MODES AND PROJECTORS OF THE VISCOUS NON-UNIFORM FLOW. HEATING CAUSED BY AN ACOUSTIC BEAM BETWEEN TWO PARALLEL PLATES

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The modes of the thermoviscous flow over a strongly non-uniform background have been obtained. Corresponding matrix projecting operators that decompose every mode, or every possible type of the hydrodynamic motion, from the overall field perturbation have been written in the explicit form. Projecting has been used in order to get dynamic nonlinear equations. Acting by the projector corresponds to the entropy mode of the system of conservation laws, while the acoustic mode which is supposed to be dominating yields in the governing equation for the acoustic heating caused by the attenuated acoustic beam. An example of calculations of heating in the viscous flow between two parallel plates has been given.

Key words: non-uniform thermoviscous flow, boundary layer, acoustic heating.

1. Introduction

Acoustic waves are not only a possible type of hydrodynamic motions. Many applications of hydrodynamics and, in particular, aerodynamics deal with slow motions (in comparison to acoustic waves): the vortex and entropy ones that are basic for hydrodynamic motions as well as acoustic waves. The complete classification of hydrodynamic motions of a free stream was made by CHU and KOVASZNAY [1] on the basis of an analysis of the linear flow. The modes have been determined as mathematical links between wave perturbations specific for every type of motion. In general, every small-amplitude disturbance in the free stream can be decomposed into three independent different types of motion: the acoustic, vortical and entropy modes. Only the first one corresponds to the pressure fluctuation propagating with the sound speed, the last two do not cause any pressure perturbation. The results by Chu and Kovasznay relate to hydrodynamic motions over a uniform background without main streams which considerably simplifies the definition of the modes.

Aerodynamics pays a great attention to possibly correct description of flow in a boundary viscous layer. It was proved experimentally that the appearance of the vorticites by the wing of an airplane may cause unstability of a flight; the engineering is aim at the reducing of the turbulence appearing in the flows with large Reynolds numbers as well. The mathematical description of the hydrodynamics in the boundary layer is extremely difficult since the background is strongly non-uniform in the normal direction to the boundary. To consider vortices, the initial system of the conservation equations is traditionally reduced: the continuity equation is of no account since the dynamics of an incompressible liquid is considered. The effect of thermal conductivity is of no account as well: neglecting the thermal conductivity makes the linear system governing the linear incompressible flow complete. Recent discussions [2, 3] suggest that these points are essentially inconsistent since an acoustic wave can propagate only over a compressible medium. Furthermore, it has been proved that the effect of thermal conductivity could not be discarded in the studies of temperature variations: this approximation is well-understood for a typical liquid like water but should be revised for other liquids [4]. The important point is that the entropy mode (slow heating) appears and may grow due to nonlinear interactions in a viscous flow. The entropy mode is a specific compound of a flow, it participates in the nonlinear interaction of modes. Ignoring this mode and going to an incompressible flow that essentially simplifies the calculations is incompatible with a correct physical description of the flow.

These reasons made author to work out governing equations in the boundary layer accounting for the most general flow as possible. The initial point is the overall system of equations of the conservation laws. The next steps are to define the modes as eigenvectors of the linear flow in order to get matrix projectors and then to investigate the nonlinear flow using those matrix projectors. The first results on this way were published in [5]. There appeared also a number of papers by the author concerning investigations of nonlinear flow in the plane and quasi-plane geometry over uniform and non-uniform media [6, 7, 8].

Our idea is to fix relations between specific perturbations (velocity components, density and pressure) for every mode following from the linear equations, and to apply these relations in the studies of nonlinear dynamics. Since modes are defined by eigenvectors correspondent to the dispersion relations of the linear problem, the links do not depend on time. Therefore, an overall field of the linear dynamics may be separated by projectors to the independent modes at any moment. Fixing the relations when going to the nonlinear flow yields in a system of coupled evolution equations for the modes in the most general case of flow. A further simplification of the equations depends on the concrete problem to be solved (e.g. the acoustic mode is dominating). When a plane boundary in a viscous flow is present, there appears a near-boundary flow like the Blasius layer, and a linearization occurs on account of the near-boundary flow as a background. The background stream in the problem relating to a motion between two parallel plates is the Couette flow.

2. Basic conservation equations taking into account equations of state in the general form

The set of equations describing hydrodynamics in a viscous heat-conducting medium has the following form:

$$\rho\left[\frac{\partial e}{\partial t} + (\mathbf{v}\boldsymbol{\nabla})e\right] + p\boldsymbol{\nabla}\mathbf{v} - \chi\Delta T = \varsigma\,(\boldsymbol{\nabla}\mathbf{v})^2 + \frac{\eta}{2}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_l}{\partial x_l}\right)^2,$$

$$\rho\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\boldsymbol{\nabla})\mathbf{v}\right] = -\boldsymbol{\nabla}p + \eta\Delta\mathbf{v} + \left(\varsigma + \frac{\eta}{3}\right)\boldsymbol{\nabla}(\boldsymbol{\nabla}\mathbf{v}), \tag{1}$$

