# Thermal Self-Action of Acoustic Beams Containing Several Shock Fronts

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Thermal self-action of an acoustic beam with one discontinuity or several shock fronts is studied in a Newtonian fluid. The stationary self-action of a single sawtooth wave with discontinuity (or some integer number of these waves), symmetric or asymmetric, is considered in the cases of self-focusing and self-defocusing media. The results are compared with the non-stationary thermal self-action of the periodic sound. Thermal self-action of a single shock wave which propagates with the various speeds is considered.

**Keywords:** thermal self-focusing of acoustic beam, acoustic wave with discontinuity, acoustic shock waves, thermal lens.

## 1. Introduction

The fact that acoustic beams may manifest thermal self-action similarly to laser beams has been pointed out in (ASKARYAN, 1966). The nonlinear transfer of acoustic energy into energy of the thermal mode leads to enlargement of a fluid's temperature in the course of sound propagation. This affects the sound speed and, as a consequence, produces thermal lenses. That is also a reason for refraction of sound rays. A beam experiences additional divergence or convergence. Thermal inhomogeneity of a fluid alters both a width of sound beam and transversal distribution of the magnitude of acoustic pressure. The speed of sound increases with enlargement of temperature in gases, and acoustic beam undergoes defocusing, while in a liquid (except for water) with a negative thermal coefficient $\delta = (\partial c/\partial T)_p/c_0 < 0$ it undergoes focusing. $c = \sqrt{C_P/C_V \beta \rho}$ denotes the infinitely small-signal sound speed in a fluid, $\rho$ and $p$ are fluid's density and pressure, $\beta = \rho^{-1} (\partial \rho/\partial p)_T$ is the isothermal compressibility, $C_P$ and $C_V$ are specific heats under constant pressure and volume, respectively, and $T$ denotes temperature. The sound speed, evaluated at unperturbed pressure and temperature, $p_0$ and $T_0$, respectively is marked by the lower index 0. The first theoretical results concerning thermal self-action of sound were reviewed in (BAKHVALOV *et al.*, 1987), and the first experiments confirming the theory of acoustic self-focusing were described in (ASSMAN, 1985; ANDREEV

*et al.*, 1985). Considerable attention was paid to the thermal self-action of quasi-harmonic sound waves due to counterparts in thermal self-action of optic waves (AKHMANOV *et al.*, 1968). Optic waves are strongly dispersive, which makes it possible to consider propagation of harmonic compounds of a waveform and their thermal self-action individually (TALANOV, 1964; 1970). On the contrary, sound alters nonlinear distortions which enrich the spectrum of a waveform as it propagates. The nonlinear self-action is especially significant in the case of intense ultrasound waves in a weakly attenuating media.

The comprehensive review by RUDENKO and SAPOZHNIKOV (2004) focuses on the thermal self-action of periodic beams containing shock fronts in weakly dispersive media with quadratic and cubic nonlinearities. The waveforms with shock fronts are of great importance in technical and medical applications of ultrasound in fluids, solids, and biological tissues (MILLER, 2012; CHAN *et al.*, 2003). Waveforms containing shock fronts represent a broad frequency spectrum signals. The joint action of nonlinearity, diffraction, and absorption makes a wave to acquire the N-shape as it propagates. The only parameters of a waveform are the peak value and its duration. Stationary waveforms with discontinuities are similar to solitons in nonlinear dispersive media, and they are of particular importance. As usual, the scale of thermal inhomogeneity is much larger than the acoustic wavelength. They are formed slowly, with the charac-

teristic time of formation much larger than the wave period. That allows to treat these inhomogeneities as almost stationary as compared with quickly propagating acoustic perturbations. The approximation of geometric acoustics which is used in the theory implies weak diffraction.

The statement of the problem of thermal self-action consists in fact of two issues: the first one is to establish acoustic pressure, and the second one is to evaluate slow variations of the background temperature in the course of sound propagation and their influence on a sound beam itself. The simplified system of equations includes the Khokhlov-Zabolotskaya-Kuznetsov [KZK] equation taking into account variation in the sound speed due to variations in temperature, and equation which describes dynamics of an excess temperature of the thermal mode. The initial equations have been derived by O.V. Rudenko and co-authors (KARABUTOV, RUDENKO, SAPOZHNIKOV, 1988; RUDENKO, SAGATOV, SAPOZHNIKOV, 1990); stationary and non-stationary thermal self-action of strictly periodic waves with discontinuities was also studied by this group of authors. Thermal self-action of a single sawtooth wave or the integer number of sawtooth waves is the subject of this study. Thermal self-action of a solitary shock wave which propagates with different speeds is also considered. Though the mathematical content reminds the one which has been developed by Rudenko et al. in studies of self-action of periodic sound beams, there is a significant distinction which concerns aperiodicity of impulses under study. We make use of instantaneous, not averaged over the sound period acoustic force of heating. It has been derived by the author in (PERELOMOVA, 2006).

