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# A RECURSIVE METHOD FOR THE DETERMINATION OF THE OUTPUT POWER SPECTRUM OF SOME NONLINEAR SYSTEMS

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This paper presents a new method for the determination of the output power spectral density of different types of cascading a linear system with memory and a nonlinear system without memory, with a stationary Gaussian input.

This method permits uniform determination of the output spectrum for each type of the systems mentioned above. The recursive form of the resulting expressions also permits fast calculation of the output spectral density for higher orders of nonlinearity.

## 1. Introduction

This paper presents a new method for the determination of the output power spectral density for different types of cascading a linear system with memory and a nonlinear no-memory system with a stationary Gaussian input. Different methods for calculating the power spectral density in the cases mentioned above have already been discussed [2, 6, 7]. They are relatively complex, however, and for each type of the cascade system a different approach has been used, becoming essentially more complicated with the increasing nonlinearity of the system.

The method presented here is based on the known [1] method for the derivation of the output power spectral density of a general nonlinear system with memory with a stationary Gaussian input. The method proposed gives simple and, equally importantly, uniform determination of the power spectral density.

## 2. The relation between the output power spectral density and the multidimensional transfer functions of cascade-systems

The starting point is a general nonlinear system with memory [1, 3, 4] described by the following set of multidimensional transfer functions

$$\{H_n(\boldsymbol{f}^n)\}, \quad n = 1, 2, \ldots, \tag{1}$$

where $\boldsymbol{f}^n = \{f_1, \ldots, f_n\}$.

The functions $H_n(f_1, \ldots, f_n)$ are symmetrical with respect to their arguments. They are generalizations of the transfer function $H_1(f)$ of a linear-memory system to include the case of a nonlinear-memory system [1, 3-5]. This generalization results directly from the Volterra series approach to the problem of a nonlinear-memory system representation.

When the input $x(t)$ is a stationary ergodic zero-mean Gaussian with the two-sided power spectral density $W_x(f)$, the two-sided power spectral density $W(f)$ of the nonlinear-memory system output $y(t)$ is given by the following expression [1]

$$W_y(f) = |b_0|^2 \delta(f) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{-\infty}^{+\infty} d\boldsymbol{f}^n \, |b_n(\boldsymbol{f}^n)|^2 \, \delta(f - \boldsymbol{f}^n) \prod_{i=1}^{n} W_x(f_i), \tag{2}$$

where

$$\int_{-\infty}^{+\infty} d\boldsymbol{f}^n = \int_{-\infty}^{-\infty} df_1 \ldots \int_{-\infty}^{+\infty} df_n, \tag{3}$$

$$\delta(f - \boldsymbol{f}^n) = \delta(f - f_1 - \ldots - f_n), \tag{4}$$

$$b_0 = \sum_{m=1}^{\infty} \frac{1}{m!\,2^m} \int_{-\infty}^{+\infty} d\boldsymbol{f}^m H_{2m}(\boldsymbol{f}^m, -\boldsymbol{f}^m) \prod_{i=1}^{m} W_x(f_i); \tag{5}$$

$$b_n(\boldsymbol{f}^n) = H_n(\boldsymbol{f}^n) + \sum_{m=1}^{\infty} \frac{1}{m!\,2^m} \int_{-\infty}^{+\infty} d\boldsymbol{k}^m H_{2m+n}(\boldsymbol{f}^n, \boldsymbol{k}_m - \boldsymbol{k}^m) \prod_{i=1}^{m} W_x(k_i), \quad n = 1, 2, \ldots, \tag{6}$$

$$H_{2m}(\boldsymbol{f}^m, -\boldsymbol{f}^m) = H_{2m}(f_1, -f_1, \ldots, f_m, -f_m), \tag{7a}$$

$$H_{2m+n}(\boldsymbol{f}^n, \boldsymbol{k}^m, -\boldsymbol{k}^m) = H_{2m+n}(f_1, \ldots, f_n, k_1, -k_1, \ldots, k_m, -k_m). \tag{7b}$$

