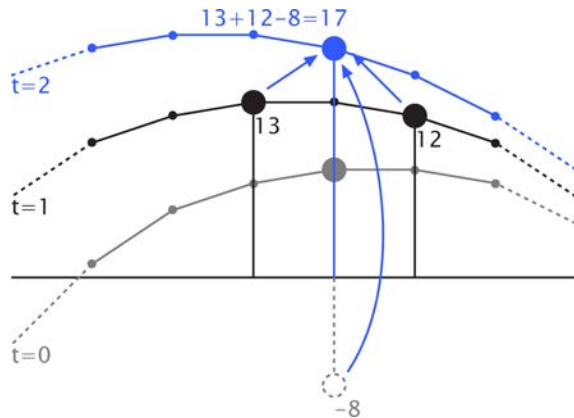

Fig. 6. Calculating one point in the FDM.

By applying finite differences to the enhanced wave equation describing the movement of the dispersive string (Fletcher, Rossing, 1998)

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} - \frac{E \cdot S \cdot K^2}{\mu} \frac{\partial^4 y}{\partial x^4}, \quad (10)$$

where E is the Young modulus of the string, S the string's cross-sectional area, K the radius of gyration and μ, as earlier, linear density, we can calculate the movement of each point of the dispersive string as

$$y(x, t + \Delta t) = y(x + \Delta x, t) + y(x - \Delta x, t)$$
$$- y(x, t - \Delta t) - Q[y(x + 2\Delta x, t)$$
$$- 4y(x + \Delta x, t) + 6y(x, t) - 4y(x - \Delta x, t)$$
$$+ y(x - 2\Delta x, t)], \quad (11)$$

where

$$Q = \frac{E \cdot S \cdot K^2}{c^4 \cdot \mu \cdot \Delta t^2}. \quad (12)$$

By introducing constraints for the movement of some points at one end of the string the effect of the sitar's bridge can be imitated in FDM. Since those points lose their kinetic energy when colliding with the bridge the string is damped. Further damping is achieved by introducing additional coefficients in Eq. (11) as proposed by Karjalainen (2002) for the non-dispersive equation:

$$y(x, t + \Delta t) = g \cdot [y(x + \Delta x, t) + y(x - \Delta x, t)]$$
$$- a \cdot y(x, t - \Delta t) - Q[y(x + 2\Delta x, t)$$
$$- 4y(x + \Delta x, t) + 6y(x, t) - 4y(x - \Delta x, t)$$
$$+ y(x - 2\Delta x, t)]. \quad (13)$$

Karjalainen (2002) calculates g and a from the two constants b and d by

$$a = 2bd - 1, \quad (14)$$

$$g = 1 - d, \quad (15)$$

where $0 \leq d \leq 1$ and $0 \leq b \leq 1$. Proposed values for the sitar string's model are $b = 0.97$ and $d = 0.00005$ to avoid a too fast decay.

3.6. Sound output

In acoustic string instruments the resonator, which is coupled to the strings through the bridge, serves as an amplifier and radiates sound waves into the air. The audio output of the string models is therefore a force exercised by the string on the bridge.

4. Analysis of the model

4.1. Graphical output

The effects of dispersion and the flat bridge are best seen in stroboscopic images of the string movement. Figure 7, depicting the vibration of an ideal string and a dispersive string, both without flat bridges, shows how dispersion alters the form of the vibrating string over time. The effect this deformation would have on the string's interaction with the bridge is obvious, especially when closely examining the snapshots taken at $t = 84$ ms and $t = 126$ ms, where the form at the left end of the dispersive string, where we would insert the flat bridge, is quite different.