## 2. The governing equations and starting points

The system of equations which describes thermal self-action in axially symmetric flow of a Newtonian fluid takes the form (RUDENKO, SAPOZHNIKOV, 2004; KARABUTOV *et al.*, 1988; RUDENKO *et al.*, 1990):

$$\frac{\partial}{\partial \tau}\left(\frac{\partial p}{\partial x} - \frac{\delta T}{c_0}\frac{\partial p}{\partial \tau} - \frac{\varepsilon}{c_0^3 \rho_0}p\frac{\partial p}{\partial \tau}\right.$$
$$\left. - \frac{b}{2c_0^3\rho_0}\frac{\partial^2 p}{\partial \tau^2}\right) = \frac{c_0}{2}\Delta_\perp p, \quad (1)$$

$$\frac{\partial T}{\partial t} - \frac{\chi}{\rho_0 C_P}\Delta_\perp T = F, \qquad (2)$$

where $x$ and $r$ are cylindrical coordinates, $x$ denotes coordinate along axis of a beam, $t$ is time, $p$ is the acoustic pressure, $\rho_0$ is the unperturbed density of a fluid, $\tau = t - x/c_0$ is the retarded time in the reference frame which moves with the sound speed $c_0$, $\Delta_\perp$ is the Laplacian with respect to the radial coordinate $r$, $\varepsilon$ is the parameter of nonlinearity, and $\chi$ denotes the coeffi-

cient of thermal conductivity. $F$ is an acoustic force of a fluid's heating due to loss in acoustic energy. Equation (1) describes an acoustic pressure in a beam which propagates in the positive direction of the axis $x$. The total attenuation $b$ is a sum of terms representing the shear viscosity $\mu$, bulk viscosity $\eta$ and thermal conductivity,

$$b = \frac{4}{3}\mu + \eta + \left(\frac{1}{C_V} - \frac{1}{C_P}\right)\chi. \qquad (3)$$

In the theory of nonlinear self-action of sound beams, $\langle F \rangle$ usually replaces $F$ in the right-hand side of Eq. (2) (RUDENKO, SOLUYAN, 2005). The angular brackets denote averaging over integer number of sound periods. $\langle F \rangle$ has been well-established for the periodic sound (RUDENKO, SOLUYAN, 2005):

$$\langle F \rangle = \frac{b}{c_0^4\rho_0^3 C_P}\left\langle\left(\frac{\partial p}{\partial \tau}\right)^2\right\rangle. \qquad (4)$$

In contrast to the KZK equation, Eq. (1) accounts for variations in the wave speed due to slow enlargement of the fluid temperature (the second term in the left-hand side of equation) (RUDENKO, 2010). The form of Eqs. (2), (4) imposes that a) an acoustic heating is a slow process as compared to fast variations of sound perturbations sound, and b) sound is periodic at any time and any distance from a transducer. The first condition is always valid but the second one is no longer valid in the case of aperiodic sound, impulses, or wave packets. Strictly speaking, it is not valid for physical conditions of transmission of periodic at a transducer sound which starts at some time and has a finite duration, that is, which is periodic inside some temporal domain. The instantaneous acoustic force has been derived by the author in (PERELOMOVA, 2006):

$$F = \frac{1}{c_0^6\rho_0^3}\left(\frac{(1/C_V - 1/C_P)\chi}{\alpha}p\frac{\partial^2 p}{\partial \tau^2}\right.$$
$$+ \left(\frac{c_0^2}{C_P}\left(\frac{4\mu}{3}+\eta\right) - D\frac{(1/C_V-1/C_P)\chi}{\alpha}\right)\left(\frac{\partial p}{\partial \tau}\right)^2$$
$$+ \left(-\frac{\varepsilon(1/C_V - 1/C_P)\chi}{2\alpha} + \frac{(\varepsilon-1)\chi}{C_V\alpha}\right.$$
$$+ 0.5\frac{\chi C_P}{\beta^2 C_V^2}\frac{\partial^2 T}{\partial p^2} + 0.5\frac{\chi}{C_P}\frac{\partial^2 T}{\partial \rho^2}$$
$$\left.\left. + \frac{\chi}{\beta C_V}\frac{\partial^2 T}{\partial p \partial \rho}\right)\frac{\partial p^2}{\partial \tau^2}\right), \qquad (5)$$

where $\alpha = -\rho^{-1}\left(\frac{\partial \rho}{\partial T}\right)_p$ is the thermal expansion, and $D$ is a coefficient expressed in terms of the partial derivatives of the internal energy of a fluid, $e$:

$$D = \frac{\alpha c_0^2}{C_P}\left(-1 + c_0^2\rho_0^2\frac{\partial^2 e}{\partial p^2} + \rho_0^2\frac{\partial^2 e}{\partial p \partial \rho}\right). \qquad (6)$$

Temperature and internal energy of a fluid in Eqs. (5), (6) are considered as functions of pressure and density. Equation (5) coincides with Eq. (18) from

(PERELOMOVA, 2006), which is rearranged recalling that the isobaric perturbations of temperature $T'$ and density $\rho'$ are connected by equality $T' = -\rho'/(\alpha\rho_0)$.

