APPLIED GENERAL TOPOLOGY

Appl. Gen. Topol. 20, no. 1 (2019), 97-108
doi:10.4995/agt.2019.10036

F-n-resolvable spaces and compactifications

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Communicated by S. García-Ferreira

Abstract

A topological space is said to be resolvable if it is a union of two disjoint dense subsets. More generally it is called n-resolvable if it is a union of n pairwise disjoint dense subsets.
In this paper, we characterize topological spaces such that their reflections (resp., compactifications) are n-resolvable (resp., exactly-n-resolvable, strongly-exactly-n-resolvable), for some particular cases of reflections and compactifications.

2010 MSC: *54B30; 54D10; 46M15.*

Keywords: *categories; functors; resolvable spaces; compactifications.*

Introduction

Let $n > 1$ be an integer. Generalizing the concept of resolvable spaces introduced by Hewitt in [16], Ceder in [6] defined a topological space X to be n-resolvable space if it has a family of n pairwise disjoints dense subsets. The latter is called exactly n-resolvable if it is n-resolvable but not $(n+1)$-resolvable and it is called strongly exactly n-resolvable denoted by SE_nR if it is n-resolvable and no empty subset of X is $(n+1)$-resolvable. SE_1R space is commonly said strongly irresolvable space (abbreviated as SI-space) or hereditarily irresolvable (see [7] and [13]).

Received 25 April 2018 – Accepted 01 February 2019

The theory of categories and functors play an enigmatic role in topology, specially the notion of reflective subcategories. Recently, some authors have been interested by particular functors like $\mathbf{T_0}$, $\mathbf{S}$, ρ and $\mathbf{FH}$.

In [10], [11] and [8], the authors have characterized topological spaces whose F-reflections are door, submaximal, nodec and resolvable.

Some papers, as [5] and [3] were interested in spaces such that their compactifications are submaximal, door and nodec. Specially in [2], K. Belaid and M. Al-Hajri have characterized topological spaces such that their one point compactifications (resp., Wallman compactifications) are resolvable.

In the first section of this paper, we characterize topological spaces such that their T_0-reflections are n-resolvable (resp., exactly n-resolvable, strongly exactly n-resolvable).

In the second section, topological spaces, such that their Tychonoff reflections and functionally Hausdorff reflections are n-resolvable (resp., exactly n-resolvable), have been characterized.

The third section of this paper is devoted to a characterization of topological spaces such that their one point compactifications (resp., Wallman compactifications) are n-resolvable (resp., exactly n-resolvable, strongly exactly n-resolvable).

1. T_0-n-resolvable spaces, T_0-exactly-n-resolvable spaces and T_0-strongly-exactly-n-resolvable spaces.

Let X be a topological space. The $\mathbf{T}_0$-reflection of X denoted by $\mathbf{T}_0(X)$ is defined as follow.

Consider the equivalence relation $\sim$ on X by: $x \sim y$ if and only if $\overline{\{x\}} = \overline{\{y\}}$, for $x, y \in X$. Then the resulting quotient space $\mathbf{T}_0(X) := X/\sim$ is a Kolmogroff space called the $\mathbf{T}_0$-reflection of X.

Recall that a continuous map $q : X \longrightarrow Y$ is said to be a quasihomeomorphism if $U \longmapsto q^{-1}(U)$ (resp., $C \longmapsto q^{-1}(C)$) defines a bijection $\mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$ (resp., $\mathcal{F}(Y) \longrightarrow \mathcal{F}(X)$), where $\mathcal{O}(X)$ (resp., $\mathcal{F}(X)$) is the collection of all open sets (resp., closed sets) of X)[15].
In particular the canonical surjection $\mu_X : X \longrightarrow \mathbf{T}_0(X)$ is an onto *quasihomeomorphism* and consequently a closed (resp., open) map, (see [4]).

In order to give the main result of this section we recall the following results introduced in [10].

Notation 1.1 ([10, Notations 2.2]). Let X be a topological space, $a \in X$ and $A \subseteq X$. We denote by:

(1) $d_0(a) := \{x \in X : \overline{\{x\}} = \overline{\{a\}}\}$.
(2) $d_0(A) = \cup[d_0(a); a \in A]$.

Remark 1.2 ([10, Remarks 2.3]). Let X be a topological space and A be a subset of X. The following properties hold.