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}(\rho\mathbf{v}) = 0.$$

The final term in the first equation of (1) is written in the Cartesian tensor notation. For details of the notations, the reader is referred to the Appendix. The system (1) is incomplete since two thermodynamic relations are necessary: $e(p, \rho)$, $T(p, \rho)$. The reason of the choice of $e(p, \rho)$ instead of the entropy $s(p, \rho)$, which is used in the most papers [9, 10], is that the internal energy is a specific feature of the chemical structure of the substance and it does not depend on the type of a possible motion (acoustic modes are known as quasi-isentropic). Following the ideas of the previous papers by the author ([11] and the papers referred there), let us use the most general form of these relations as an expansion in the Fourier series:

$$\rho_0 e' = E_1 p' + \frac{E_2 p_0}{\rho_0}\rho' + \frac{E_3}{p_0}p'^2 + \frac{E_4 p_0}{\rho_0^2}\rho'^2 + \frac{E_5}{\rho_0}p'\rho' + \dots,$$

$$T' = \frac{\Theta_1}{\rho_0 C_v}p' + \frac{\Theta_2 p_0}{\rho_0^2 C_v}\rho' + \dots \tag{2}$$

The two-dimensional viscous flow over the half-space $z > 0$ along the x-axes relates to the two-component velocity vector in the coordinates (x, z):

$$\mathbf{v} = (u, w) + \mathbf{u}_0, \tag{3}$$

where $\mathbf{u}_0$ means the background flow and (u, w) mark field perturbations. Accordingly to the geometry of problem and boundary conditions,

$$\mathbf{u}_0 = (U_0(z), 0) \qquad \text{and} \qquad U_0(0) = 0. \tag{4}$$

The system (1), (2) taking into account (3), (4) yields in the following equations written with an accuracy up to quadratic nonlinear terms:

$$\frac{\partial p'}{\partial t} + U_0\frac{\partial p'}{\partial x} + c^2\rho_0\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) - \frac{\chi}{E_1}\left(\frac{\Theta_1}{\rho_0 C_v}\Delta p' + \frac{\Theta_2 p_0}{\rho_0^2 C_v}\Delta\rho'\right)$$

$$= \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right)(\alpha p' + \beta c^2\rho') - u\frac{\partial p'}{\partial x} - w\frac{\partial p'}{\partial z}$$

$$+ \frac{\varsigma}{E_1}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right)^2 + \frac{\eta}{2E_1}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_l}{\partial x_l}\right)^2, \tag{5}$$

$$\rho_0 \left(\frac{\partial u}{\partial t} + U_0 \frac{\partial u}{\partial x} + w \frac{\partial U_0}{\partial z} \right) + \frac{\partial p'}{\partial x} - \eta \Delta u - \left(\varsigma + \frac{\eta}{3} \right) \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 w}{\partial x \partial z} \right)$$

$$= -\rho_0 u \frac{\partial u}{\partial x} - \rho_0 w \frac{\partial u}{\partial z} + \frac{\rho'}{\rho_0} \frac{\partial p}{\partial x}$$

$$\rho_0 \left(\frac{\partial w}{\partial t} + U_0 \frac{\partial w}{\partial x} \right) + \frac{\partial p'}{\partial z} - \eta \Delta w - \left(\varsigma + \frac{\eta}{3} \right) \left(\frac{\partial^2 u}{\partial x \partial z} + \frac{\partial^2 w}{\partial z^2} \right) \qquad \text{(5)}$$
[cont.]

$$= -\rho_0 w \frac{\partial w}{\partial z} - \rho_0 u \frac{\partial w}{\partial x} + \frac{\rho'}{\rho_0} \frac{\partial p}{\partial z}$$

$$\frac{\partial \rho'}{\partial t} + U_0 \frac{\partial \rho'}{\partial x} + \rho_0 \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) = -\rho' \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) - u \frac{\partial \rho'}{\partial x} - w \frac{\partial \rho'}{\partial z},$$

where

$$\alpha = \frac{1}{E_1} \left(-1 + 2 \frac{1 - E_2}{E_1} E_3 + E_5 \right) \quad \text{and} \quad \beta = \frac{1}{1 - E_2} \left(1 + E_2 + 2E_4 + \frac{1 - E_2}{E_1} E_5 \right)$$

are constants, c is the small-signal sound velocity (equation of state for a perfect gas gives the quantities $\alpha = -\gamma = -C_p/C_v$, $\beta = 0$, $c = \sqrt{\gamma p_0/\rho_0}$). Quadratic nonlinear equations for a flow in media other than a perfect gas can be obtained by letting γ be equal to $B/A + 1$. The first equation of (5) follows from the energy balance and the continuity equation.

3. Modes as the basic types of linear motion

The initial system (5) contains four dynamic equations and therefore there are four independent modes of the linear flow: two acoustic ones, vorticity and entropy modes that obviuosly differ from those over the uniform background. The starting point of the investigations of vortices (linear vortices are called also waves of Tollmienn–Schlichting) is to consider an incompressible liquid as already mentioned in the introduction. Formally, if the thermal conductivity is ignored, the left-hand linear terms of the three first equations of (5) do not involve perturbations of density, so the vortex mode may be defined on the basis of these three equations. Note that the right-hand quadratic terms of all these equations *do include* perturbations of density that leads to inconsistency when going to the nonlinear dynamics. If the thermal conductivity is strong, the corresponding linear term proportional to the perturbation of density could not be neglected as well. Excluding the continuity equation leads to a principle impossibility to consider changes of flow related to the entropy mode.

