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Selection principles and covering properties in bitopological spaces

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ABSTRACT

Our main focus in this paper is to introduce and study various selection principles in bitopological spaces. In particular, Menger type, and Hurewicz type covering properties like: Almost p-Menger, star p-Menger, strongly star p-Menger, weakly p-Hurewicz, almost p-Hurewicz, star p-Hurewicz and strongly star p-Hurewicz spaces are defined and corresponding properties are investigated. Relations between some of these spaces are established.

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1. INTRODUCTION

Our main focus in this paper is to introduce and study various selection principles, by using p-open covers in bitopological spaces. We will deal with variations of the following classical selection principles originaly studied in topological spaces:

Let $\mathcal{A}$ and $\mathcal{B}$ be sets whose elements are families of subsets of an infinite set X and $\mathcal{O}$ denotes the family of all open covers of a topological space (X, τ). Then:

$\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis:

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For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(U_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, U_n is a member of $\mathcal{U}_n$, and $\{U_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$ (see [12]).

The covering property $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$ is called the *Rothberger (covering) property, and topological spaces with the Rothberger property are called Rothberger spaces.*

$\mathsf{S}_{\mathsf{fin}}(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis:

> For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$, and $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is an element of $\mathcal{B}$.

The property $\mathsf{S}_{\mathsf{fin}}(\mathcal{O}, \mathcal{O})$ is called the *Menger (covering) property.*

$\mathsf{U}_{\mathsf{fin}}(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis:

> For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$, and the family $\{\cup\mathcal{V}_n : n \in \mathbb{N}\}$ is a γ-cover of X.

The property $\mathsf{U}_{\mathsf{fin}}(\mathcal{O}, \mathcal{O})$ is called the *Hurewicz (covering) property.* An indexed family $\{A_n : n \in \mathbb{N}\}$ is a γ-cover of X if for every $x \in X$ the set $\{n \in \mathbb{N} : x \notin A_n\}$ is finite.

The properties of Menger and Hurewicz were defined in [3].

The concept of bitopological spaces was introduced by Kelly [4] in 1969. For details on the topic we refer the reader to see [2]. According to Kelly, a bitopological space is a set endowed with two topologies which may be independent of each other. Some mathematicians studied bitopological spaces with some relation between the two topologies, but here we consider bitopological spaces in the sense of Kelly.

In 2011, Kočinac and Özçağ introduced and studied in [8] the selective versions of separability in bitopological spaces. In particular, they investegated these properties in function spaces endowed with two topologies with one topology of pointwise convergence and the other with compact-open topology. In 2012, Kočinac and Özçağ [9], reviewed some known results of selection principles in the context of bitopology. They defined three versions of the Menger property in a bitopological space (X, τ_1, τ_2), namely, δ_2-Menger, $(1, 2)$-almost Menger, and $(1, 2)-$weakly Menger. These results are mainly related to function spaces and hyperspaces endowed with two arbitrary topologies. They proposed some possible lines of investigation in the areas. In 2016, Özçağ and Eysen in [11] introduced the notion of almost Menger property and almost γ-set in bitopological spaces. Our focus in this paper is to continue study of selection principles in bitopological spaces.

2. Preliminaries

Throughout this paper a space (X, τ_1, τ_2) is an infinite bitopological space (called here bispace X) in the sense of Kelly. For a subset U of X, $Cl_i(U)$ (resp. $Int_i(U)$) will denote the closure (resp. interior) of U in (X, τ_i), $i = 1, 2$, respectively. We use the standard bitopological notion and terminology as in [2].

A subset F of a bispace X is said to be:

(i) i-open if F is open with respect to τ_i in X, F is called open in X if it is both 1-open and 2-open in X, or equivalently, $F \in (\tau_1 \cap \tau_2)$;
(ii) i-closed if F is closed with respect τ_i in X, F is called closed in X if it is both 1-closed and 2-closed in X, or equivalently, $X \backslash F \in (\tau_1 \cap \tau_2)$;
(iii) i-clopen if F is both i-closed and i-open set in X, F is called clopen in X if it is both 1-clopen and 2-clopen.
(iv) τ_i-regular open if F is regular open set with respect to τ_i.
(v) τ_i-regular closed if F is regular closed set with respect to τ_i.