The importance of the flat bridge can be seen in Fig. 8. Since each snapshot is taken several periods after a preceding one, even the first snapshot after the excitation in the right image already shows the effect of the string's periodic collision with the bridge.
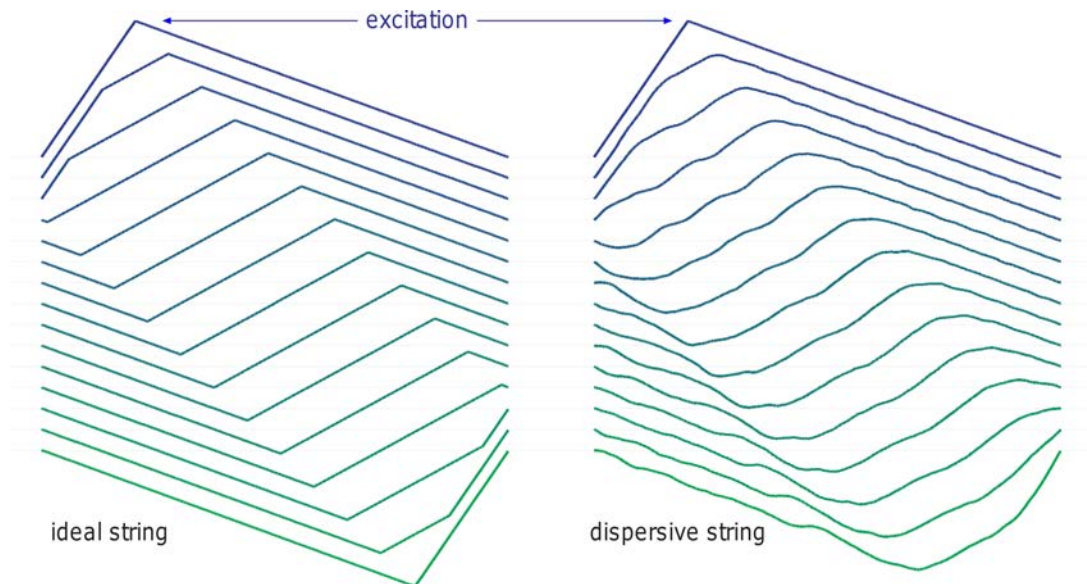
Fig. 7. Stroboscopic images of the ideal string and the dispersive string; time interval between the snapshots $\Delta t = 14$ ms.
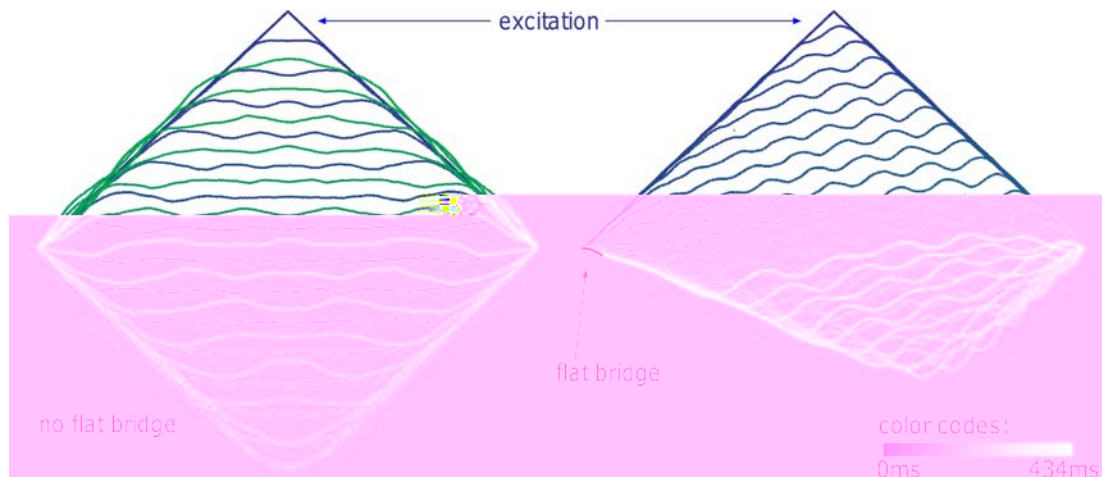

Fig. 8. Stroboscopic images of the dispersive string without vibrational constraints (left) and with the flat bridge (right), $\Delta t = 14$ ms.

4.2. Sound analysis

When the string's interaction with the bridge and dispersion is modelled the same phenomena which were observed in the sitar can be observed in the model.

1. *Descending formants*
   The importance of dispersion can be verified in the model: descending formants, as in the sound of the real instrument, can only be observed in dispersive models. The flat bridge's effect on the sound spectrum is much weaker if dispersion is neglected (see Fig. 9).

2. *Energy flow between partials*
   In a non-dispersive model with a flat bridge energy flow between the harmonics can only be ob-served at the very beginning of the sound. A similar complexity in the structure of the harmonics' envelopes to the one observed in the sitar sound can only be found in the model with a flat bridge and dispersion (see Fig. 10).