Now, the forms and properties of multidimensional transfer functions will be considered for different types of cascading the linear-memory system *LI* and the nonlinear no-memory system *NB*. The formulae for the two-sided output power spectral density will then follow for each type of the cascade-system. The cascade-systems considered here consist of the linear-memory systems *LI* and *LI'* with the transfer functions $H(f)$ and $H'(f)$, respectively, and of the nonlinear no-memory system $NB$ whose nonlinearity is given by the power series

$$y(t) = \sum_{n=1}^{\infty} a_n x^n(t), \tag{8}$$

with $x(t)$ and $y(t)$ being the system input and output, respectively, and $a_n$ denote the coefficients of the series.

A. *The system LI-NB-LI*

The multidimensional transfer functions (1) of the *LI-NB-LI* cascade-system (Fig. 1) are given by [4, 5, 8]

$$H_n(\boldsymbol{f}^n) = a_n H'\Big(\sum_{i=1}^{n} f_i\Big) \prod_{j=1}^{n} H(f_j). \tag{9}$$

It can be easily shown that the multidimensional transfer functions (9) have the following properties

$$H_{2m}(\boldsymbol{f}^m, -\boldsymbol{f}^m) = a_{2m} H'(0) \prod_{i=1}^{m} H(f_i) H(-f_i), \quad m = 1, 2, \ldots, \tag{10}$$

$$H_{2m+n}(\boldsymbol{f}^n, \boldsymbol{k}^m, -\boldsymbol{k}^m) = a_{2m+n} H'\Big(\sum_{i=1}^{n} f_i\Big) \prod_{j=1}^{n} H(f_j) \prod_{l=1}^{m} H(k_l) H(-k_l),$$
$$m = 1, 2, \ldots \tag{11}$$

Using (9)-(11) in (2), (5), (6), the two-sided output power spectral density of the *LI-NB-LI* cascade-system can be expressed as (see the Appendix)

$$W_y(f) = c_0 |H'(0)|^2 \delta f + |H'(f)|^2 \sum_{n=1}^{\infty} \frac{c_n}{n!} \int_{-\infty}^{+\infty} d\boldsymbol{f}^n \, \delta(f - \boldsymbol{f}^n) \prod_{i=1}^{n} W_x(f_i) |H(f_i)|^2, \tag{12}$$

where

$$c_n = \Big| a_n + \sum_{m=1}^{\infty} \frac{1}{m! 2^m} a_{2m+n} I^m \Big|^2, \quad a_0 = 0, \ n = 0, 1, \ldots, \tag{13}$$

$$I = \int_{-\infty}^{+\infty} W_x(f) |H(f)|^2 df.$$

B. *The system LI-NB*

The two-sided output power spectral density for the cascade-system *LI-NB* follows from (12), if $H'(f) = 1$ for all values of the argument $f$, and is given by

$$W_y(f) = c_0 \delta(f) + \sum_{n=1}^{\infty} \frac{c_n}{n!} \int_{-\infty}^{+\infty} d\boldsymbol{f}^n \, \delta(f - \boldsymbol{f}^n) \prod_{i=1}^{n} W_x(f_i) |H(f_i)|^2. \tag{14}$$

The coefficients $c_n$ in formula (14) are defined by (13).

C. *The system NB-LI*

Assumption that $H(f) = 1$ for all values of the argument $f$ in expressions (12) and (13) yields the two-sided output power spectral density for the cascade-system *NB-LI*

$$W_y(f) = d_0|H'(0)|^2\delta(f)+|H'(f)|^2\sum_{n=1}^{\infty}\frac{d_n}{n!}\int_{-\infty}^{+\infty}df^n\,\delta(f-f^n)\prod_{i=1}^{n}W_x(f_i), \quad (15)$$

where

$$d_n = \left|a_n+\sum_{m=1}^{\infty}\frac{1}{m!2^m}a_{2m+n}J^m\right|^2, \quad J = \int_{-\infty}^{+\infty}W_x(f)df, \quad n = 0,1,2,\ldots, a_0 = 0. \quad (16)$$

D. *The system NB*

The two-sided output power spectral density for the system *NB* can be obtained from (12), assuming that $H(f) = H'(f) = 1$ for all $f$, and is given by

$$W_y(f) = d_0\delta(f)+\sum_{n=1}^{\infty}\frac{dn}{n!}\int_{-\infty}^{+\infty}df^n\prod_{i=1}^{n}W_x(f_i)\,\delta(f-f^n), \quad (17)$$

where the coefficients $d_n$ are expressed by (16).