A bitopological space X is said to be (i,j)-regular $(i,j = 1,2,\ i \neq j)$ if, for each point $x \in X$ and each τ_i-open (i-open) set V of X containing x, there exists an i-open set U such that $x \in U \subseteq Cl_j(U) \subseteq V$. X is said to be pairwise regular if it is both $(1,2)$-regular and $(2,1)$-regular.

A cover $\mathcal{U}$ of a bispace X is said to be a p-open cover if it is $\tau_1\tau_2$-open and $\mathcal{U} \cap (\tau_1 \backslash \phi) \neq \phi$ and $\mathcal{U} \cap (\tau_2 \backslash \phi) \neq \phi$, where $\mathcal{U}$ is $\tau_1\tau_2$-open if $\mathcal{U} \subset \tau_1 \cup \tau_2$. p$-\mathcal{O}$ denotes the family of all p-open covers of X. A p-open cover $\mathcal{U}$ of a bispace X is a p-ω-cover [9] if $X \notin \mathcal{U}$ and each finite subset of X is contained in a member of $\mathcal{U}$. $\mathcal{U}$ is a p-γ-cover if it is infinite and each $x \in X$ belongs to all but finitely many elements of $\mathcal{U}$. The symbols p-Ω and p-Γ denote the family of all p-ω-covers and p-γ-covers of a bispace respectively.

Definition 2.1 ([9])**.** A bispace X is called:

(1) p-Lindelöf if every p-open cover has a countable subcover.
(2) d-paracompct if every dense family of subsets of X has a locally finite refinement.
(3) p-metacompact if every p-open cover $\mathcal{U}$ of X has a point-finite p-open refinement $\mathcal{V}$ (that is, every point of X belongs to at most finitely many members of $\mathcal{V}$).
(4) p-metaLindelöf if every p-open cover $\mathcal{U}$ of X has a point-countable, p-open refinement $\mathcal{V}$.
(5) *p-Menger* if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of p-open covers of X, there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is a p-open cover of X. $A \subset X$ is p-Menger in X if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of covers of A by p-open sets in X, there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $A \subset \bigcup_{n\in\mathbb{N}} \mathcal{V}_n$.
(6) *p-Rothberger* if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of p-open covers of X, there is a sequence $(U_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $U_n \in \mathcal{U}_n$ and $\{U_n : n \in \mathbb{N}\}$ is a p-open cover of X.
(7) p-Hurewicz (or simply pairwise Hurewicz), if it satisfies: For each sequence $(U_n : n \in \mathbb{N})$ of elements of p-$\mathcal{O}$, there is a sequence $(V_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, V_n is a finite subset of U_n, and for each $x \in X$, for all but finitely many n, $x \in \cup V_n$.

A bispace X is called a p-space, if every countable intersection of open sets is open in X.

3. p-Menger and related bispaces

Following the fact that every p-ω-cover of X is a p-open cover of X, we state the following theorem:

Theorem 3.1.

(1) *If a bispace X is p-Menger then X satisfies* $\mathsf{S}_{\mathsf{fin}}(p\text{–}\Omega, p\text{–}\mathcal{O})$.
(2) *If a bispace X is p-Rothberger then X satisfies* $\mathsf{S}_1(p\text{–}\Omega, p\text{–}\mathcal{O})$.
(3) *If a bispace X is p-Hurewicz then X satisfies* $\mathsf{U}_{\mathsf{fin}}(p\text{–}\Omega, p\text{–}\mathcal{O})$.

In [7], the notion of almost Menger topological space was introduced, and in [5] Kocev studied this class of spaces. We make use of this concept and define almost p-Menger and almost p-Rothberger bispaces with the help of p-open covers.

Definition 3.2. A bitopological space (X, τ_1, τ_2) is *almost p-Menger* if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of p-open covers of X there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and

$$\bigcup_{\mathrm{n}\in\mathbb{N}}\{\mathrm{Cl_i(V)} : \mathrm{V} \in \mathcal{V}_\mathrm{n};\ \mathrm{i} = \left\{\begin{array}{l} 1 \text{ if } V \in \tau_1 \\ 2 \text{ if } V \in \tau_2 \end{array}\right\}\} = \mathrm{X}$$

Definition 3.3. A bitopological space (X, τ_1, τ_2) is *almost p-Rothberger* if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of p-open covers of X there exists a sequence $(U_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $U_n \in \mathcal{U}_n$ and

$$\bigcup_{\mathrm{n}\in\mathbb{N}}\{\mathrm{Cl_i(U_n)}\ ;\ \mathrm{i} = \left\{\begin{array}{l} 1 \text{ if } U_n \in \tau_1 \\ 2 \text{ if } U_n \in \tau_2 \end{array}\right\}\} = \mathrm{X}$$

We note that every p-Menger (resp. p-Rothberger) bispace is almost p-Menger (resp. almost p-Rothberger).