(i) $d_0(A) = \mu_X^{-1}(\mu_X(A))$.

(ii) $d_0(d_0(A)) = d_0(A)$.
(iii) $A \subseteq d_0(A) \subseteq \overline{A}$ and consequently $\overline{d_0(A)} = \overline{A}$.
(iv) In particular if A is open (resp., closed), then $d_0(A) = A$.

The following definitions are natural.

Definition 1.3. A topological space X is called *T_0-n-resolvable* (resp., *T_0-exactly-n-resolvable*, *T_0-strongly-exactly-n-resolvable*) if its T_0-reflection is n-resolvable (resp., exactly-n-resolvable, strongly-exactly-n-resolvable).

Before giving the characterization of T_0-n-resolvable spaces, let us introduce the following definition.

Definition 1.4. A family $\{A_i : i \in I\}$ of subsets of a topological space X is called pairwise d_0-disjoint if and only if $d_0(A_i) \cap d_0(A_j) = \varnothing$, for any $i \neq j \in I$.

By Remarks 1.2 (*iii*), a pairwise d_0-disjoint family is a pairwise disjoint family.

The following result characterise T_0-n-resolvable spaces.

Theorem 1.5. *Let X be a topological space. Then the following statements are equivalent:*

(1) *X is a T_0-n-resolvable space;*
(2) *X have a dense pairwise d_0-disjoint family with cardinality n.*

Proof. (1)$\Longrightarrow$(2)

Suppose that X is a T_0-n-resolvable space. Then $T_0(X)$ has a dense pairwise disjoint family $\{\mu_X(A_i);\ 1 \leq i \leq n\}$, where A_1,..., A_n are subsets in X. So applying μ_X^{-1}, one can see easily that $\{d_0(A_i) :\ 1 \leq i \leq n\}$ is a family of pairwise disjoint subsets of X.

Now since μ_X is an onto quasihomeomorphism then, by [10, Lemma 2.16], we have:

$$\forall 1 \leq i \leq n \ \ X = \mu_X^{-1}(\mathbf{T}_0(X)) = \mu_X^{-1}\left(\overline{\mu_X(A_i)}\right) = \overline{\mu_X^{-1}(\mu_X(A_i))} = \overline{d_0(A_i)}.$$

Therefore $\{A_i;\ 1 \leq i \leq n\}$ is a dense pairwise d_0-disjoint family of X.

(2)$\Longrightarrow$(1)

Suppose that X has a dense pairwise d_0-disjoint family $\{A_i;\ 1 \leq i \leq n\}$ with cardinality n. Then, for any $1 \leq i \neq j \leq n$, the condition $d_0(A_i) \cap d_0(A_j) = \varnothing$ implies immediately that $\mu_X(A_i) \cap \mu_X(A_j)) = \varnothing$.

Now, let $1 \leq i \leq n$. The density of $d_0(A_i)$ in X shows that:

$$T_0(X) = \mu_X(X) = \mu_X(\overline{d_0(A_i)}) = \mu_X(\overline{\mu_X^{-1}(\mu_X(A_i))}) = \mu_X(\mu_X^{-1}\left(\overline{\mu_X(A_i)}\right)) = \overline{\mu_X(A_i)}.$$

Therefore, $\{\mu_X(A_i) :\ 1 \leq i \leq n\}$ is a dense pairwise disjoint family of subsets of $T_0(X)$. □

Remark 1.6. Clearly every T_0-n-resolvable space is a n-resolvable space. The converse does not hold, indeed:

Let X be a subset of cardinality n $(n > 1)$ equipped with the indiscreet topology. Clearly the family $\{\{x\};\quad x \in X\}$ is composed by disjoint dense subsets of X and thus X is n-resolvable. But $T_0(X)$ is a one point which is not 2-resolvable. Remark that in this case $d_0(\{x\}) = X$, for any $x \in X$ and consequently, $d_0(A) = X$ for any subset A of X, therefore there is no d_0-disjoint family of X with cardinality greater or equal to 2.

The following result is an immediate consequence of the previous theorem.

Corollary 1.7. *Let X be a topological space. X is a T_0-exactly-n-resolvable space if and only if* $\max\{\mid \mathcal{F} \mid \quad \mathcal{F}$ *is a dense* d_0- *disjoint family of* $X\} = n$.