3. *Young-Helmholtz law is not applicable*
   Different modelled strings have been "plucked" at one fifth of their length. Only in the case of ideal strings the Young-Helmholtz law is fully applicable. Even dispersion introduces some limitations because the ratio between the fundamental frequency and the frequency of the higher partials is not an integer. In string models with flat bridges, the Young-Helmholtz law is not valid, as it is in the case of the sitar (see Fig. 11).
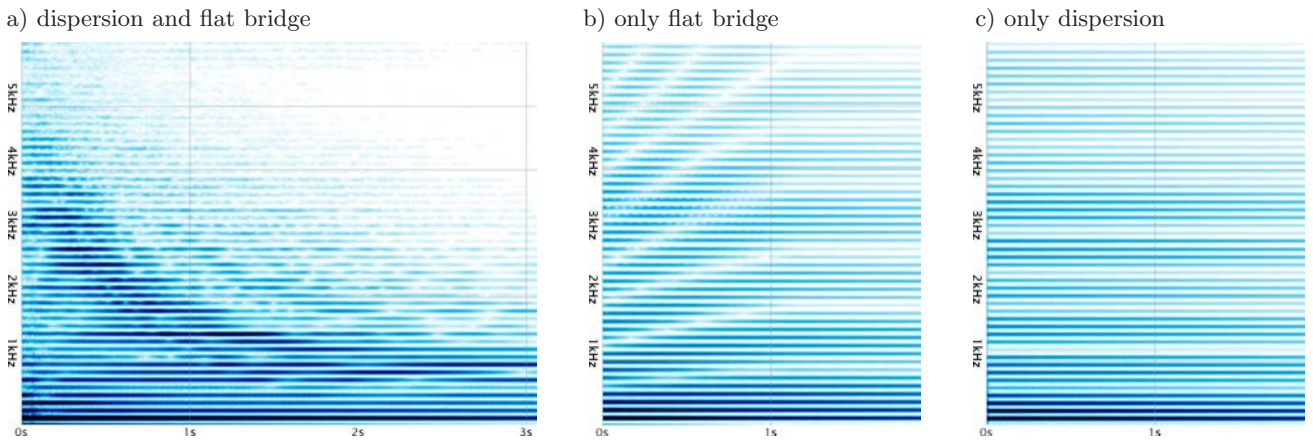
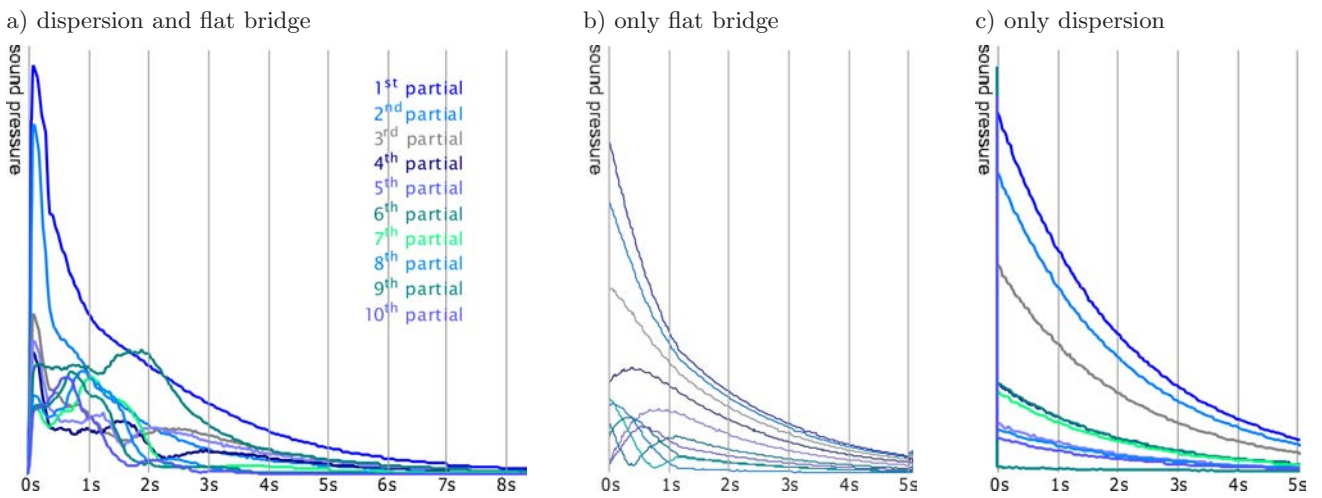a) dispersion and flat bridge   b) only flat bridge   c) only dispersion

Fig. 9. Spectrograms of different models.

a) dispersion and flat bridge   b) only flat bridge   c) only dispersion

Fig. 10. Partial's envelopes of different models.