In this study, we focus attention on the aperiodic waveforms. Among them, one period (or integer number of periods) of the sawtooth wave is of the most interest. These waveforms are defined exclusively by their magnitude which depends on the distance from a transducer and from the axis of a beam and plays a similar role as the amplitude of a single harmonics in optics. Another important waveform is the solitary shock wave which may propagate with the speed which differs from the linear sound speed. In spite of the fact that Eq. (2) with the acoustic source (5) reflects all reasons for the Newtonian attenuation, it is fairly difficult for analytical description of thermal self-focusing in the thermoconducting fluid. We consider a fluid without thermal conduction; some brief comments regarding thermoconducting flows will be given in the Concluding Remarks.

The approximation of the geometrical acoustics is successful when the acoustic nonlinearity is important and a beam is slightly divergent. For the validity of approximation of geometrical acoustics, diffraction should be insignificant over the characteristic length of the self-focusing. The acoustic pressure may be found in the form which follows from the theory of geometrical acoustics (RUDENKO, SAPOZHNIKOV, 2004),

$$p = p(x, r, \theta), \qquad \theta = \tau - \psi(x, r)/c_0, \qquad (7)$$

where $\psi$ denotes eikonal. Substituting it into Eq. (1), we arrive at the following equations in the case of short wavelengths, which are small as compared with the scale of thermal inhomogeneities:

$$\frac{\partial p}{\partial x} - \frac{\varepsilon}{c_0^3 \rho_0} p \frac{\partial p}{\partial \theta} - \frac{b}{2c_0^3 \rho_0} \frac{\partial^2 p}{\partial \theta^2} + \frac{\partial \psi}{\partial r} \frac{\partial p}{\partial r} + \frac{\Delta_\perp \psi}{2} p = 0, \quad (8)$$

$$\frac{\partial \psi}{\partial x} + \frac{1}{2}\left(\frac{\partial \psi}{\partial r}\right)^2 + \delta T = 0. \quad (9)$$

This set of equations, along with Eq. (2), is the famous starting point in the studies of thermal self-focusing of acoustic beams in Newtonian fluids (RUDENKO, SAPOZHNIKOV, 2004; KARABUTOV et al., 1988; RUDENKO et al., 1990).

## 3. Sawtooth wave consisting of one of some integer number of shock fronts

Each period of a sawtooth wave whose shock front has a finite width described by the formula (RUDENKO, 2010):

$$p(x, r, \theta) = A(x, r)\left(-\frac{\omega\Theta}{\pi} + \tanh\left(\frac{\varepsilon\Theta}{b}A(x, r)\right)\right) + LP_0. \quad (10)$$

$2\pi/\omega$ denotes the characteristic duration of the waveform. The amplitude of the shock wave $A(x, r)$ varies with coordinates, and

$$\Theta = \theta + \frac{\pi(x/x_s + G)}{\omega}L$$
$$= \theta + \left(\frac{P_0}{A(x, r)} + G - 1\right)\frac{\pi}{\omega}L,$$

where

$$x_s = \frac{\pi c_0^3 \rho_0}{\varepsilon\omega P_0} \qquad (11)$$

denotes the distance at which a break of initially sinusoidal planar wave occurs. $P_0$ is the initial peak acoustic pressure at the axis of a beam and $L$, $G$ are arbitrary dimensionless constants correspondent to the physical meaning of the problem. Equation (10) recalls the exact solution of the Burgers equations which describes the planar nonlinear wave. One period of this planar wave is determined in the domain

$$-\pi + \pi(1 - G)L < \omega\theta < \pi + \pi(1 - G)L, \qquad (12)$$

with $A$ being a function of coordinate $x$:

$$A(x) = \frac{P_0}{1 + x/x_s}. \qquad (13)$$

Equation (10) possesses a sawtooth profile in the limit when $b\omega/(c_0^2 \rho_0)$ tends to zero. The domain of distances where a shock belongs to the interval $[-\pi + \pi(1 - G)L, \pi + \pi(1 - G)L]$, is determined by inequality $|LP_0| \leq A(x)$. In the case of $L = 0$, a shock wave is symmetric, and it propagates along axis $Ox$ with the speed $c_0$. Substituting Eq. (10) into Eq. (8) and allowing $b\omega/(c_0^2 \rho_0) \to 0$ yields the transport equation for the amplitude $A(x, r)$:

$$\frac{\partial A}{\partial x} + \frac{A^2}{x_s P_0} + \frac{\partial \psi}{\partial r} \frac{\partial A}{\partial r} + \frac{\Delta_\perp \psi}{2} A = 0. \qquad (14)$$

The acoustic force of heating in the case of symmetric periodic shock wave in the limit $b\omega/(c_0^2 \rho_0) \to 0$ has been obtained by Rudenko and co-authors. It has been used in the evaluations of thermal self-action of the sound (RUDENKO et al., 1990):

$$\langle F \rangle = \frac{2\varepsilon\omega}{3\pi\rho_0^3 c_0^4 C_P} A^3. \qquad (15)$$