E. *The system LI-LI*

If $a_1 = 1$ and $a_n = 0$, $n = 2, 3, \ldots$, in (13), then

$$W_y(f) = |H(f)|^2|H'(f)|^2W_x(f). \quad (18)$$

F. *The system LI*

The well-known formula

$$W_y(f) = |H(f)|^2W_x(f) \quad (19)$$

results from (12) if $a_1 = 1$, $a_n = 0$, $n = 2, 3, \ldots$ and $H'(f) = 1$ for all $f$.

### 3. A recursive method for the calculation of power spectral density

The calculation of multiple integrals is necessary when formulae (12)-(17) of the previous section are used for the calculation of the power spectral density. This is rather troublesome, especially when computer-calculations are used. To avoid this inconvenience, a recursive method for the calculation of the power spectral density mentioned will be presented in this section.

The following observation is the basis for the recursive method. For $n = 2, 3, \ldots$ let

$$B_n(f) = \int_{-\infty}^{+\infty}df^n\,\delta(f-f^n)\prod_{i=1}^{n}A(f_i),$$

then

$$B_n(f) = \underbrace{A(f)*\ldots*A(f)}_{n} = B_{n-1}(f)*B_1(f), \tag{20}$$

where

$$B_1(f) = A(f) = |H(f)|^2 W_x(f)$$

and $*$ denotes convolution.

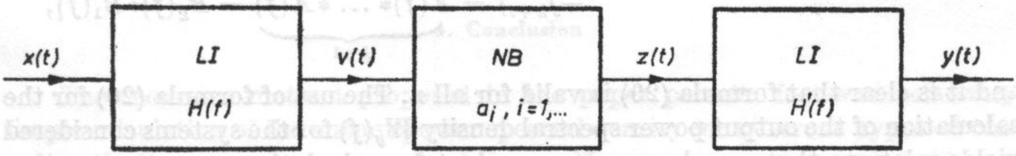


Fig. 1. The *LI-NB-LI* cascade-system

Property (20) can be proven by induction. For $n = 2$

$$B_2(f) = \int_{-\infty}^{+\infty} df^2\, \delta(f-f^2) \prod_{i=1}^{2} A(f_i).$$

Since

$$\delta(f-f^n) = \delta(f-f_1)*\ldots*\delta(f-f_n), \quad n = 2, 3, \ldots, \tag{21}$$

then

$$\delta(f-f^2) = \int_{-\infty}^{+\infty} \delta(l-f_1)\,\delta(f-f_2-l)\,dl.$$

Hence

$$B_2(f) = \int_{-\infty}^{+\infty} dl \int_{-\infty}^{+\infty} A(f_1)\,\delta(l-f_1)\,df_1 \int_{-\infty}^{+\infty} A(f_2)\,\delta(f-l-f_2)\,df_2 =$$

$$= A(f)*A(f) = B_1(f)*B_1(f).$$

Now, suppose that for $n = k$

$$B_k = \int_{-\infty}^{+\infty} df^k\, \delta(f-f^k) \prod_{i=1}^{k} A(f_i) = \underbrace{A(f)*\ldots*A(f)}_{k} = B_{k-1}(f)*B_1(f).$$