Let us define the modes accordingly to the relations of specific variables following from the *overall* linearized system of dynamic equations (5). Using the right-hand part of the system (5) as a basis of the modes definition and introducing the follwing two non-dimensional functions

$$V_0(z) = U_0(z)/U_\infty, \qquad \phi(z) = V_{0z}(z) l_0, \qquad \text{(6)}$$

we rewrite it in the non-dimensional quantities x_*, z_*, t_*, u_*, w_*, p_*, ρ_* (see Appendix).

In the new variables (asterisks will be omitted everywhere below in the text) a linear analogue of (4) is obtained:

$$\frac{\partial p}{\partial t} + V_0(z)\frac{\partial p}{\partial x} + \varepsilon^{-2}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) - \delta_1\varepsilon\Delta p - \delta_2\varepsilon^{-1}\Delta\rho = 0,$$

$$\frac{\partial u}{\partial t} + V_0(z)\frac{\partial u}{\partial x} + \phi w + \frac{\partial p}{\partial x} - \mathrm{Re}^{-1}\Delta u - R^{-1}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 w}{\partial x\partial z}\right) = 0,$$

$$\frac{\partial w}{\partial t} + V_0(z)\frac{\partial w}{\partial x} + \frac{\partial p}{\partial z} - \mathrm{Re}^{-1}\Delta w - R^{-1}\left(\frac{\partial^2 u}{\partial x\partial z} + \frac{\partial^2 w}{\partial z^2}\right) = 0,$$

$$\frac{\partial \rho}{\partial t} + V_0(z)\frac{\partial \rho}{\partial x} + \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) = 0 \tag{7}$$

with constant parameters δ_1, δ_2, ε, R and Re being the Reynolds number determined in Appendix.

The definition of the modes on the basis of (7) could not be proceeded in a direct way, i.e. by determining the dispersion relations and further links of every mode due to the compexity of the problem: V_0, ϕ are functions of z. Since the geometry of the viscous flow over the boundary supposes strong non-uniformity in the vertical direction, all disturbances may be thought not on the basis of plane waves but as functions like $\psi(z)\exp(i\omega t - ik_x x)$.

3.1. Acoustic modes

The potential flow imposes two acoustic modes with $\partial u/\partial z - \partial w/\partial x = 0$. In the limit of $\mathrm{Re}^{-1} = 0$, $R^{-1} = 0$, $\chi = 0$, $\phi = 0$, (7) yields naturally the dispersion relation for the acoustic modes:

$$\left(\varepsilon^2\frac{\partial^2}{\partial t^2} - \Delta\right)p = 0. \tag{8}$$

Then, two acoustic modes are determined by links of specific perturbations:

$$A_1 = \begin{pmatrix} p_{A1} \\ u_{A1} \\ w_{A1} \\ \rho_{A1} \end{pmatrix} = \begin{pmatrix} 1 \\ \varepsilon\partial_x\Delta^{-1/2} \\ \varepsilon\partial_z\Delta^{-1/2} \\ \varepsilon^2 \end{pmatrix} p_{A1},$$

$$A_2 = \begin{pmatrix} p_{A2} \\ u_{A2} \\ w_{A2} \\ \rho_{A2} \end{pmatrix} = \begin{pmatrix} 1 \\ -\varepsilon\partial_x\Delta^{-1/2} \\ -\varepsilon\partial_z\Delta^{-1/2} \\ \varepsilon^2 \end{pmatrix} p_{A2}. \tag{9}$$

Note that taking into account the linear thermoviscous terms proportional to R^{-1}, Re^{-1}, χ result in the corrected eigenvectors that differ from (9) by operators proportional to these small values. They will be obtained in the Sec. 6 below.

3.2. The Tollmienn–Schlichting (vortex) mode

Formally, the limit of an incompressible fluid ($\rho = 0$) corresponds to the vorticity mode. The first relation for the velocity components of vortex flow is obvious ($\boldsymbol{\nabla}\mathbf{v} = 0$):

$$\partial u/\partial x + \partial w/\partial z = 0. \tag{10}$$

An expression for pressure perturbation follows from the last two equations of (7):

$$2\phi\partial w/\partial x + \Delta p = 0. \tag{11}$$

Both (10) and (11) define the Tollmienn–Schlichting (TS) mode due to links of the specific perturbations of pressure and velocity components:

$$\text{TS} = \begin{pmatrix} p_{\text{TS}} \\ u_{\text{TS}} \\ w_{\text{TS}} \\ \rho_{\text{TS}} \end{pmatrix} = \begin{pmatrix} 1 \\ 0.5\partial_x^{-2}\partial_z\phi^{-1}\Delta \\ -0.5\partial_x^{-1}\phi^{-1}\Delta \\ 0 \end{pmatrix} p_{\text{TS}}. \tag{12}$$

The symbols of the operators are listed in the Appendix.

3.3. The entropy mode

The last type of a possible motion is the entropy mode related to a slow isobaric heating:

$$En = \begin{pmatrix} p_E \\ u_E \\ w_E \\ \rho_E \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \rho_E. \tag{13}$$

As a reffering value, perturbation of density is chosen since all other perturbations are equal to zero. Corrections due to the thermoviscosity will be obtained in the Sec. 6 below.