A single "period" given by Eq. (10) can no longer be considered as periodic at any time. Equations (4), (15) are no longer valid either. Substitution Eq. (10) into Eq. (5) allows to evaluate a perturbation of the background temperature in accordance to Eq. (2) when $\chi = 0$ by simple integration from $-\pi/\omega$ till any $\Theta$ after passing of the shock front (that is, $\Theta > 0$). The limit of temperature $T$ when $b\omega/(c_0^2 \rho_0)$ tends to zero

is a function of coordinates $x$, $r$ but it does not depend on $\Theta$ and constants $L$ and $G$:

$$T = T_0 + \frac{4\varepsilon A^3}{3\rho_0^3 c_0^4 C_P} = T_0 + \frac{2\pi}{\omega}\langle F\rangle. \qquad (16)$$

This equation describes variation in temperature caused by one period of the sawtooth wave. Variation in temperature equals $\frac{2\pi n}{\omega}\langle F\rangle$ for integer number of periods $n$. Evaluations are readily simplified by assuming the parabolic wave front, that is,

$$\psi(x, r, t) = \psi_0(x, t) + \frac{r^2}{2}\frac{\partial}{\partial x}\ln f(x, t) \qquad (17)$$

and with allowance for power series of temperature $T$ in the transversal coordinate in the leading order (RUDENKO, SAPOZHNIKOV, 2004),

$$T = T_0 - \frac{r^2}{2}T_2(x, t). \qquad (18)$$

Equation (17) reflects the sphericity of the wave front at any $x$ and $t$, only its curvature may vary during propagation of a beam. The unknown function of two variables, $f$, is responsible for these variations, and $\psi_0(x, t)$ is the phase shift of the wavefront at the axis of a beam. In accordance to Eqs. (9), (17), evolution of eikonal $\psi$ is described by equation

$$\frac{1}{f}\left(\frac{\partial^2 f}{\partial x^2}\right) = \delta T_2. \qquad (19)$$

The solution of Eq. (14) with account for Eq. (17) takes the form (RUDENKO, SOLUYAN, 2005)

$$A(x, r) = \frac{P_0}{f}\Phi\left(\frac{r}{a_0 f}\right)\left[1 + \frac{1}{x_s}\Phi\left(\frac{r}{a_0 f}\right)\int_0^x \frac{\mathrm{d}x'}{f(x')}\right]^{-1}. \quad (20)$$

The function $\Phi$ describes the initial transversal distribution, $A(x = 0, r) = P_0\Phi\left(\frac{r}{a_0}\right)$, where $a_0$ denotes the initial beam's radius at $x = 0$. Following Rudenko and co-authors (RUDENKO, SAPOZHNIKOV, 2004), we will consider the Gaussian beams at a transducer, for which

$$\Phi(\xi) = \exp(-\xi^2).$$

Making use of Eqs. (18), (19) and performing expansion of $A$ in powers of transversal coordinate $r$ in the vicinity of a beam axis, one arrives at the equation:

$$\left[1 + \int_0^z \frac{\mathrm{d}z'}{f(z')}\right]^4 f^4\left(\frac{\mathrm{d}^2 f}{\mathrm{d}z^2}\right) = \Pi, \qquad (21)$$

$$z = \frac{x}{x_s}, \qquad \Pi = \frac{8\delta M\pi^2 c_0^4}{a_0^2\varepsilon C_P\omega^2}, \qquad (22)$$

where $M = P_0/\rho_0 c_0^2$ is the initial Mach number. In the case of the waveform containing $n$ shock fronts, $\Pi$ should be replaced by $\Pi_n$,

$$\Pi_n = \frac{8n\delta M\pi^2 c_0^4}{a_0^2\varepsilon C_P\omega^2}. \qquad (23)$$

Equation (21) differs from equation which describes the stationary self-focusing of a periodic beam (Eq. (20) from RUDENKO, SAPOZHNIKOV, 2004) not only by its form, but also in essence. Rudenko and Sapozhnikov considered stationary self-focusing of a periodic beam in a medium with non-zero thermal conductivity exclusively. The parameters of the Eq. (20) from (RUDENKO, SAPOZHNIKOV, 2004) depend on the coefficient of thermal conductivity $\chi$; they do not include the initial beam's radius $a_0$. That is due to expansion in the series in powers of $r$ of Laplacian of temperature, not perturbation of temperature.