Then

$$B_{k-1}(f) = B_k(f)*B_1(f).$$

Indeed, since

$$B_{k+1}(f) = \int_{-\infty}^{+\infty} df^{k+1}\, \delta(f-f^{k+1}) \prod_{i=1}^{k+1} A(f_i)$$

and

$$\delta(f-f^{k+1}) = \int\limits_{-\infty}^{+\infty} \delta(l-f^k)\,\delta(f-l-f_{k+1})\,dl,$$

then

$$B_{k+1}(f) = \int\limits_{-\infty}^{+\infty} dl \int\limits_{-\infty}^{+\infty} df^k\,\delta(l-f^k) \prod_{i=1}^{k} A(f_i) \int\limits_{-\infty}^{+\infty} df_{k+1} A(f_{k+1})\,\delta(f-l-$$

$$-f_{k+1}) = \underbrace{A(f)*\ldots*A(f)}_{k+1} = B_k(f)*B_1(f),$$

and it is clear that formula (20) is valid for all $n$. The use of formula (20) for the calculation of the output power spectral density $W_y(f)$ for the systems considered yields relations that are clear and convenient for calculations.

A. *The system LI-NB-LI*

Application of formula (20) in (12) yields

$$W_y(f) = c_0|H'(0)|^2\,\delta(f) + |H'(f)|^2\left[c_1 B_1(f) + \sum_{n=2}^{\infty} \frac{c_n}{n!} B_{n-1}(f)*B_1(f)\right]. \tag{22}$$

The coefficients $c_n$ are given by (13).

B. *The system LI-NB*

Application of formula (20) in (14) gives

$$W_y(f) = c_0\,\delta(f) + c_1 B_1(f) + \sum_{n=2}^{\infty} \frac{c_n}{n!} B_{n-1}(f)*B_1(f), \tag{23}$$

with the coefficients $c_n$ expressed by (13).

C. *The system NB-LI*

It follows from equation (15) that

$$W_y(f) = d_0|H'(0)|^2\,\delta(f) + |H'(f)|^2\left[d_1 B_1(f) + \sum_{n=2}^{\infty} \frac{d_n}{n!} B_{n-1}(f)*B_1(f)\right], \tag{24}$$

where the coefficients $d_n$ are given by (16).

D. *The system NB*

The resulting form of equation (17) is

$$W_y(f) = d_0\,\delta(f) + d_1 B_1(f) + \sum_{n=2}^{\infty} \frac{d_n}{n!} B_{n-1}(f)*B_1(f) \tag{25}$$

and the coefficients $d_n$ are expressed by (16).

E. *The system LI-LI*

Equation (18) becomes

$$W_y(f) = |H'(f)|^2 B_1(f). \tag{26}$$

F. *The system LI*

Equation (19) yields

$$W_y(f) = B_1(f). \tag{27}$$

## 4. Conclusion

The recursive method proposed in the paper permits fast and relatively simple calculation of the output power spectral density for different types of cascading the liner-memory systems and the nonlinear no-memory system (with any nonlinearity order) with Gaussian inputs. The considerations in section 2 permit uniform calculation of the output power spectral density for any cascade-system. The recursive form of the final formulae, as given in section 3, permits the avoidance of troublesome complexity of formulae that arises with increasing the nonlinearity order. The calculation of multiple integrals may also be avoided.

The cascade-systems mentioned are commonly used models of nonlinear devices. These models can be implemented, for example, for the analysis of a modulator or a detector cascaded with a linear filter — the well known subsystems of radio-devices.

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## Appendix

1. *Derivation of formula (9)*

The relation between the input $x(t)$ and output $y(t)$ of a general nonlinear system with memory can be expressed as follows [1]

$$y(t) = \sum_{n=1}^{\infty} y_n(t), \tag{A1.a}$$

$$y_n(t) = \int_{-\infty}^{+\infty} d\mathbf{s}^n h_n(\mathbf{s}^n) \prod_{i=1}^{n} x(t-s_i), \tag{A1.b}$$

where $h_n(\mathbf{s}^n)$ is an $n$-dimensional inverse Fourier transform of the function $H_n(\mathbf{f}^n)$, i.e. the $n$-dimensional impulse response of the system.