Before giving a characterization of a T_0-strongly-exactly-n-resolvable space we need the following lemma.

Lemma 1.8. *Let X be a topological space and S a subset of X. Then* $\mu_X(S) \simeq \mu_S(S)$.

Proof. S is a subset of X then, the following diagram is commutative.

$$\begin{array}{ccc} S & \xrightarrow{\ i\ } & X \\ {\scriptstyle \mu_S}\downarrow & \square & \downarrow{\scriptstyle \mu_X} \\ \mathbf{T}_0(S) & \xrightarrow[\mathbf{T}_0(i)]{} & \mathbf{T}_0(X) \end{array}$$

- $T_0(\mathbf{i}) : T_0(S) \longrightarrow T_0(\mathbf{i})(T_0(S))$ is bijective. In fact it is enough to show that $T_0(\mathbf{i})$ is one-to-one.

Let x, y two elements of S such that $T_0(\mathbf{i})(\mu_S(x)) = T_0(\mathbf{i})(\mu_S(y))$. Then, $\mu_X(\mathbf{i}(x)) = \mu_X(\mathbf{i}(y))$ and thus $\mu_X(x) = \mu_X(y)$. Hence, we get $\overline{\{x\}}^S = \overline{\{x\}} \cap S = \overline{\{y\}} \cap S = \overline{\{y\}}^S$, as desired.

- $T_0(\mathbf{i})$ is an open map. Indeed, let $\widetilde{U}$ be an open set of $T_0(S)$. Then, there exists V an open set of X such that $\mu_S^{-1}(\widetilde{U}) = V \cap S$. Thus

$$\begin{array}{rcl} T_0(\mathbf{i})(\widetilde{U}) & = & T_0(\mathbf{i})(\mu_S(V \cap S)) \\ & = & \mu_X(\mathbf{i}(V \cap S)) \\ & = & \mu_X(V \cap S) \end{array}$$

So, let us show that $\mu_X(V \cap S) = \mu_X(V) \cap T_0(\mathbf{i})(T_0(S))$. Indeed:

$$\begin{array}{rcl} \mu_X(V \cap S) & \subseteq & \mu_X(V) \cap \mu_X(S) \\ & = & \mu_X(V) \cap \mu_X(\mathbf{i}(S)) \\ & = & \mu_X(V) \cap T_0(\mathbf{i})(\mu_S(S)) \\ & = & \mu_X(V) \cap T_0(\mathbf{i})(T_0(S)) \end{array}$$

Which gives the first inclusion.

Conversely, let $x \in \mu_X(V) \cap T_0(\mathbf{i})(T_0(S))$. Then there exist $y \in V$ and $t \in S$ such that $\mu_X(y) = x = T_0(\mathbf{i})(\mu_S(t)) = \mu_X(\mathbf{i}(t)) = \mu_X(t)$. Thus, $\overline{\{y\}} = \overline{\{t\}}$.

Since $y \in V$, then $V \cap \{t\} \neq \varnothing$. So, $x = \mu_X(t) \in \mu_X(V \cap S)$ which proves that $\mu_X(V) \cap T_0(\mathbf{i})(T_0(S)) \subseteq \mu_X(V \cap S)$ which gives the second inclusion as desired.

- $\mu_X(S) \simeq \mu_S(S)$.

According to the above, we conclude that $T_0(\mathbf{i})$ is an homeomorphism from $T_0(S)$ to $T_0(\mathbf{i})(T_0(S))$. Then, $\mu_X(S) = \mu_X(\mathbf{i}(S)) = T_0(i)(\mu_S(S)) = T_0(i)(T_0(S)) \simeq T_0(S) = \mu_S(S)$. □

Theorem 1.9. *Let X be a topological space. Then the following statements are equivalent:*

(1) *X is a T_0-strongly-exactly-n-resolvable space.*
(2) *X is T_0-n-resolvable and for any subset S of X, S is not T_0-$(n+1)$-resolvable.*

Proof. (1)$\Longrightarrow$(2)

Let S be a subset of X. Since X is T_0-strongly-exactly-n-resolvable, $\mu_X(S)$ is not $(n+1)$-resolvable. Then, by Lemma 1.8, $\mu_S(S) = T_0(S)$ is not $(n+1)$-resolvable. Therefore, X is a T_0-n-resolvable space in which every subset S of X, is not T_0-$(n+1)$-resolvable.