4. Projecting operators

Every mode is completely defined by one of the specific perturbations – pressure or density or velocity components since there are strict relations between them. They form a complete basis, no other type of motion can exist. In the linear flow, overall perturbation can be decoupled into modes by the orthogonal projectors.

The overall perturbation is a sum of modes which, taking into account (9), (12) and (13), looks:

$$\psi = \begin{pmatrix} p \\ u \\ w \\ \rho \end{pmatrix} = \begin{pmatrix} p_{A1} + p_{A2} + p_{TS} + p_E \\ u_{A1} + u_{A2} + u_{TS} + u_E \\ w_{A1} + w_{A1} + w_{TS} + w_E \\ \rho_{A1} + \rho_{A1} + \rho_{TS} + \rho_E \end{pmatrix}$$

$$= \begin{pmatrix} p_{A1} + p_{A2} + p_{TS} + 0 \\ Hp_{A1} - Hp_{A2} + Kp_{TS} + 0 \\ Mp_{A1} - Mp_{A2} + Qp_{TS} + 0 \\ \varepsilon^2 p_{A1} + \varepsilon^2 p_{A2} + 0 + \rho_E \end{pmatrix} = L \begin{pmatrix} p_{A1} \\ p_{A2} \\ p_{TS} \\ \rho_E \end{pmatrix}. \qquad (14)$$

L denotes a matrix operator including the elements: $H = \varepsilon\Delta^{-1/2}\partial_x$, $M = \varepsilon\Delta^{-1/2}\partial_z$, $K = 0.5\partial_x^{-2}\partial_z\phi^{-1}\Delta$, $Q = -0.5\partial_x^{-1}\phi^{-1}\Delta$.

The inverse to the L matrix yields immediately in matrix projectors being the tensor products as follows:

$$P_{A1} = \begin{pmatrix} 1 \\ \varepsilon\partial_x\Delta^{-1/2} \\ \varepsilon\partial_z\Delta^{-1/2} \\ \varepsilon^2 \end{pmatrix}$$
$$\left(\frac{1}{2} \quad -\partial_x^2\Delta^{-1}\phi\partial_z\Delta^{-1} + 0.5\partial_x\varepsilon^{-1}\Delta^{-1/2} \quad \partial_x^3\Delta^{-1}\phi\Delta^{-1} + 0.5\varepsilon^{-1}\Delta^{-1/2}\partial_z \quad 0 \right),$$

$$P_{A2} = \begin{pmatrix} 1 \\ -\varepsilon\partial_x\Delta^{-1/2} \\ -\varepsilon\partial_z\Delta^{-1/2} \\ \varepsilon^2 \end{pmatrix}$$
$$\left(\frac{1}{2} \quad -\partial_x^2\Delta^{-1}\phi\partial_z\Delta^{-1} - 0.5\partial_x\varepsilon^{-1}\Delta^{-1/2} \quad \partial_x^3\Delta^{-1}\phi\Delta^{-1} - 0.5\varepsilon^{-1}\Delta^{-1/2}\partial_z \quad 0 \right),$$

$$\qquad (15)$$

$$P_{TS} = \begin{pmatrix} 1 \\ 0.5\partial_x^{-2}\partial_z\phi^{-1}\Delta \\ -0.5\partial_x^{-1}\phi^{-1}\Delta \\ 0 \end{pmatrix} \left(0 \quad 2\partial_x^2\Delta^{-1}\phi\partial_z\Delta^{-1} \quad -2\partial_x^3\Delta^{-1}\phi\Delta^{-1} \quad 0 \right),$$

$$P_{En} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \left(-\varepsilon^2 \quad 2\varepsilon^2\partial_x^2\Delta^{-1}\phi\partial_z\Delta^{-1} \quad -2\varepsilon^2\partial_x^3\Delta^{-1}\phi\Delta^{-1} \quad 1 \right).$$

The projectors possess all properties of orthogonal projectors and their sum is a unit matrix operator since all eigenvectors of the linear system are taken into account.

$$
\begin{aligned}
P_{\text{TS}} + P_{A1} + P_{A2} + P_{En} &= I, \\
P_{\text{TS}} \cdot P_{A1} = P_{\text{TS}} \cdot P_{A2} &= \ldots = 0, \\
P_{\text{TS}} &= P_{\text{TS}} \cdot P_{\text{TS}}, \quad \ldots
\end{aligned}
\tag{16}
$$

where I and 0 are unit and zero matrices. In the linear flow, the projectors separate every mode from the overall perturbation, for example:

$$
P_{\text{TS}} \begin{pmatrix} p \\ u \\ w \\ \rho \end{pmatrix} = \begin{pmatrix} p_{\text{TS}} \\ u_{\text{TS}} \\ w_{\text{TS}} \\ \rho_{\text{TS}} \end{pmatrix},
\tag{17}
$$

and so on. Moreover, when the projector is acting on the system of dynamic equations (7), one gets a linear evolution equation for the mode corresponding to this projector, in fact three equations for every referring value. For the first (rightwards) acoustic mode the linear evolution equation is:

$$
\partial p_{A1}/\partial t + V_0(z)\partial p_{A1}/\partial x + \varepsilon^{-1}\Delta^{1/2}p_{A1} = 0.
\tag{18}
$$