Rudenko and co-authors considered also the non-stationary self-action, for which thermal conduction is unimportant. It occurs at times much smaller than the characteristic time $t_0$,

$$t_0 = \frac{a_0^2 C_P\varepsilon\omega}{4|\delta|M\pi c_0^4} = \frac{2\pi\varepsilon}{\omega|\Pi|}. \qquad (24)$$

The equation which determines function $f$ in this case, takes the form (we reproduce Eq. (22) from (RUDENKO, SAPOZHNIKOV, 2004)):

$$\left[1 + \int_0^z \frac{\mathrm{d}z'}{f(z')}\right]^4 f^5\frac{\partial}{\partial\theta}\left(\frac{1}{f}\frac{\partial^2 f}{\partial z^2}\right) = \pm 1, \qquad (25)$$

where $\theta = t/t_0$. The sign plus in the right-hand side of Eq. (25) corresponds to positive $\delta$ and defocusing medium, and the sign minus relates to the case of self-focusing medium with negative $\delta$. Results of numerical simulations of Eq. (21), the case of stationary self-focusing of a single shock wave in a fluid without thermal conduction and Eq. (25) the case of non-stationary self-focusing of a periodic wave in a fluid without thermal conduction (RUDENKO, SAPOZHNIKOV, 2004), are shown in Fig. 1 for the planar at a transducer wave which corresponds to the initial conditions

$$f(z = 0) = 1, \qquad \frac{\partial f}{\partial z}(z = 0) = 0. \qquad (26)$$

In the non-stationary simulations for periodic sound in accordance to Eq. (25), $f(\theta = 0) = 1$. Dotted lines refer to different dimensionless times $\theta = t/t_0$. The bold lines represent numerical simulations of Eq. (21) at different values of $\Pi$. The graphs show a dimensionless amplitude of a beam at the axis,

$$\frac{A(z, r = 0)}{P_0} = \frac{1}{f}\left(1 + \int_0^z \frac{\mathrm{d}z'}{f(z')}\right)^{-1} \qquad (27)$$
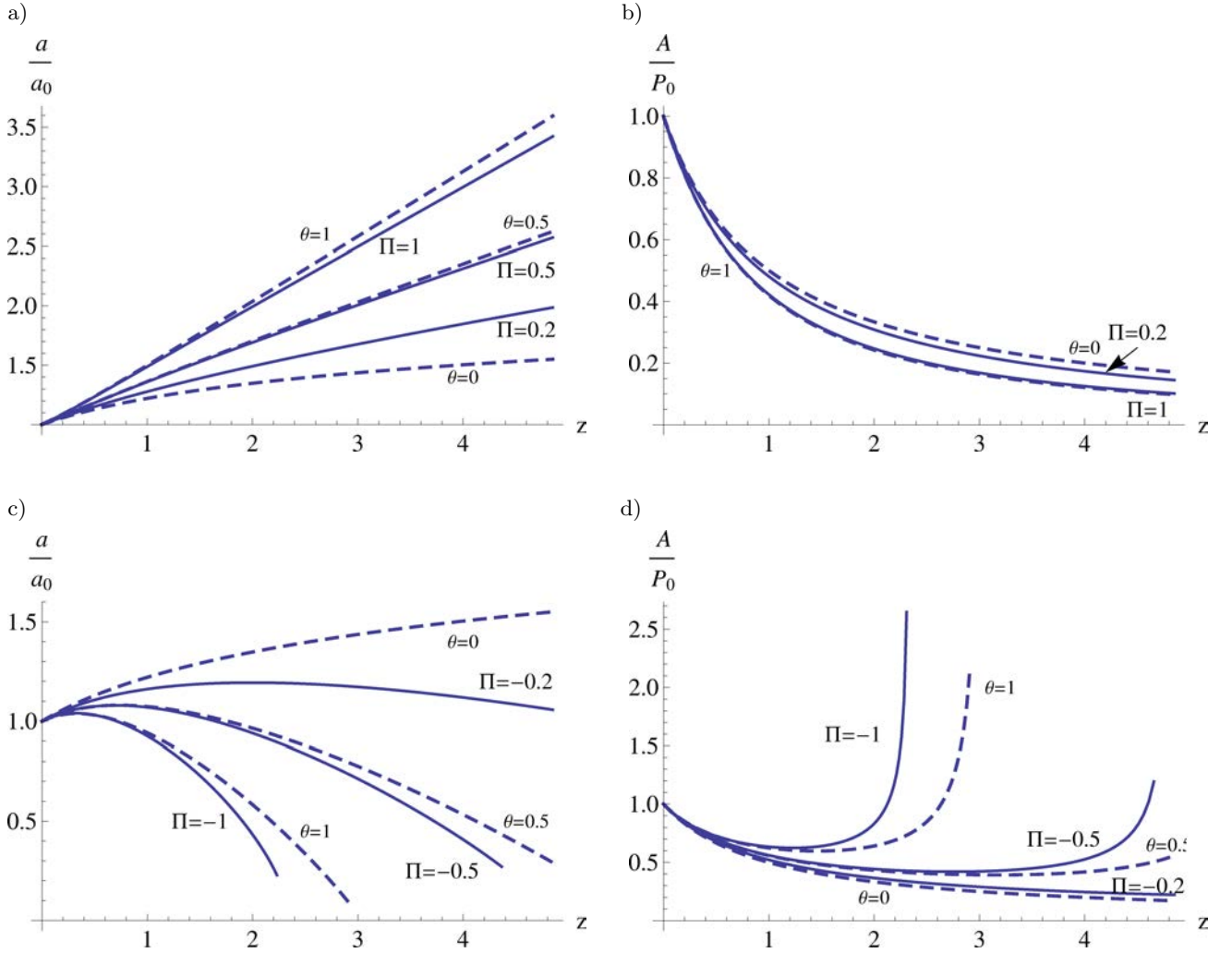
a)

b)

c)

d)