(2)$\Longrightarrow$(1)

Let $\mu_X(S)$ be a subset of $T_0(X)$, where S be a subset of X. By hypothesis, S is not $T_0 - (n+1)$-resolvable that is $T_0(S) = \mu_S(S)$ is not $(n+1)$-resolvable. Using Lemma 1.8, $\mu_X(S)$ is not $(n+1)$-resolvable. So that every subset $\mu_X(S)$ of $T_0(X)$ is not $(n+1)$-resolvable and thus $T_0(X)$ is strongly-exactly-n-resolvable. □

2. $\boldsymbol{\rho}$-n-resolvable spaces and $\boldsymbol{FH}$-n-resolvable spaces

Recall that a T_1 topological space X is called Tychonoff if for any closed subset F of X and for any $x \in X$ not in F there exists a real continuous map f from X to $\mathbb{R}$ (we write $f \in \mathbf{C}(X)$) such that $f(x) = 0$ and $f(F) = \{1\}$. We say that F and x are completely separated. In particular two distinct points in a given Tychonoff space X are said to be completely separated if x and $\{y\}$ are completely separated.

A T_1 topological space in which every two distinct points are completely separated, is called functionally Hausdorff space.

Give a topological space X. We define the equivalence relation $\sim$ on X by $x \sim y$ if and only if $f(x) = f(y)$ for all $f \in \mathbf{C}(X)$.

On the one hand, the set of equivalence classes $X/\sim$ equipped with the quotient topology, is a functionally Hausdorff space called the **FH**-reflection of X.

On the other hand, consider ρ_X the canonical surjection map from X to $X/\sim$. Then for any continuous map f_α from X to $\mathbb{R}$, there exists a unique map $\rho(f_\alpha)$ from $X/\sim$ to $\mathbb{R}$ satisfying $\rho(f_\alpha)(\rho_X(x)) = f(x)$, for any $x \in X$. So, $X/\sim$ equipped with the the topology whose closed sets are of the form $\cap[\boldsymbol{\rho}(f_\alpha)^{-1}(F_\alpha) : \alpha \in I]$, where $f_\alpha : X \longrightarrow \mathbb{R}$ (resp., F_α) is a continuous map (resp., a closed subset of $\mathbb{R}$), is a a Tychonoff space (see for instance [22]) called the ρ-reflection of X.

We need to introduce and recall some definitions, notations and results.

Notation 2.1 ([10, Notation 3.1]). Let X be a topological space, $a \in X$ and A a subset of X. We denote by:

(1) $d_{\boldsymbol{\rho}}(a) := \cap[f^{-1}(f(\{a\})) : f \in \mathbf{C}(X)]$.
(2) $d_{\boldsymbol{\rho}}(A) := \cup[d_{\boldsymbol{\rho}}(a) : a \in A]$.

Definition 2.2. Let X be a topological space. X is called:

(1) *ρ-n-resolvable* (resp., **FH**-*n-resolvable*) space if its ρ-reflection (resp., **FH**-reflection) is a n-resolvable space.
(2) *ρ-exactly-n-resolvable* (resp., **FH**-*exactly-n-resolvable*) space if its ρ-reflection (resp., **FH**-reflection) is an exactly-n-resolvable space.
(3) *ρ-strongly-exactly-n-resolvable* (resp., **FH**-*strongly-exactly-n-resolvable*) space if its ρ-reflection (resp., **FH**-reflection) is a strongly-exactly-n-resolvable space.

Recall that for a given topological space X and $A \subseteq X$, A is called a zero-set if there exists $f \in C(X)$ such that $A = f^{-1}(\{0\})$. The complement of a zero-set is called a cozero-set.

A space is Tychonoff if and only if the family of zero-sets of the space is a base for the closed sets (equivalently, the family of cozero-sets of the space is a base for the open sets)(see [22, Proposition 1.7]). In [10] it is showen that a closed (resp., open) subset of $\boldsymbol{\rho}(X)$ is of the form $\cap[\boldsymbol{\rho}(f)^{-1}(\{0\}) : f \in H]$ (resp., $\cup[\boldsymbol{\rho}(f)^{-1}(\mathbb{R}^{\star}) : f \in H]$) , where H is a collection of continuous maps from X to $\mathbb{R}$.