A subset of a bitopological space is said to be dense if it is dense with respect to both topologies.

Proposition 3.4. *If a bispace X contains a dense subset which is p-Menger in X, then X is almost p-Menger.*

Proof. Let A be a p-Menger dense subset of a bispace X and let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of p-open covers of X. Since A is p-Menger in X therefore there exist finite sets $\mathcal{V}_n$, $n \in \mathbb{N}$ such that $A \subset \bigcup_{n\in\mathbb{N}}\{V : V \in \mathcal{V}_n\} \subset \bigcup_{n\in\mathbb{N}}\{\mathrm{Cl}_i(V) : V \in \mathcal{V}_n\ ;\ \mathrm{i} = \left\{\begin{array}{l} 1 \text{ if } V \in \tau_1 \\ 2 \text{ if } V \in \tau_2 \end{array}\right\}\}$. Since A is dense in X, we have

$$X = \bigcup_{n\in\mathbb{N}}\{\mathrm{Cl}_i(V) : V \in \mathcal{V}_n;\ \mathrm{i} = \left\{\begin{array}{l} 1 \text{ if } V \in \tau_1 \\ 2 \text{ if } V \in \tau_2 \end{array}\right\}\}$$

□

The following theorem shows: when an almost p-Menger bispace becomes p-Menger?

Theorem 3.5. *Let X be a pairwise regular bispace. If X is an almost p-Menger, then X is a p-Menger bispace.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of p-open covers of X. Since X is a pairwise regular bispace, by definition there exists for each n a p-open cover $\mathcal{V}_n$ of X such that $\mathcal{V}'_n = \{\mathrm{Cl}_i(V) : V \in \mathcal{V}_n;\ \mathrm{i} = \left\{\begin{array}{l} 1 \text{ if } \ V \in \tau_1 \\ 2 \text{ if } \ V \in \tau_2 \end{array}\right\}\}$ form a refinement of $\mathcal{U}_n$. By assumption, there exists a sequence $(\mathcal{W}_n : n \in \mathbb{N})$ such that for each n, $\mathcal{W}_n$ is a finite subset of $\mathcal{V}_n$ and $\bigcup(\mathcal{W}'_n : n \in \mathbb{N})$ is a cover of X, where $\mathcal{W}'_n = \{\mathrm{Cl}_i(W) : W \in \mathcal{W}_n;\ \mathrm{i} = \left\{\begin{array}{l} 1 \text{ if } \ W \in \tau_1 \\ 2 \text{ if } \ W \in \tau_2 \end{array}\right\}\}$. For every $n \in \mathbb{N}$ and every $W \in \mathcal{W}_n$ we can choose $U_W \in \mathcal{U}_n$ such that $\mathrm{Cl}_i(W) \subset U_W$. Let $\mathcal{U}'_n = \{U_W : W \in \mathcal{W}_n\}$. We shall prove that $\mathcal{U}'_n$ is a p-open cover of X. Let $x \in X$. There exists $n \in \mathbb{N}$ and $\mathrm{Cl}_i(W) \in \mathcal{W}'_n$ such that $x \in \mathrm{Cl}_i(W)$. By construction, there exists $U_W \in \mathcal{U}'_n$ such that $\mathrm{Cl}_i(W) \subset U_W$. Hence, $x \in U_W$. □

Theorem 3.6. *A bispace X is almost p-Menger if and only if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of covers of X by τ_i-regular closed sets (i =1 or i = 2), there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is a cover of X.*

Proof. Let X be an almost p-Menger bispace. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of covers of X by τ_i-regular closed sets (i =1 or i = 2), $(\mathcal{U}_n : n \in \mathbb{N})$ is a sequence of p-open covers of X. By assumption, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is a cover of X, where $\mathrm{Cl}_i(V) = V$ for all $V \in \mathcal{V}_n$; i= $\left\{\begin{array}{l} 1 \text{ if } \ V \in \tau_1 \\ 2 \text{ if } \ V \in \tau_2 \end{array}\right\}$.