a) dispersion and flat bridge   b) only flat bridge

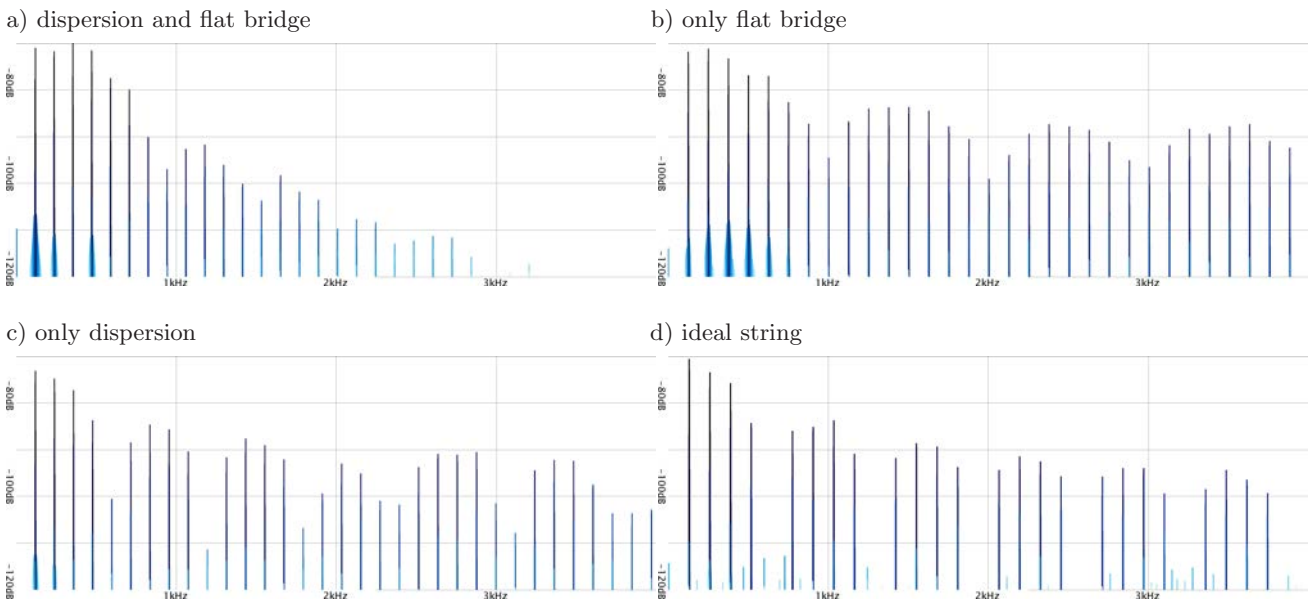c) only dispersion   d) ideal string

Fig. 11. Spectra of the output of different models when the string is plucked at one fifth of its length.
Note that only in the ideal string every fifth partial is absent.

5. Conclusions

A real time physical model of the sitar string has been proposed. Audio output of the model shows important characteristic features that are also found in the sound of the sitar. Not only the flat bridge but also the dispersion on the string plays an important role in the sound formation of the instrument. Neglecting dispersion when describing the vibration of strings periodically colliding with objects is an oversimplification and abstraction with a very restricted validity.

Acknowledgments

References

1. BERTRAND D. (1992), *Les chevalets "plats" de la lutherie de l'Inde* [in French], Editions de la Maison des sciences de lŐhomme, Paris.

2. BURRIDGE R., KAPPRAFF J., MORSHEDI C. (1982), *The Sitar String, a Vibrating String with a One-Sided Inelastic Constraint*, SIAM Journal on Applied Mathematics, **42**, 6, 1231–1251.

3. FLETCHER N., ROSSING T. (1998), *The physics of musical instruments*, 2nd Ed., Springer, New York.

4. KARJALAINEN M. (2002), *1-D Digital Waveguide Modeling for Improved Sound Synthesis*, Proceedings of the IEEE International Conference on Acoustics, Speech and Signal Processing, 1869–1872, Orlando.

5. KRISHNASWAMY A., SMITH J. (2003), *Methods for simulating string collisions with rigid spatial obstacles*, IEEE Worhshop on Applications of Signal Processing to Audio and Acoustics, 233–236, New Paltz.

6. LIEBER E. (1992), *Dispersion und Dämpfung von Transversalschwingungen gespannter Saiten und ihre Bedeutung in der musikalischen Akustik* [in German], Fortschritte der Akustik (DAGA 1992), 253–256.

7. RAMAN C. (1921), *On some Indian stringed Instruments*, Proceedings of the Indian Association for the Cultivation of Science, **7**, 29–33.

8. RAUHALA J., VÄLIMÄKI V. (2006), *Dispersion Modeling in Waveguide Piano Synthesis using tunable Allpass Filters*, Proc. of the 9th Int. Conference on Digital Audio Effects (DAFx-06), 71–76, Montreal.

9. SIDDIQ S. (2010), *Physical Modelling of the sitar string*, Proceedings of the Second ViennaTalk, Sept. 19–21, 2010, University of Music and Performing Arts Vienna, Austria, 137–140.

10. SMITH J. (1992), *Physical Modeling using Digital Waveguides*, Computer Music Journal, **16**, 4, 74–91.

11. TAGUTI T. (2007), *Numerical Simulation of Vibrating String Subject to Sawari Mechanism*, 19th International Congress on Acoustics, 1–6, Madrid.

12. VALETTE C., CUESTA C. (1993) *Mécanique de la corde vibrante* [in French], Hermès, Paris.

13. VYASARAYANI C., BIRKETT S., MCPHEE J. (2009), *Modeling the dynamics of a vibrating string with a finite distributed unilateral constraint: Application to the sitar*, Journal of the Acoustical Society of America, **125**, 6, 3673–3682.