Definition 2.3 ([10, Definition 3.14])**.** Let X be a topological space, a subset V of X is called:

(i) a *functionally open* subset of X (for short F-open) if and only if $d_{\rho}(V)$ is open in X.
(ii) a *functionally dense* subset of X (for short F-dense) if and only if for any F-open subset W of X, $d_{\boldsymbol{\rho}}(V)$ meets $d_{\boldsymbol{\rho}}(W)$.
(iii) $\boldsymbol{\rho}$-dense, if $g(V) \neq \{0\}$ for every nonzero continuous map g from X to $\mathbb{R}$.

Definition 2.4. Let X be a topological space and $\{A_i : i \in I\}$ be a family of subsets of X. We say that this family is pairwise $d_{\boldsymbol{\rho}}$-disjoint if and only if $d_{\boldsymbol{\rho}}(A_i) \cap d_{\boldsymbol{\rho}}(A_j) = \varnothing$, for any $i \neq j \in I$.

Theorem 2.5. *Let X be a topological space. Then the following statements are equivalent:*

(i) *X is **FH**-n-resolvable.*
(ii) *X have a F-dense pairwise $d_{\boldsymbol{\rho}}$-disjoint family with cardinality n.*

Proof. $(i) \Longrightarrow (ii)$

Suppose that X is an **FH**-n-resolvable space. Then, there exists a family $\{\rho_X(A_1), ..., \rho_X(A_n)\}$ of dense pairwise disjoint subsets of $\mathbf{FH}(X)$.

Now, applying ρ_X^{-1}, we see easily that the family $\{A_1, ..., A_n\}$ is pairwise $d_{\boldsymbol{\rho}}$-disjoint. Finally, the equality $\overline{\boldsymbol{\rho}_X(A_i)} = \mathbf{FH}(X)$ means that A_i is a F-dense subset of X. Therefore, $\{A_1, ..., A_n\}$ is pairwise $d_{\boldsymbol{\rho}}$-disjoint family of X with cardinality n.

$(ii) \Longrightarrow (i)$

Conversely, let $\{A_i : 1 \leq i \leq n\}$ be a family of F-dense pairwise $d_{\boldsymbol{\rho}}$-disjoint subsets of X. Then on the one hand, for every $1 \leq i \leq n$, $\boldsymbol{\rho}_X(A_i)$ is a dense subset of $\mathbf{FH}(X)$ and on the other hand, $\forall\ 1 \leq i \neq j \leq n$, we have

$$\begin{array}{rcl} d_{\boldsymbol{\rho}}(A_i) \cap d_{\boldsymbol{\rho}}(A_j)) & = & \boldsymbol{\rho}_X^{-1}(\boldsymbol{\rho}_X(A_1)) \cap \boldsymbol{\rho}_X^{-1}(\boldsymbol{\rho}_X(A_j))) \\ & = & \boldsymbol{\rho}_X^{-1}(\boldsymbol{\rho}_X(A_i) \cap \boldsymbol{\rho}_X(A_j))) \\ & = & \varnothing \end{array}$$

Therefore, $\{\rho_X(A_1), ..., \rho_X(A_n)\}$ is a family of dense pairwise disjoint subsets of $\mathbf{FH}(X)$. □

By the same way as in Theorem 2.5, the following result is immediate.

Theorem 2.6. *Let X be a topological space. Then the following statements are equivalent:*

(i) X is $\boldsymbol{\rho}$-n-resolvable.

(ii) X have a $\boldsymbol{\rho}$-dense and pairwise $d_{\boldsymbol{\rho}}$-disjoint family of cardinality n.

Remark 2.7. Since every F-dense subset is a $\boldsymbol{\rho}$-dense subset (see [10, Remarks 3.15]), then by Theorem 2.6, every $\mathbf{FH}$-n-resolvable space is $\boldsymbol{\rho}$-n-resolvable.

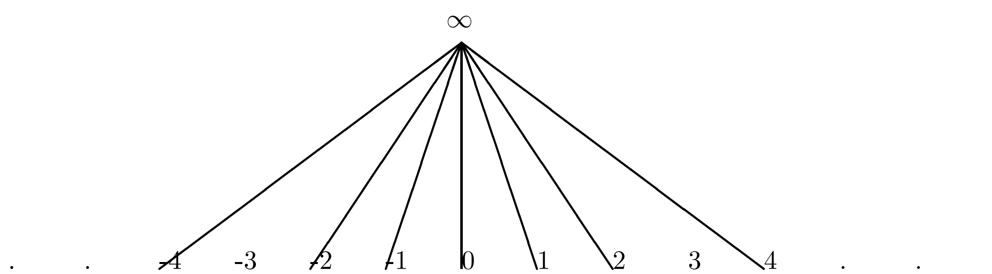
The following results are immediate.