Conversely, let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of p-open covers of X. Let $(\mathcal{U}'_n : n \in \mathbb{N})$ be a sequence defined by $\mathcal{U}'_n = \{\mathrm{Cl}_i(U) : U \in \mathcal{U}_n\}$. Then each $\mathcal{U}'_n$ is a cover of X by τ_i-regular closed sets. Thus there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}'_n$ and $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is a cover of X. By construction, for each $n \in \mathbb{N}$ and $V \in \mathcal{V}_n$ there exists $U_V \in \mathcal{U}_n$ such that $V = \mathrm{Cl}_i(U_V)$. Hence, $\bigcup_{n\in\mathbb{N}}\{\mathrm{Cl}_i(U_V) : V \in \mathcal{V}_n\} = X$. So, X is an almost p-Menger bispace. □

3.1. **Star p-Menger bispaces.** A number of results in the literature shows that many topological properties can be defined and studied in terms of star covering properties. In particular, such a method is also used in investigation of selection principles for topological spaces. This investigation was initiated by Kočinac in [6] and then studied in many papers. We extend this idea for bitopological spaces.

Let A be a subset of a topological space $(X,\ \tau)$ and $\mathcal{U}$ be a collection of subsets of X. Then

$$\mathrm{St}(A,\mathcal{U}) = \cup\{U \in \mathcal{U} : U \cap A \neq \varnothing\},$$

$$\mathrm{St}^{n+1}(A,\mathcal{U}) = \cup\{U \in \mathcal{U} : U \cap \mathrm{St}^{n}(A,\mathcal{U}) \neq \varnothing\}.$$

We usually write $\mathrm{St}(x,U)$ for $\mathrm{St}(\{x\},\mathcal{U})$.

Definition 3.7 ([6])**.** In a topological space (X,τ),

(1) $\mathsf{S_{fin}}^*(\mathcal{A},\mathcal{B})$ denotes the selection hypothesis: For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$, and $\bigcup_{n\in\mathbb{N}}\{St(V,\mathcal{U}_n) : V \in \mathcal{V}_n\}$ is an element of $\mathcal{B}$.

(2) $\mathsf{SS_{fin}}^*(\mathcal{A},\mathcal{B})$ denotes the selection hypothesis: For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of X such that $\{St(F_n,\mathcal{U}_n) : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

The symbols $\mathsf{S_{fin}}^*(\mathcal{O},\mathcal{O})$ and $\mathsf{SS_{fin}}^*(\mathcal{O},\mathcal{O})$ denotes the *star-Menger property* and *strongly star-Menger property*, respectively in topological spaces.

In a similar way we introduce the following definition for bitopological spaces.

Definition 3.8. A bitopological space (X,τ_1,τ_2) is said to have:

(1) the *star p-Menger property* if it satisfies $\mathsf{S_{fin}}^*(\mathrm{p}-\mathcal{O},\mathrm{p}-\mathcal{O})$.
(2) the *strongly star p-Menger property* if it satisfies $\mathsf{SS_{fin}}^*(\mathrm{p}-\mathcal{O},\mathrm{p}-\mathcal{O})$.

Theorem 3.9. *Every strongly star p-Menger, p-metacompact bispace is p-Menger bispace.*

Proof. Let X be a strongly star p-Menger p-metacompact bispace. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of p-open covers of X. For each $n \in \mathbb{N}$, let $\mathcal{V}_n$ be a point-finite p-open refinement of $\mathcal{U}_n$. Since X is strongly star p-Menger, there is a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of X such that $\bigcup_{n\in\mathbb{N}}\mathrm{St}(F_n,\mathcal{V}_n) = X$.