A perturbation of density of the entropy mode satisfies the linear equation:

$$
\partial \rho_{En}/\partial t + V_0(z)\partial \rho_{En}/\partial x = 0.
\tag{19}
$$

The acting of P_{TS} at (7) yields in the well-known well-known equation for the TS mode, when rewritten for the new variable such as the stream function Ψ ($u = \partial\Psi/\partial y$, $w = -\partial\Psi/\partial x$):

$$
\Delta\partial\Psi/\partial t + V_0\Delta\partial\Psi/\partial x - \partial\Psi/\partial x \cdot \partial\phi/\partial z - \text{Re}^{-1}\Delta^2\Psi = 0.
\tag{20}
$$

When Ψ is sought in the form $\Psi = \Phi(z)\exp(i\omega t - ik_x x)$, the well-known Orr–Sommerfeld (OS) equation follows from (20):

$$
(V_0(z) - c)\left(\partial_z^2\Phi - k_x^2\Phi\right) - \Phi\partial_z\phi = \frac{i}{\text{Re}\,k_x}\left(\partial_z^4\Phi - 2k_x^2\partial_z^2\Phi + k_x^4\Phi\right).
\tag{21}
$$

That equation is an initial point of the laminar flow stability and determines an eigenvalue $\Phi(z)$ and complex phase velocity $c = \omega/k_x = c_r + ic_i$ for every pair (k_x, Re). The sign of c_i is namely a criterion of the flow stability: a negative value corresponds to the growth of perturbation and thereby to the non-stability of the flow.

5. Coupling dynamic equations for nonlinear flow

In the non-dimensional quantities, taking into account the nonlinear terms of the second order, the dynamic equations are as follows:

$$
\begin{aligned}
\frac{\partial p}{\partial t} + V_0\frac{\partial p}{\partial x} + \varepsilon^{-2}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) - \delta_1\varepsilon\Delta p - \delta_2\varepsilon^{-1}\Delta\rho &= \widetilde{\psi}_1, \\
\frac{\partial u}{\partial t} + V_0\frac{\partial u}{\partial x} + \phi w + \frac{\partial p}{\partial x} - \mathrm{Re}^{-1}\Delta u - R^{-1}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 w}{\partial x\partial z}\right) &= \widetilde{\psi}_2, \\
\frac{\partial w}{\partial t} + V_0\frac{\partial w}{\partial x} + \frac{\partial p}{\partial z} - \mathrm{Re}^{-1}\Delta w - R^{-1}\left(\frac{\partial^2 u}{\partial x\partial z} + \frac{\partial^2 w}{\partial z^2}\right) &= \widetilde{\psi}_3, \\
\frac{\partial\rho}{\partial t} + V_0\frac{\partial\rho}{\partial x} + \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) &= \widetilde{\psi}_4,
\end{aligned}
\tag{22}
$$

with a vector of the second-order nonlinear elements $\widetilde{\psi}$ standing for the right-hand side:

$$
\widetilde{\psi} = \begin{pmatrix}\widetilde{\psi}_1 \\ \widetilde{\psi}_2 \\ \widetilde{\psi}_3 \\ \widetilde{\psi}_4\end{pmatrix} = \begin{pmatrix}
-u\dfrac{\partial p}{\partial x} - w\dfrac{\partial p}{\partial z} + \left(\dfrac{\partial u}{\partial x} + \dfrac{\partial w}{\partial z}\right)\left(\alpha p + \varepsilon^{-2}\beta\rho\right) \\[2ex]
+\dfrac{\mathrm{Re}^{-1}}{E_1}\left(\dfrac{\partial v_i}{\partial x_k} + \dfrac{\partial v_k}{\partial x_i} - \dfrac{2}{3}\delta_{ik}\dfrac{\partial v_l}{\partial x_l}\right)^2 \\[2ex]
-u\dfrac{\partial u}{\partial x} - w\dfrac{\partial u}{\partial z} + \rho\dfrac{\partial p}{\partial x} \\[2ex]
-u\dfrac{\partial w}{\partial x} - w\dfrac{\partial w}{\partial z} + \rho\dfrac{\partial p}{\partial z} \\[2ex]
-u\dfrac{\partial\rho}{\partial x} - w\dfrac{\partial\rho}{\partial z} - \rho\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial w}{\partial z}\right)
\end{pmatrix}.
\tag{23}
$$

Acting by every projector at the nonlinear system (22) results in a set of coupled nonlinear equations. The reader is referred to the papers [6–8, 11] for the detail applications of projecting to the different problems of hydrodynamics. It should be noted that in the nonlinear right-hand side all values present a sum of specific perturbations of all modes:

$$
\begin{aligned}
p &= 2\partial_x^2\Delta^{-1}(\phi\Psi) + p_{A1} + p_{A2}, \\
u &= \partial_z\Psi + \varepsilon\Delta^{-1/2}\partial_x p_{A1} - \varepsilon\Delta^{-1/2}\partial_x p_{A2}, \\
w &= -\partial_x\Psi + \varepsilon\Delta^{-1/2}\partial_z p_{A1} - \varepsilon\Delta^{-1/2}\partial_z p_{A2}, \\
\rho &= \varepsilon^2\left(p_{A1} + p_{A2}\right) + \rho_{En}.
\end{aligned}
\tag{24}
$$

The equations governing a flow may be written algorithmically by projecting of the basic system into a specific evolution equation for every mode. Final equations are very difficult to solute since they are general differential equations with multipliers and operators being functions of z.

