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When is the super socle of $C(X)$ prime?

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Dedicated to professor O.A.S. Karamzadeh who is not only a "role model" for us, he is also so for many other mathematics teachers and students, alike, in this country

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Abstract

Let $SC_F(X)$ denote the ideal of $C(X)$ consisting of functions which are zero everywhere except on a countable number of points of X. It is generalization of the socle of $C(X)$ denoted by $C_F(X)$. Using this concept we extend some of the basic results concerning $C_F(X)$ to $SC_F(X)$. In particular, we characterize the spaces X such that $SC_F(X)$ is a prime ideal in $C(X)$ (note, $C_F(X)$ is never a prime ideal in $C(X)$). This may be considered as an advantage of $SC_F(X)$ over $C_F(X)$. We are also interested in characterizing topological spaces X such that $C_c(X) = \mathbb{R} + SC_F(X)$, where $C_c(X)$ denotes the subring of $C(X)$ consisting of functions with countable image.

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1. Introduction

We refer the reader to [7] and [12] for necessary background concerning X, and $C(X)$, the ring of all real-valued continuous functions on a space X. All topological spaces X in this note are infinite completely regular Hausdorff, unless otherwise mentioned. $C_F(X)$, the socle of $C(X)$, is the sum of all minimal ideals of $C(X)$ which is also the intersection of all essential ideals in $C(X)$. An

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ideal in a commutative ring is essential if it intersects every nonzero ideal of the ring nontrivially, see [14], where the socle of $C(X)$ is topologically characterized. It is folklore that one of the main objectives of working in the context of $C(X)$ is to characterize topological properties of a given space X in terms of a suitable algebraic properties of $C(X)$. It turns out, in this regard, the ideal $C_F(X)$, plays an appropriate role in the literature, see for examples [2], [3], [5], [6], [8], [9] [14], [15], and [17]. Motivated by this role of $C_F(X)$, the concept of the super socle, which contains $C_F(X)$, is introduced in [11], see also [16]. We are going to extend some of the basic results of the socle of $C(X)$ (i.e., $C_F(X)$) to the super socle of $C(X)$ (i.e., $SC_F(X)$). An outline of this article is as follows:
In Section 2, the concept of the super socle and some preliminary results concerning this ideal, which are frequently used in the subsequent sections, are given. In the next section, we are going to investigate the primeness of the super socle in $C(X)$. This may be considered as an advantage of $SC_F(X)$ over $C_F(X)$ (note, $C_F(X)$ is never a prime ideal in $C(X)$). We also characterize topological spaces X such that $C_c(X) = \mathbb{R} + SC_F(X)$, where $C_c(X)$ denotes the subring of $C(X)$ consisting of functions with countable image, see [9, Proposition 6.6], [10].

2. Preliminaries

We begin with the definition of the super socle of $C(X)$ which is motivated by [11, Proposition 3.3].

Definition 2.1. The set $S = \{f \in C(X) : X \setminus Z(f) \text{ is countable}\}$ is called the super socle of $C(X)$ and it is denoted by $SC_F(X)$.

One can easily see that $SC_F(X)$ is a z-ideal in $C(X)$. Clearly $C_F(X) \subseteq SC_F(X)$, by [11, Proposition 3.3].
It is trivial to see that a point in a space X is isolated if and only if it has a finite neighborhood. Motivated by this, the next two definitions are natural and are also needed.

Definition 2.2. An element $p \in X$ is called a countably isolated point if p has a countable neighborhood. The set of countably isolated points of X is denoted by $I_c(X)$.

Definition 2.3. If every point of X is countably isolated, then X is called a countably discrete space. Thus, X is a countably discrete space in case $I_c(X) = X$.

We recite the following results which are in [11].

Proposition 2.4 ([11], Proposition 2.4)**.** *For any space X, $I_c(X) = \bigcup\{coz(f) : f \in SC_F(X)\}$.*

Corollary 2.5 ([11], Corollary 2.5)**.** *For any space X, we have the following:*
(a) $SC_F(X)$ is not a zero ideal if and only if X has a countably isolated point.
(b) $SC_F(X)$ is a free ideal if and only if X is countably discrete.

We remind the reader that an ideal I is said to be regular if for every $a \in I$ there exists $b \in I$ such that $a = aba$.