The corresponding relation can be written for the *LI-NB-LI* cascade-system (Fig. 1). Aplication of the configuration of the system (Fig. 1) and consideration that $h_1(\cdot)$ and $h_1'(\cdot)$ are the inverse Fourier transforms of $H(f)$ and $H'(f)$, respectively, yields

$$y(t) = \sum_{n=1}^{\infty} y_n(t), \tag{A2.a}$$

$$y_n(t) = \int_{-\infty}^{+\infty} d\mathbf{s}^n \int_{-\infty}^{+\infty} dr\, a_n h_1'(r) \prod_{i=1}^{n} h(u_i - r) \prod_{i=1}^{n} x(t-u_i), \tag{A2.b}$$

where $u_i = r + s_i$.

Comparison of formulae (A1) and (A2) gives the expression describing the $n$-dimensional impulse response of the *LI-NB-LI* cascade-system:

$$h_n(\mathbf{u}^n) = a_n \int_{-\infty}^{+\infty} h_1'(r) \prod_{i=1}^{n} h_1(u_1 - r)\, dr. \tag{A3}$$

The $n$-dimensional Fourier transform of formula (A3) yields equation (9):

$$H_n(\mathbf{f}^n) = a_n H_n'\Big(\sum_{i=1}^{n} f_i\Big) \prod_{i=1}^{n} H_1(f_i).$$

Formulae (10) and (11) follow directly from (9), after substitution of the relevant arguments and indices.

2. *Derivation of formula (12)*

Application of property (10) in (5) yields

$$b_0 = H'(0) \sum_{m=1}^{\infty} \frac{a_{2m}}{m!\,2^m} I^m. \tag{A4}$$

Combination of formulae (9) and (11) with (6) gives

$$b_n(f^n) = a_n H'\Big(\sum_{j=1}^{n} f_j\Big)\prod_{i=1}^{n} H(f_i)+$$

$$+\sum_{i=1}^{\infty}\frac{1}{m!2^m}\int_{-\infty}^{+\infty} dk^m \prod_{i=1}^{m} W_x(k_i)\prod_{j=1}^{n} H(f_j)H'\Big(\sum_{l=1}^{n} f_l\Big)a_{2m+n}\times$$

$$\times\prod_{p=1}^{m} H(k_p)H(-k_p) = \prod_{i=1}^{n} H(f_i)H'\Big(\sum_{j=1}^{n} f_j\Big)\Big[a_n+\sum_{m=1}^{\infty}\frac{1}{m!2^m}a_{2m+n}I^m\Big]. \quad \text{(A5)}$$

Application of expressions (A4) and (A5) in formula (2) and the use of the relation $f=\sum_{i=1}^{n} f_i$ leads to

$$W_y(f) = |H'(0)|^2\Big|\sum_{m=1}^{\infty}\frac{1}{m!2^m}a_{2m}I^m\Big|^2\delta(f)+$$

$$+\sum_{n=1}^{\infty}\frac{1}{n!}\int_{-\infty}^{+\infty} df^n\prod_{i=1}^{n} W_x(f_i)\Big|\prod_{j=1}^{n} H(f_j)H'\Big(\sum_{l=1}^{n} f_l\Big)\Big[a_n+\sum_{m=1}^{\infty}\frac{1}{m!2^m}a_{2m+n}I^m\Big]\Big|^2\times$$

$$\times\delta(f-f^n) = |H'(o)|^2\Big|\sum_{m=1}^{\infty}\frac{1}{m!2^m}a_{2m}I^m\Big|^2\delta(f)+$$

$$+\Big|a_1+\sum_{m=1}^{\infty}\frac{1}{m!2^m}a_{2m+1}I^m\Big|^2 W_x(f)|H(f)|^2|H'(f)|^2+$$

$$+\sum_{n=2}^{\infty}\frac{1}{n!}\Big|a_n+\sum_{m=1}^{\infty}\frac{1}{m!2^m}a_{2m+n}I^m\Big|^2\int_{-\infty}^{+\infty} df^{n-1}\prod_{i=1}^{n-1} W_x(f_i)|H(f_i)|^2\times$$

$$\times W_x\Big(f-\sum_{j=1}^{n-1} f_j\Big)\Big|H\Big(f-\sum_{j=1}^{n-1} f_j\Big)\Big|^2|H'(f)|^2.$$

The last expression can be written in the desired form of (12).