Corollary 2.8. *Let X be a topological space. X is a $\mathbf{FH}$-exactly-n-resolvable space if and only if* $\max\{|\ \mathcal{F}\ |\quad \mathcal{F}$ *is F-dense and $d_{\boldsymbol{\rho}}-$disjoint family of X*$\} = n$.

Corollary 2.9. *Let X be a topological space. X is a $\boldsymbol{\rho}$-exactly-n-resolvable space if and only if* $\max\{|\ \mathcal{F}\ |\quad \mathcal{F}$ *is $\boldsymbol{\rho}$-dense and $d_{\boldsymbol{\rho}}-$disjoint family of X*$\} = n$.

Remark 2.10. Regarding Lemma 1.8, this result does not subsist in the case of the functors FH and ρ as showing by the following example.

Consider the Alexandroff space $X = \mathbb{Z} \cup \{\infty\}$ such that $\overline{\{n\}} = \{n\}$, for every $n \in \mathbb{Z}$ and $\overline{\{\infty\}} = X$. It is clear that every real continuous map from X is constant and thus $FH(X) = \rho(X)$ is a one point space. Now, consider $S = \mathbb{Z}$, then $FH(S) = \rho(S) = S$, but $\rho_X(S)$ is a one point. One can illustrates this situation by the following picture.



Question 2.11. *The Theorem 1.9 is an immediate consequence of Lemma 1.8 which is not valuable in the case of the functors FH and ρ as showing by Remark 2.10. Hence the following question is immediate. Are* **FH***-strongly-exactly-n-resolvable (resp., ρ-strongly-exactly-n-resolvable) spaces equivalent to FH-n-resolvable (resp., ρ-n-resolvable) in which every subset S of X, is not FH-$(n+1)$-resolvable (resp., ρ-$(n+1)$-resolvable)?*

3. n-resolvable spaces and compactifications

Definition 3.1. A *compactification* of a topological space X is a pair $(K(X), e)$ where $K(X)$ is a compact space and e an embedding of X as a dense subset of $K(X)$.

Remark 3.2. In many cases, e will be an inclusion map, so that $X \subseteq K(X)$. In other cases , we can agree to write X when mean $e(X)$, so that we can again write $X \subseteq K(X)$. Whenever one of this situations occurs we say simply that $K(X)$ is a compactification of X, and think of $K(X)$ as containing X as a dense subspace.

Lemma 3.3 ([2, Lemma 2.1])**.** *Let X be a topological space and $K(X)$ be a compactification of X and A be a subset of $K(X)$. If X is an open set of $K(X)$ Then the following statements are equivalent:*

1. *A is a dense subset of $K(X)$.*
2. *$A \cap X$ is a dense subset of X.*

Using Lemma 3.3, the following proposition is immediate.

Proposition 3.4. *Let X be a topological space and $K(X)$ be a compactification of X. If X is an open set of $K(X)$ Then the following statements are equivalent:*

(1) X *is* n*-resolvable.*
(2) $K(X)$ *is* n*-resolvable.*

Recall that for a topological space X, the set $\widetilde{X} = X \cup \{\infty\}$ with the topology whose members are the open sets of X and all subsets U of $\widetilde{X}$ such that $\widetilde{X} \setminus U$ is a closed compact subset of X, is called the Alexandroff extension of X (or the one-point compactification of X).

Now, regarding Proposition 3.4, we get immediately the following result.

Corollary 3.5. *Let X be a non compact topological space Then the following statements are equivalent:*

(1) *The one point compactification $\widetilde{X}$ of X is n-resolvable.*
(2) *X is n-resolvable.*

We turn our attention to spaces such that their Wallman compactifications are n-resolvable spaces.

First, let us recall the construction of Wallman compactification of T_1-space (a concept introduced, in 1938, by Wallman [23]).

Let $\mathcal{P}$ be a class of subsets of a topological space X wich is closed under fnite intersections and finite unions.