6. Heating caused by losses in energy of an acoustic beam

As a limit that simplifies considerably the mathematics, let us go to a quasi-plane viscous flow over non-uniform background. The longitudinal derivative ∂_x is supposed to be much larger than the transversal one ∂_z (in other words, k_x is much larger than k_z). That allows to present the Laplace operator as a series of the small value k_z/k_x: $\Delta = \partial_x^2(1 + \partial_z^2/\partial_x^2)$; thus $\Delta^{1/2} \approx \partial_x(1 + 0.5\partial_z^2/\partial_x^2)$. This is a standard procedure in the theory of acousctics: the equation governing an acoustic beam, i.e. the Khokhlov–Zabolotskaya–Kuznetsov (KZK) equation, is obtained in this way [9, 10]. The corrected acoustic and entropy modes are as follows (the vortex one keeps the same as before in Eq. (12)):

$$A_1 = \begin{pmatrix} 1 \\ \varepsilon(1 - 0.5\,(\partial_z/\partial_x)^2 - \partial_x(B/2 - \delta_1 - \delta_2)) \\ \varepsilon\partial_z/\partial_x \\ \varepsilon^2(1 + \partial_x(\delta_1 + \delta_2)) \end{pmatrix} p_{A1},$$

$$A_2 = \begin{pmatrix} 1 \\ \varepsilon(-1 + 0.5\,(\partial_z/\partial_x)^2 - \partial_x(B/2 - \delta_1 - \delta_2)) \\ -\varepsilon\partial_z/\partial_x \\ \varepsilon^2(1 - \partial_x(\delta_1 + \delta_2)) \end{pmatrix} p_{A2}, \qquad (25)$$

$$En = \begin{pmatrix} 0 \\ \varepsilon\delta_2\partial_x \\ 0 \\ 1 \end{pmatrix} \rho_E.$$

B means the overall attenuation due to thermoviscous phenomena:

$$B = \delta_1 + \delta_2 + \varepsilon\left(R^{-1} + \mathrm{Re}^{-1}\right). \qquad (26)$$

All calculations before are proceeded with an accuracy $O(k_z/k_x, B)$. The corresponding projectors can be corrected accordingly to the new definition of modes in the thermoviscous flow (25). Similar calculations relating to the quasi-plane flow over a uniform background were proceeded by the author in the paper [11]. In particular, the entropy projector has the form:

$$P_{En} = \begin{pmatrix} 0 \\ \varepsilon\delta_2\partial_x \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} -\varepsilon^2 & 2\varepsilon^2\partial_x^{-2}\phi\partial_z - \partial_x(\delta_1 + \delta_2) & -2\varepsilon^2\partial_x^{-1}\phi & 1 \end{pmatrix}. \qquad (27)$$

The projector P_{En}, when acting on system (22), yields an evolution equation for the density of the entropy mode caused by the progressive acoustic waves if only inputs of the rightwards progressive acoustic mode are hold in the right-hand side of the nonlinear vector. For the flow over an ideal gas without thermal conductivity $E_1 = (\gamma - 1)^{-1}$, $\delta_1 = 0$, $\delta_2 = 0$, it reads in the leading order:

$$\frac{\partial \rho_{En}}{\partial t} + V_0(z)\frac{\partial \rho_{En}}{\partial x} = (\gamma - 1)\,\varepsilon^3 \left(p_{A1}\frac{\partial p_{A1}}{\partial x} - 0.5 B p_{A1}\frac{\partial^2 p_{A1}}{\partial x^2} - B\left(\frac{\partial p_{A1}}{\partial x}\right)^2 \right)$$

$$+ \varepsilon^4 \phi(z) \int p_{A1}\frac{\partial^2 p_{A1}}{\partial x \partial z}\,dx. \qquad (28)$$

In flows over media other than perfect gas, γ should be replaced by $B/A + 1$. The pressure of acoustic rightwards propagating wave is a solution of the nonlinear evolution equation as follows:

$$\frac{\partial p_{A1}}{\partial t} + \left(V_0(z) + \varepsilon^{-1}\right)\frac{\partial p_{A1}}{\partial x} + \frac{\gamma + 1}{2}\varepsilon p_{A1}\frac{\partial p_{A1}}{\partial x} - \frac{\varepsilon^{-1}B}{2}\frac{\partial^2 p_{A1}}{\partial x^2}$$

$$+ \frac{\varepsilon^{-1}}{2}\int \frac{\partial^2 p_{A1}}{\partial z^2}\,dx = 0, \qquad (29)$$

which in fact is analogous to the KZK equation with transferring over the background flow term $V_0(z)\dfrac{\partial p_{A1}}{\partial x}$. More precisely, Eq. (29) corresponds to one of the branches of the KZK equation, the rightwards beam, whereas the KZK equation satisfies both the acoustic modes and includes temporal derivatives of the second order. The acoustic field must satisfy the corresponding boundary conditions as well.