Fig. 1. Dimensionless width of a beam $a/a_0$ and amplitude of acoustic pressure at the axis of a beam $A(z, r = 0)/P_0$ *versus* dimensionless distance from a transducer $z$. Curves (a,b) correspond to self-defocusing and curves (c,d) to self-focusing of initially planar wave. Solid lines relate to solutions of Eq. (21) for a single sawtooth waveform at different parameters $\Pi$, and dotted lines relate to solutions of Eq. (25) for periodic everywhere sound at different dimensionless times $\theta = t/t_0$.

and its characteristic dimensionless width (referring to the level where the magnitude decreases $e$ times),

$$\frac{a(z)}{a_0} = f \sqrt{\ln\left(e + (e-1)\int_0^z \frac{\mathrm{d}z'}{f(z')}\right)}. \qquad (28)$$

The non-stationary curves at $\theta = 0$ cover with these for a single shock wave at $\Pi = 0$.

## 4. Stationary shock wave

### 4.1. Stationary shock wave which propagates with the linear sound speed

The waveform which resembles a stationary solution of the planar Burgers equation takes the form (RUDENKO, SOLUYAN, 2005):

$$p(x, r, \theta) = A(x, r) \tanh\left(\frac{\varepsilon\theta A(x, r)}{b}\right). \qquad (29)$$

The relative transport equation for the pressure step $A(x, r)$ with allowing the duration of the shock front to tend to zero, takes the limiting form:

$$\frac{\partial A}{\partial x} + \frac{\partial \psi}{\partial r}\frac{\partial A}{\partial r} + \frac{\Delta_\perp \psi}{2}A = 0, \qquad (30)$$

which has a solution

$$A(x, r) = \frac{P_0}{f}\Phi\left(\frac{r}{a_0 f}\right). \qquad (31)$$

An excess temperature coincides with that given by Eq. (16). Equations (18), (19) yield an equation which determines the unknown function $f$,

$$f^4 \left(\frac{\mathrm{d}^2 f}{\mathrm{d}z^2}\right) = \Pi, \qquad (32)$$
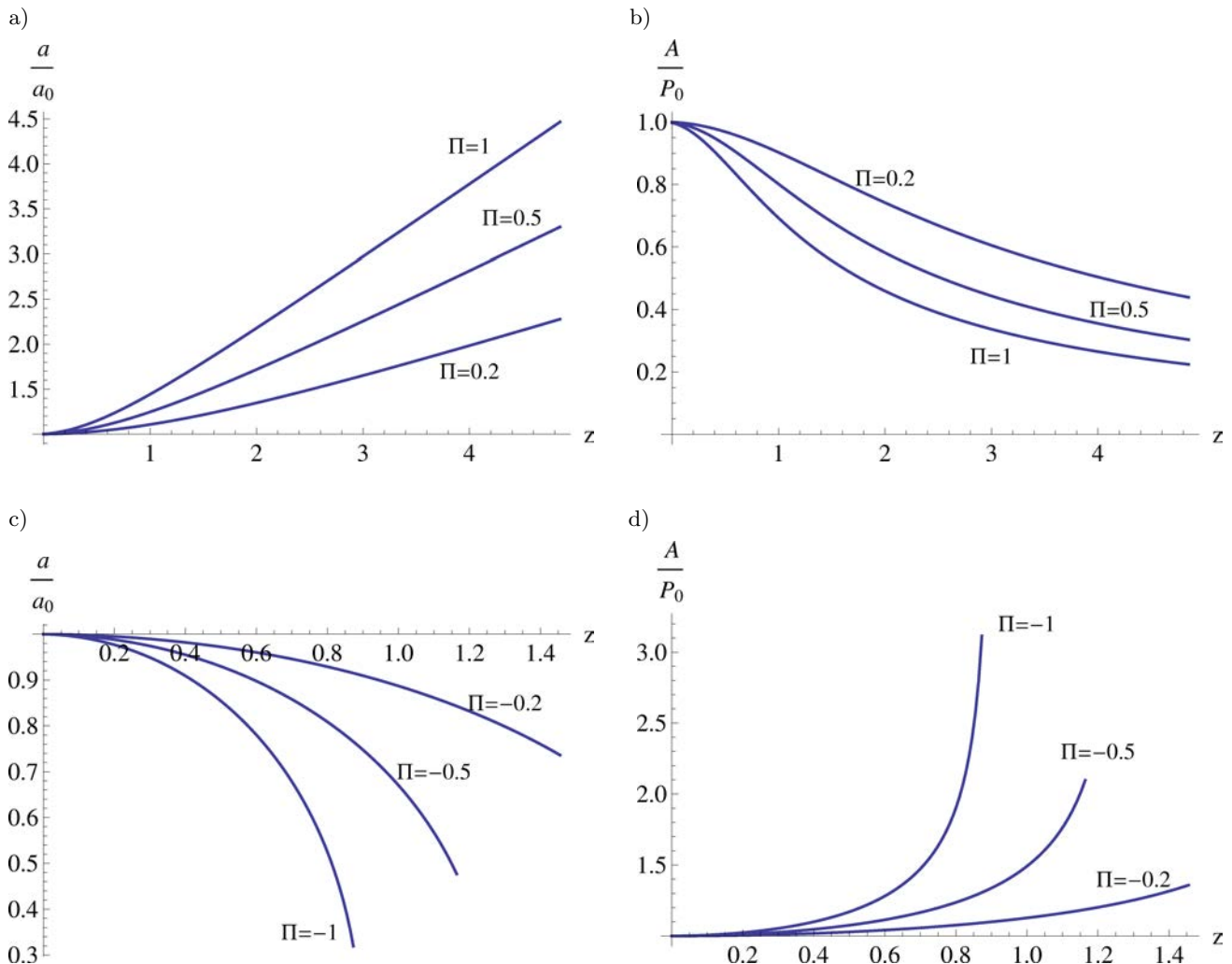
a)