A $\mathcal{P}$-filter on X is a collection $\mathcal{F}$ of nonempty elements of $\mathcal{P}$ with the properties:

(a) $P_1, P_2 \in \mathcal{F}$ implies $P_1 \cap P_2 \in \mathcal{F}$.
(b) $P_1 \in \mathcal{F}$ $P_1 \subseteq P_2$ implies $P_2 \in \mathcal{F}$.

A $\mathcal{P}$-ultrafilter is a maximal $\mathcal{P}$-filter. When $\mathcal{P}$ is the class of closed sets of X, then the $\mathcal{P}$-filters are called closed filters.

The points of the Wallman compactification wX of a space X are the closed ultrafilters on X. For each closed set $D \subseteq X$, define D^* to be the set $D^* = \{\mathcal{A} \in wX : D \in \mathcal{A}\}$, if $D \neq \varnothing$ and $\varnothing^* = \varnothing$. Thus $\mathcal{C} = \{D^* : D$ is a closed set of $X\}$ is a base for the closed sets of a topology on wX.

Let U be an open subset of X. We define $U^* = \{\mathcal{A} \in wX : F \subseteq U$ for some F in $\mathcal{A}\}$. It is easily seen that the class $\{U^* : U$ is an open set of $X\}$ is a base for the open sets of the topology of wX. The following properties of wX are frequently useful:

Proposition 3.6. *For $x \in X$, let $w_X(x) = \{A \mid A$ is a closed set of X and $x \in A\}$. Then w_X is an embedding of X into wX. Thus, if $x \in X$, then $w_X(x)$ will be identified to x.*

Proposition 3.7. *If $U \subset X$ is open, then $wX \setminus U^* = (X \setminus U)^*$.*

Proposition 3.8. *If $D \subset X$ is closed, then $wX \setminus D^* = (X \setminus D)^*$.*

Proposition 3.9. *If U_1 and U_2 are open in X, then $(U_1 \cap U_2)^* = U_1^* \cap U_2^*$ and $(U_1 \cup U_2)^* = U_1^* \cup U_2^*$.*

In [19], Kovar has characterized space with finite Wallman compactification remainder as following:

Proposition 3.10. *Let X be a T_1-space, wX the Wallman compactification of X and k a finite number. Then the following statements are equivalent:*

1. *$Card(wX - X) = k$.*
2. *There exists a collection of k pairwise disjoint non compact closed sets of X and every family of non compact pairwise disjoint closed sets of X contain at most k elements.*

The following proposition follows immediately from Proposition 3.10.

Proposition 3.11. *Let X be a T_1-space and $k \in \mathbb{N}$ such that every family of non compact pairwise disjoint closed sets of X contain at most k elements. Then X is n-resolvable if and only if wX is n-resolvable.*

Corollary 3.12 ([2, corollary 3.5])**.** *Let X be a T_1-space, wX be the Wallman compactification of X and U be an open set of X. Then the following statements are equivalent:*

1. *$U \subsetneq U^*$.*
2. *There exists a non compact closed set F of X such that $F \subseteq U$.*

Definition 3.13. Let X be a T_1-topological space. Then X is said to be w-n-resolvable, if its Wallman compactification is n-resolvable.

Before characterizing w-n-resolvable spaces, let us introduce the useful definition.

Definition 3.14. We said that a finite family of subsets $\{D_i,\ i \in I\}$ of a topological space $(X, \mathcal{O}(X))$ satisfies the property $(\mathcal{P})$ if:

for every $(i, O) \in J = I \times \{O \in \mathcal{O}(X) :\ O \cap D_i = \varnothing\}$, there exists a non compact closed subset $F_{O,i} \subset O$ with $\{F_{O,i} :\ (i, O) \in J\}$ is a family of pairwise disjoint subsets of X.

Now, let us give one of the main result of this section.

Theorem 3.15. *Let X be a T_1- topological space, Then the following statements are equivalent:*

1. *X is w-n-resolvable.*
2. *X is a partition of a family of n subsets satisfying $(\mathcal{P})$.*

Proof. Let X be a w-n-resolvable space. Then there exist n pairwise disjoint dense subsets A_1,A_2,..., A_n of wX such that $wX = A_1 \cup A_2 \cup ... \cup A_n$. We denote $D_i = A_i \cap X$. It is clear that the family $\{D_i; 1 \le i \le n\}$ is a partition of X.