6.1. Application to a viscous flow between two parallel plates

The final equation governing acoustic heating is Eq. (28) with an acoustic source in the right-hand side which is a solution of Eq. (29) modified by the transferring term (modified KZK). It is known that the analytical solution of the KZK equation is very difficult for and the solution of the modified Eq. (29) is even much more difficult.

As an acoustic wave, a non-diffracting beam caused by a plane transducer of radius d placed between two parallel plates (d is dimensionless since the dimensional radius divided by the distance l_0 between parallel plane plates) is considered. The pressure of the acoustic beam is a result of the product of the transversal Gaussian function and a periodic solution of the Burgers equation for the plane wave of amplitude P_0 with a transferring term:

$$p(x, z, t) = P_0 \exp\left(-z^2/d^2 - 0.5 B x k_{ak}^2\right) \sin(k_{ak}(x - t(V_0(z) + \varepsilon^{-1}))). \qquad (30)$$

Here, k_{ak} is the characteristic longitudinal wavenumber of the acoustic wave (dimensionless value which equals to dimensional wavenumber multiplied by the distance l_0 between the plates). The acoustic wavelength is supposed to be much smaller than both the radius of transducer and distance between plates:

$$k_{ak} \gg 1, \qquad k_{ak} \gg d^{-1}. \tag{31}$$

According to the first relation of (31) the variations of the background flow are supposed to be slow in comparison with the pressure variations in the acoustic beams and therefore the transferring term is involved in equation (30) in a simple way. Simultaneously, $1 \gg d$. The latter condition allows to consider an unbounded beam.

The background viscous flow between two parallel plates is a solution of the linearized Navier-Stokes equation and is a so-called Couette flow:

$$V_0(z) = z + 0.5 - 0.5\mathrm{Re}^{-1}\frac{\partial p_0}{\partial x}\left(z + 0.5\right)\left(z - 0.5\right). \tag{32}$$

The dimensionless transversal co-ordinates $z = 0.5$, $z = -0.5$ correspond to the upper plate and to the lower one, $\dfrac{\partial p_0}{\partial x} = $ const is a longitudinal gradient of the background pressure. Formula (32) yields a following expression for $\phi(z)$: $\phi(z) = 1 - z\mathrm{Re}^{-1}\partial p_0/\partial x$. The velocity of the Couette flow tends to zero at the lower plate and to 1 at the upper one (U_∞ in dimensional quantities). Figure 1 a demonstrates the geometry of the background flow.
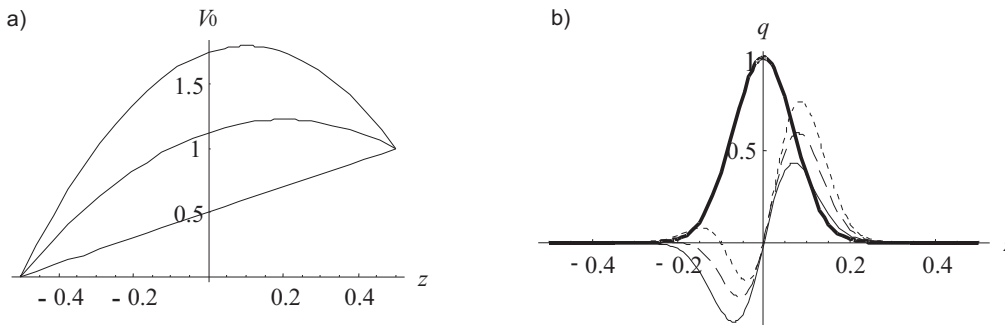

Fig. 1. a) Geometry of the background flow (the Couette flow) when $\mathrm{Re}^{-1}\partial p_0/\partial x$ equals to 0, 5, 10 (from the lower curve to the upper one); b) Heating caused by a non-diffracting beam: the thin line shows the heat release over the uniform background. The solid line denotes an additional heating due to the non-uniformity of the background viscous flow when $\mathrm{Re}^{-1}\partial p_0/\partial x$ equals to zero. The dashed and dropped curves correspond to additional heating when the values of $\mathrm{Re}^{-1}\partial p_0/\partial x$ equal to 5 and 10.

Temporal averaging over a period of the acoustic wave gives in the leading order an equation for the heat production (isobaric decrease of density) with an averaged acoustic source:

$$-q = \left\langle \frac{\partial \rho_E}{\partial t} \right\rangle = -0.5 BP_0^2 \exp(-Bxk_{ak}^2 - 2z^2/d^2)$$

$$\cdot \varepsilon^3 \left(k_{ak}^2 \left(\gamma - 1 \right) + \varepsilon \phi(z)z/d^2 \right). \qquad (33)$$

The value q denotes the dimensionless release of acoustic energy in the unit time and in the unit volume. The last term in the brackets caused by the presence of the non-uniform background flow expresses additional losses of the acoustic energy due to the non-uniformity. After temporal averaging, the acoustic source in the right-hand side of equation (33) does not depend on time and the losses of energy are proportional to the time.