b)



c)



d)



Fig. 2. Dimensionless width of a beam $a/a_0$ and magnitude of acoustic pressure at the axis of a beam $A(z, r = 0)/P_0$ *versus* dimensionless distance from a transducer $z$. Curves (a,b) correspond to defocusing and curves (c,d) to the self-focusing of a single shock stationary wave which is planar at the transducer.

with dimensionless coordinate $z$ and coefficient $\Pi$ determined by Eq. (22).

The graphs show the magnitude of a beam at the axis and its characteristic width,

$$\frac{A(z, r = 0)}{P_0} = \frac{1}{f}, \qquad \frac{a(z)}{a_0} = f \qquad (33)$$

as functions of a dimensionless distance from a transducer $z$ at various $\Pi$.

### 4.2. Solitary shock wave which propagates with a speed different from the linear sound speed

A magnitude of the stationary waveform which was considered in the previous subsection (Eq. (29)) tends to finite but non-zero values at both infinities, $\theta \to \infty$ and $\theta \to -\infty$. There exist other stationary waveforms propagating with speed $\widetilde{c}$ different from the linear sound speed $c_0$. We consider an acoustic pressure

which exhibits a functional form of the retarded time $\tau$, $\tau = t - x/\widetilde{c}$, and $\mu x$, where $\mu$ is a small parameter. The coordinate $\mu x$ is the so-called slow scale correspondent to the retarded time $\tau$. This is a standard way to derive simplified wave equations valid at the leading order with respect to the powers of $\mu$ (RUDENKO, SOLUYAN, 2005; HAMILTON *et al.*, 1997). The Burgers equation may be readily rearranged into the following one:

$$\frac{\partial p}{\partial x} + \left( \frac{1}{c_0} - \frac{1}{\widetilde{c}} \right) \frac{\partial p}{\partial \tau} - \frac{\varepsilon}{c_0^2 \widetilde{c} \rho_0} p \frac{\partial p}{\partial \tau}$$
$$- \frac{b}{2 c_0 \widetilde{c}^2 \rho_0} \frac{\partial^2 p}{\partial \tau^2} = 0. \quad (34)$$

Its stationary solution is

$$p = A \left( 1 + \tanh \left( \frac{A \varepsilon (A \varepsilon + c_0^2 \rho_0) \tau}{b c_0^2 \rho_0} + \frac{\ln(\varepsilon)}{2} \right) \right), \quad (35)$$

where $A$ is some constant. An acoustic pressure (35) propagates with the speed which may exceed the linear sound speed $c_0$, if $A > 0$, or with the smaller speed, if $A < 0$:

$$\widetilde{c} = c_0 + \frac{A\varepsilon}{2c_0\rho_0} \qquad (36)$$

and tends to zero when $\tau$ tends to minus infinity ($A > 0$), or when $\tau$ tends to plus infinity ($A < 0$). Making use of all steps of the scheme which was reported by Rudenko and co-authors, that is: considering $A$ as a function of $x$ and $r$, choosing a new variable $\theta = \tau - \psi(x,r)/\widetilde{c}$ while going to the non-planar geometry of a flow, allowing duration of a shock front tend to zero, and considering a Gaussian perturbation at a transducer, one finally arrives at the equation for the unknown function $f$, which determines $A(x,r)$ by means of Eq. (31) and the equation as follows:

$$f^4 \left( \frac{d^2 f}{dz^2} \right) = \Pi \left( 1 + \frac{4\varepsilon M}{3f} \right). \qquad (37)$$

It differs from the equation which describes the symmetric shock wave propagating with the speed $c_0$, Eq. (32). In view of that $\varepsilon M$ is typically much less than one, and $f$ takes values in the vicinity of the unit, the difference in solutions is not noticeable. The conclusion is that a beam diverges slower for positive $\Pi$ and converges slower for negative $\Pi$ in the case of the shock wave in the form (35) as compared with the shock wave in the form (29).