Theorem 2.6 ([11], Theorem 2.10)**.** *The following are equivalent for an uncountable space X.*
(1) *X is the one-point Lindelöffication of some uncountable discrete space.*
(2) *$SC_F(X)$ is a regular ideal, and $SC_F(X) = O_x$ for some $x \in X$.*

3. The primeness of $SC_F(X)$ in $C(X)$

We recall that $C_F(X)$ is never a prime ideal in $C(X)$, see [1], [8, Proposition 1.2]. Our aim in this section is to investigate the primeness of the super socle in $C(X)$. We begin with an example to show that $SC_F(X)$ can be a prime ideal (even a maximal ideal). This may be considered as an advantage of $SC_F(X)$ over $C_F(X)$.

Example 3.1. Let $X = Y \cup \{x_0\}$ be the one-point Lindelöffication of a countably discrete space Y, then we claim that $C(X) = \mathbb{R} + SC_F(X)$ and this shows that $SC_F(X)$ is a maximal ideal in $C(X)$. To see this, let $f \in C(X)$, then we consider two cases. First, let $x_0 \in Z(f)$, then for all $n \in \mathbb{N}$, $x_0 \in f^{-1}(\frac{-1}{n}, \frac{1}{n})$. Therefore for all $n \in \mathbb{N}$, $X \setminus f^{-1}(\frac{-1}{n}, \frac{1}{n})$ is countable. So $X \setminus Z(f)$ is countable (note, $Z(f) = \bigcap\limits_{n\in\mathbb{N}} f^{-1}(-\frac{1}{n}, \frac{1}{n})$), i.e., $f \in SC_F(X) \subseteq \mathbb{R} + SC_F(X)$. Now, let $x_0 \notin Z(f)$, then there exists $0 \neq r \in \mathbb{R}$, such that $f(x_0) = r$. Put $g = f - r$, hence $x_0 \in Z(g)$, then by what we have already proved $g \in SC_F(X)$ and so $f \in \mathbb{R} + SC_F(X)$.

In the next theorem, we characterize the spaces X such that $SC_F(X)$ is a prime ideal in $C(X)$. We need the following well-known lemma, which is in [12, 4I. 4]. Let us first emphasize that since every prime ideal P in $C(X)$ is contained in a unique maximal ideal, hence it is either free or it is in the fixed maximal ideal M_x for a unique $x \in X$.

Lemma 3.2. *Let P be a fixed prime ideal in $C(X)$. Then there exists $x \in X$ with $O_x \subseteq P \subseteq M_x$.*

Theorem 3.3. *Let $SC_F(X)$ be a prime ideal in $C(X)$. Then X is either a countably discrete space or the one-point Lindelöffication of a countably discrete space.*

Proof. If $SC_F(X)$ is a free ideal in $C(X)$, then by Corollary 2.5, X is a countably discrete space. If not, then by Lemma 3.2, there exists $x \in X$ such that $O_x \subseteq SC_F(X)$. Consequently, in view of Theorem 2.6, X is either countable or the one-point Lindelöffication of an uncountable discrete space (i.e., a countably discrete space), hence we are done. □

The following corollary is now immediate.

Corollary 3.4. *Let X have at least one non-countably isolated point. Then $SC_F(X)$ is a prime ideal if and only if X is the one-point Lindelöffication of a countably discrete spaces.*

Finally, motivated by Example 3.1 and [9, Proposition 6.6], we are interested in characterizing topological spaces X such that $C_c(X) = \mathbb{R} + SC_F(X)$, where $C_c(X)$ denotes the subring of $C(X)$ consisting of functions with countable image, see [9], [10]. First, we need the following definition.

Definition 3.5. A space X is called cocountably-disconnected if whenever Y is a clopen subset of X, either Y or $X \setminus Y$ is countable.

Clearly, connected spaces, one-point Lindelöffication of countably discrete spaces and $X = Y \cup I_c(X)$, where $I_c(X) = X \setminus Y$ is the countable set of countably isolated points of X and Y is connected (e.g., $X = (0, \frac{1}{2}) \cup \mathbb{N}$ as a subspace of $\mathbb{R}$, or more generally a free union of a connected space with a countable space) are some examples of cocountably-disconnected spaces.

Finally, we conclude this article with our main result which is the counterpart of [9, Proposition 6.6]. Before stating this main result we should remind the reader that in [10], it is shown that to study $C_c(X)$, we may, without loss, consider X to be zero dimentional space.