An illustration of the heat release is presented in Fig. 1b: the thick line shows the heat release by the first term in brackets, while the thin line shows the heat release by the second term in brackets of the formula (33) corresponding to different values of $\mathrm{Re}^{-1}\partial p_0/\partial x$. The illustration is an approximate one: the relative amplitudes of the curves depend on $B, k_{ak}, \varepsilon, V_0(0)$ accordingly to (30) and (33) and can be evaluated for every concrete flow. The value $d = 0.1$ was taken for the calculations. Because of the background non-uniformity, the heating in the upper half of the space is more intense, while it is less intense in the lower space. The integrated release of energy in a unit cross section and in a unit time decreases when $\mathrm{Re}^{-1}\partial p_0/\partial x > 0$ and remains unchanged when $\mathrm{Re}^{-1}\partial p_0/\partial x = 0$.

7. Conclusions

Thermoviscous flows over a strongly non-uniform background belong to the most difficult problems of hydrodynamics. The author has worked out an algorithmic procedure to distinguish different types of hydrodynamic motions with the help of projecting in this complex problem. The dynamic equation for every mode is a result of acting of a corresponding projector on the differential system of the conservation laws. Matrix projectors have been derived in an explicit form.

In particular, the equations governing the acoustic beam and acoustic heating are derived in the Sec. 6. Examples of calculations of the acoustic heating caused by a strongly attenuated non-diffracting beam propagating over the Couette background flow with different curvatures are presented. It has been shown that in the case of a viscous flow between two plane parallel plates, where the losses of energy by a strongly attenuated non-diffracting acoustic beam is the reason of heating, the main flow non-uniformity results in an additional heating.

The possibilities of the method overcome the application to acoustic heating. This method gives a complete set of governing equations for every mode. The basic difficulty is rather mathematical one, i.e. it is the absence of analytical solutions for the complicated differential equation in partial derivatives.

Appendix

ρ	density (ρ_0 denotes unperturbed value, $\rho' = \rho - \rho_0$ denotes an excess density),
p	pressure (p_0 denotes unperturbed value, $p' = p - p_0$ denotes an excess pressure),
e	internal energy per unit mass (e' denotes an excess quantity),
T	temperature (T' denotes an excess quantity),
η	shear viscosity,
ς	bulk viscosity,
χ	thermal conductivity,
x_i	space coordinates ($x_1 = x$, $x_2 = z$),
$\mathbf{v}$	velocity,
v_i	the component of $\mathbf{v}$ in direction x_i: $v_1 = u$, $v_2 = w$,
$\mathbf{u}_0 = (U_0(z), 0)$	velocity of the background flow,
δ_{ik}	the Kronecker delta, equal to unity for $i = j$ and zero otherwise,
C_p, C_v	the specific heats per unit mass at constant pressure and volume, relatively,
ω	circular frequency,
k	wavenumber with components k_x, k_y,
$E_1, \ldots E_5$	dimensionless coefficients in the Taylor series of internal energy,
Θ_1, Θ_2	dimensionless coefficients in the Taylor series of temperature,
$c = \sqrt{\dfrac{p_0(1 - E_2)}{\rho_0 E_1}}$	a small-signal sound velocity,
$P_{A1}, P_{A2}, P_{TS}, P_{En}$	matrix projecting operators distinguishing every mode (rightwards acoustic, leftwards acoustic, vortex and entropy) from the overall flow,
U_∞	characteristic velocity of the background flow,
l_0	characteristic scale of the background flow,
Ψ	stream function of a vortex flow.

Dimensionless quantities

$V_0(z) = U_0(z)/U_\infty$	velocity of the background viscous flow,
$\phi(z) = V_{0z}(z) l_0$	transversal gradient of velocity of the background viscous flow,
$x_* = x/l_0$	longitudinal coordinate,
$z_* = z/l_0$	transversal coordinate,
$u_* = u/U_\infty$	longitudinal compound of velocity perturbation,

$w_* = w/U_\infty$ transversal compound of velocity perturbation,

$t_* = tU_\infty/l_0,$

$\rho_* = \rho'/\rho_0,$

$p_* = p'/\rho_0 U_\infty^2,$

$\varepsilon = U_\infty/c,$

$\mathrm{Re} = U_\infty l_0 \rho_0/\eta$ is the Reynolds number,

$R = U_\infty l_0 \rho_0/(\eta/3 + \varsigma)$ viscous coefficient,

$$\delta_1 = \frac{\Theta_1}{E_1 C_V \rho_0 c l_0}\chi,$$

$$\delta_2 = \frac{\Theta_2}{(1-E_2)C_V \rho_0 c l_0}\chi$$
       coefficients associated with thermal conductivity.

Operators

∂_x, ∂_z denote $\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial z}$ relatively,

$\partial_x^{-1}, \partial_z^{-1}$ denote $\int dx, \int dz,$

Δ the Laplacian, correspondent in the Fourier space to the operator $-k_x^2 - k_z^2,$

$\Delta^{1/2}$ an operator, correspondent in the Fourier space to the operator $\left(-k_x^2 - k_z^2\right)^{1/2},$

$\Delta^{-1/2}$ an operator, correspondent in the Fourier space to the operator $\left(-k_x^2 - k_z^2\right)^{-1/2}.$

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