Let O be a nonempty open subset of X such that $O \cap D_i = \varnothing$. The density of A_i in wX gives an element $\mathcal{F}_i \in O^* \cap A_i$. By Corollary 3.12, there exists a non compact closed subset $G_{(i,O)} \subset O$ such that $G_{(i,O)} \in \mathcal{F}_i$.

Now, if i' is distinct from i and O' is a given nonempty open subset of X such that $O' \cap D_{i'} = \varnothing$, by the same way, there exists an element $\mathcal{F}_{i'} \in O'^* \cap A_{i'}$ and consequently there exists a non compact closed subset $G_{(i',O')} \subset O'$ such that $G_{(i',O')} \in \mathcal{F}_{i'}$. Since $A_i \cap A_{i'} = \varnothing$, then $\mathcal{F}_i \neq \mathcal{F}_{i'}$. Thus, there exist a closed subsets $F_i \in \mathcal{F}_i$ and $F_{i'} \in \mathcal{F}_{i'}$ such that $F_i \cap F_{i'} = \varnothing$. Let $F_{(i,O)} = G_{(i,O)} \cap F_i$ and $F_{(i',O')} = G_{(i',O')} \cap F'_i$. It is clear that $F_{(i,O)} \in \mathcal{F}_i \in wX \setminus X$ and $F_{(i',O')} \in \mathcal{F}_{i'} \in wX \setminus X$. Hence, $F_{(i,O)}$ and $F_{(i',O')}$ are non compact closed subsets (see [2, Lemma 3.4]), which are disjoint.

Conversely, let $\{D_i; 1 \leq i \leq n\}$ be a partition of X by n subsets satisfying $(\mathcal{P})$. For every $1 \leq i \leq n$, set $A_i = D_i \cup \{\mathcal{F} \in wX - X : \ F_{(i,O)} \in \mathcal{F}\}$, where O is an open subset of X such that $O \cap D_i = \varnothing$ (it is clearly seen that if $A_i = D_i$, then D_i is dense in wX). Clearly, by construction, A_i is a dense subset of wX for every $1 \leq i \leq n$.

To finish, let us show that the family $\{A_i : \ 1 \leq i \leq n\}$ are pairwise disjoint. So, suppose the existence of $1 \leq i \neq j \leq n$ such that $A_i \cap A_j \neq \varnothing$. Since $D_i \cap D_j = \varnothing$, then $A_i \cap A_j \cap (wX - X) \neq \varnothing$. By construction of A_i and A_j, there exist an ultrafilter $\mathcal{F}_i \in A_i$ and $\mathcal{F}_j \in A_j$ such that $\mathcal{F}_i = \mathcal{F}_j$. Furthermore, there exist open subsets O, O' and non compact closed subsets $F_{(i,O)} \in \mathcal{F}_i$ and $F_{(j,O')} \in \mathcal{F}_j$ such that $O \cap D_i = \varnothing$, $O' \cap D_j = \varnothing$, $F_{(i,O)} \subset O$ and $F_{(j,O')} \subset O'$. Hence, by the property $(\mathcal{P})$, $F_{(i,O)} \cap F_{(j,O')} = \varnothing$ and consequently $\mathcal{F}_i \neq \mathcal{F}_j$, which leads to a contradiction. □

As an immediate consequence of Theorem 3.15, for the particular case when $n = 2$, we have the following corollary.

Corollary 3.16 ([2, Theorem 3.6])**.** *Let X be a T_1- topological space, Then the following statements are equivalent:*

(1) *X is w-resolvable.*
(2) *X is a partition of two subsets $\{D_1, D_2\}$ and for each nonempty open subset $O \subseteq D_i$ ($i \in \{1, 2\}$), there exists a non compact closed subset F such that $F \subseteq O$.*

To close this section the following result is immediate.

Corollary 3.17. *Let X be a T_1-topological space. X is w-exactly-n-resolvable if and only if*
$\max\{\mid \mathcal{F} \mid ; \mathcal{F}$ *is a partition of* X*, of* n *dense subsets satisfying* $(\mathcal{P})\} = n$.

Acknowledgements. *The authors gratefully acknoweledge helpful corrections, comments and suggestions of the referee. This paper is supported by the LATAO LR11ES12.*

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