Theorem 3.6. *Let X be a zero-dimentional space. Then $C_c(X) = \mathbb{R}+SC_F(X)$ if and only if X is cocountably-disconnected and if $X = Y \cup I_c(X)$, where $I_c(X) = X \setminus Y$ is the set of countably isolated points, every function in $C_c(X)$ is constant on Y.*

Proof. First, let X be a cocountably-disconnected space with the above properties. We are to show that $C_c(X) \subseteq \mathbb{R} + SC_F(X)$. If X is connected, $SC_F(X) = (0)$, $C_c(X) = \mathbb{R}$, and we are done. Hence we may assume that X is disconnected, which in turn, implies that the set of countably isolated points of X is nonempty, for X is cocountably-disconnected. Now we consider two cases. Case I: Let $I_c(X)$ be finite. Since X is cocountably-disconnected, we infer that $Y = X \setminus I_c(X)$ must be connected. Therefore for each $f \in C_c(X)$, f is constant, say r, on Y (note, since $f \in C_c(X)$, it is constant on Y automatically and no need to make use of our assumption that f is constant on Y). Now let $f \in C_c(X)$ and define $g \in SC_F(X)$ with $g(Y) = 0$, $g(x) = r - f(x)$ for all $x \in I_c(X)$. Hence $f = r+g$ and we are done. Case II: Let us assume that $I_c(X)$ is an infinite set. Let $f \in SC_F(X)$, hence $f(X) = \{r_1, r_2, \cdots, r_n, \cdots\} \subseteq \mathbb{R}$. Thus $X = \bigcup_{i=1}^{\infty} A_i$, where $A_i = f^{-1}(r_i)$ for each $i \in \mathbb{N}$. By above hypothesis, $f(Y) = r_k$, for some $k \in \mathbb{N}$ and since X is a zero-dimensional space, there exists a clopen set U_k such that $U_k \subseteq X \setminus A_k$. Hence $Y \subseteq A_k \subseteq X \setminus U_k$ and we infer that U_k is countable (note, $X \setminus U_k$ is uncountable). Now if we define $g \in SC_F(X)$ with $g(X \setminus U_k) = 0$, $g(x) = f(x) - r_k$ for all $x \in U_k$, we have $f = g + r_k$ and we are through in this case, too. Conversely, let $C_c(X) = \mathbb{R} + SC_F(X)$. If X is connected, we are trivially done. Therefore we put $X = A \cup B$, where A, B are two disjoint clopen subsets of X. We claim that $I_c(X) \neq \varnothing$, for otherwise $SC_F(X) = (0)$, hence $C_c(X) = \mathbb{R}$. But the function f, with $f(A) = \{0\}$, $f(B) = \{1\}$ which is in $C_c(X) \setminus \mathbb{R}$ leads us to a contradiction, hence we must have $SC_F(X) \neq (0)$. Now we claim that X is cocountably-disconnected. To see this, let $X = A \cup B$, where A, B are two uncountable disjoint clopen subsets of X and obtain a contradiction. But if f is the function as above, i.e., $f(A) = \{0\}$, $f(B) = \{1\}$, then $f \in C_c(X)$ but $f \notin \mathbb{R} + SC_F(X)$ (note, if $f = r + g$ with $r \in \mathbb{R}$, $g \in SC_F(X)$, then either $g(A) \neq 0$ or $g(B) \neq 0$ which is impossible, for g must vanish everywhere except on a countable subset of X). Finally, we must show that each $f \in SC_F(X)$ is constant on Y. To see this, let $x_1, x_2 \in Y$ with $f(x_1) \neq f(x_2)$ and obtain a contradiction. Since $C_c(X) = \mathbb{R} + SC_F(X)$, we must have $f = r + g$, for some $r \in \mathbb{R}$, $g \in SC_F(X)$. But by Proposition 2.4, g is non-zero only on some countably isolated points. Whereas either $g(x_1) \neq 0$ or $g(x_2) \neq 0$, which is absurd. □

In [11, Theorem 2.10], it is observed that, in fact, $SC_F(X)$ is a maximal ideal in $C(X)$ when X is the one-point Lindelöffication of an uncountable discrete space. We conclude this note with the following related remark.

Remark 3.7. Let X be a zero-dimensional cocountably-disconnected space, with $X \setminus I_c(X)$ a singleton and $f(I_c(X))$ a countable set for any $f \in C(X)$. Then in view of this theorem and Corollary 3.4, X is the one-point Lindelöffication of a countably discrete space. Moreover in this case $SC_F(X)$ becomes a maximal ideal in $C(X)$, too (note, $C_c(X) = C(X) = \mathbb{R} + SC_F(X)$). In comparison to the latter equality, let us recall that if X is the one-point compactification of a discrete space then $C^F(X) = \mathbb{R} + C_F(X)$, see [9, Theorem 6.7]. Moreover, in this case $C_F(X)$ is a unique proper essential ideal in $C^F(X)$ (consequently, a maximal ideal in $C^F(X)$), where $C^F(X)$ is the subalgebra of $C_c(X)$, a fortiori of $C(X)$, whose elements have finite images, see [13] for general rings with the latter property .

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