## 5. Concluding remarks

This study considers thermal self-action of waveforms with finite number of shock fronts propagating in a Newtonian fluid. The conclusions are valid for shock waves with temporal profiles containing discontinuities or steep shock fronts of finite width much smaller than the characteristic duration of a perturbation ($b\omega/(c_0^2\rho_0) \ll 1$), where $2\pi/\omega$ is a duration of an impulse. The solitary shock waves which propagate with different speeds, are also considered. Figure 1 reveals some peculiarities of thermal self-action of one (or integer number of) shock wave(s) which may propagate in a Newtonian fluid. The width of a beam increases or decreases depending on the sign of the thermal coefficient $\delta$. The magnitude of the acoustic pressure at the axis of a beam depends on the signs of $\delta$ and $\Pi$. Zero value of $\Pi$ coincides with the beginning of the non-stationary self-action of the harmonic at the transducer sound beam. The difference of the width of a beam and its peak pressure increases with enlargement of the absolute value of $\Pi$, as compared with that in the case of non-stationary self-action of a periodic beam (evaluations in accordance to formulas of (RUDENKO, SAPOZHNIKOV, 2004)). The nonlinear broadening of a beam is followed by flattening of the transversal beam profile due to stronger absorption near the axis (the so-called isotropization of the directional distribution, which has been discovered in the (KARABUTOV et al., 1988). The effects relating to self-refraction of a monopolar shock wave originate from dependence of shock speed on the magnitude of acoustic pressure. They may exceed the effects relating to the thermal self-action: the variation in speed of the shock front is proportional to the pressure magnitude $A(x,r)$, while variations in sound speed due to thermal effects of sawtooth wave are proportional to $A(x,r)^3$ (RUDENKO, SAPOZHNIKOV, 2004). It may be readily discovered that speed of the shock wave given by Eq. (10) equals $c_0 + \frac{\varepsilon L P_0}{\rho_0 c_0}$ in the leading order. It is independent on coordinate, hence there is no self-refraction due to the shock sawtooth wave in the form Eq. (10).

Equation (21) and relative evaluations depend on dimensionless coefficient $\Pi$, which in turn is expressed in terms of thermodynamical properties of a medium, sound frequency and initial width of a beam. That allows to consider a wide variety of physical conditions of transmission which correspond to a concrete quantity of a dimensionless parameter. For example, series of 300 shock waves in acetone which are emitted by three centimeter transducer with the power of 20 W (this matches $M = 10^{-4}$), corresponds to $\Pi = 1$.

Similarly to the thermal self-actions, other inertial self-action may occur by means of formation of the hydrodynamic streams in a medium due to loss of momentum of an intense sound wave ("acoustic streaming"). This mechanism always leads to additional divergence because the drift caused by streaming makes the speed of sound to increase in the central part of a beam; this occurs in the course of propagation of any waveform in a viscous fluid, periodic or not. In this study, we assume that the thermal self-action takes place in a static medium. The effects associated with the inertial self-action of the sawtooth waves in Newtonian fluids were discussed in (KARABUTOV et al., 1988). The nonlinear effects of ultrasound are of great importance in medical and biological applications, particularly in medical diagnostics and therapy (GURBATOV et al., 2011; O'BRIEN, 2007).

The conclusions of the previous sections concern the case of a medium without thermal conductivity. This is the most simple case which allows to evaluate an excess temperature of a medium by a simple integration of the acoustic force of heating. The general equation which governs the excess temperature in the case of the sawtooth waveform containing $n$ shock fronts may be readily derived by making use of Eq. (5), expanding temperature in the series in the paraxial region,

$$T = T_0 - \frac{r^2}{2}T_2(x,t) - \frac{r^4}{4}T_4(x,t) \qquad (38)$$

and equating multipliers by the even powers of $r$ in Eq. (2). For an ideal gas in the limiting case $b\omega/(c_0^2\rho_0) \to 0$, $T_2$ takes the form

$$T_2 = \frac{8n\varepsilon P_0^3}{a_0^2 c_0^2 C_P f^5 \rho_0^3 \left(1 + \frac{1}{x_s}\int\limits_0^x \frac{\mathrm{d}x'}{f(x')}\right)^4} + \frac{8\chi T_4 \cdot t}{C_P \rho_0},$$

$$T_4 = \frac{8n\varepsilon P_0^3 \left(-3 + \frac{1}{x_s}\int\limits_0^x \frac{\mathrm{d}x'}{f(x')}\right)}{a_0^4 c_0^4 C_P f^7 \rho_0^3 \left(1 + \frac{1}{x_s}\int\limits_0^x \frac{\mathrm{d}x'}{f(x')}\right)^5}.$$

$$(39)$$

The preliminary evaluations reveal that thermal conduction makes thermal self-action of acoustic beam weaker. That is to be expected, because the thermal conduction smoothes the non-uniformity of a medium's temperature.

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