As $\mathcal{V}_n$ is a point-finite refinement and each F_n is finite, elements of each F_n belongs to finitely many members of $\mathcal{V}_n$ say $V_{n_1}, V_{n_2}, V_{n_3}, \ldots, V_{n_k}$. Let $\mathcal{V}'_n = \{V_{n_1}, V_{n_2}, V_{n_3}, \ldots, V_{n_k}\}$. Then $\mathrm{St}(F_n,\mathcal{V}_n) = \bigcup\mathcal{V}'_n$ for each $n \in \mathbb{N}$. We have that $\bigcup_{n\in\mathbb{N}}(\bigcup\mathcal{V}'_n) = X$. For every $V \in \mathcal{V}'_n$ choose $U_V \in \mathcal{U}_n$ such that $V \subset U_V$. Then, for every n, $\mathcal{W}_n := \{U_V : V \in \mathcal{V}'_n\}$ is a finite subfamily of $\mathcal{U}_n$ and $\bigcup_{n\in\mathbb{N}}\bigcup\mathcal{W}_n = X$, that is X is p-Menger bispace. □

Theorem 3.10. *Every strongly star p-Menger, p-metaLindelöf bispace is Lindelöf bispace.*

Proof. Let X be a strongly star p-Menger p-metaLindelöf bispace. Let $\mathcal{U}$ be a p-open cover of X and let $\mathcal{V}$ be a point-countable, p-open refinement of $\mathcal{U}$. Since X is strongly star p-Menger, there is a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of X such that $\bigcup_{n\in\mathbb{N}}\mathrm{St}(F_n,\mathcal{V}_n) = X$.

For every $n \in \mathbb{N}$, denote by $\mathcal{W}_n$ the collection of all members of $\mathcal{V}$ which intersects F_n. Since $\mathcal{V}$ is point-countable and F_n is finite, $\mathcal{W}_n$ is countable. So, the collection $\mathcal{W} = \bigcup_{n\in\mathbb{N}} \mathcal{W}_n$ is a countable subfamily of $\mathcal{V}$ and is a cover of X. For every $W \in \mathcal{W}$ pick a member $U_W \in \mathcal{U}$ such that $W \in U_W$. Then $\{U_W : W \in \mathcal{W}\}$ is a countable subcover of $\mathcal{U}$. Hence, X is Lindelöf bispace. □

Definition 3.11. A bispace X is an *almost star p-Menger* if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of p-open covers of X, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\{Cl_i(St(\cup\mathcal{V}_n,\mathcal{U}_n)) : n \in \mathbb{N}; i = 1 \text{ or } \mathrm{i} = 2\}$ is a cover of X.

Theorem 3.12. *A bispace X is an almost star p-Menger if and only if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of covers of X by τ_i-regular open sets there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\{\mathrm{Cl}_i(\mathrm{St}(\cup\mathcal{V}_n,\mathcal{U}_n)) : n \in \mathbb{N}\}$ is a cover of X.*

Proof. Since every cover by τ_i-regular open sets is p-open, necessity follows.

Conversely, let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of p-open covers of X. Let $\mathcal{U}'_n = \{\mathrm{Cl}_i(U) : U \in \mathcal{U}_n \text{ and } \mathrm{i} = \left\{ \begin{array}{l} 1 \text{ if } U \in \tau_1 \\ 2 \text{ if } U \in \tau_2 \end{array} \right\} \}$. Then $\mathcal{U}'_n$ is a cover of X by τ_i-regular open sets. Then by assumption there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}'_n$ and $\{\mathrm{Cl}_i(\mathrm{St}(\cup\mathcal{V}_n,\mathcal{U}'_n)) : n \in \mathbb{N}\}$a cover of X.

First we shall prove that $\mathrm{St}(U,\mathcal{U}_n) = \mathrm{St}(Cl_i(U),\mathcal{U}_n)$ for all $U \in \mathcal{U}_n$. It is obvious that $\mathrm{St}(U,\mathcal{U}_n) \subset \mathrm{St}(\mathrm{Cl}_i(U),\mathcal{U}_n)$ since $U \subset \mathrm{Cl}_i(U)$. Let $x \in \mathrm{St}(\mathrm{Cl}_i(U),\mathcal{U}_n)$. Then there exists some $U' \in \mathcal{U}_n$ such that $x \in U'$ and $U' \cap \mathrm{Cl}_i(U) \neq \varnothing$. Then $U' \cap \mathrm{Cl}_i(U) \neq \varnothing$ implies that $x \in \mathrm{St}(U,\mathcal{U}_n)$. Hence, $\mathrm{St}(\mathrm{Cl}_i(U),\mathcal{U}_n) \subset \mathrm{St}(U,\mathcal{U}_n)$.

For each $V \in \mathcal{V}_n$ we can find $U_V \in \mathcal{U}_n$ such that $V = \mathrm{Cl}_i(U_V)$. Let $\mathcal{V}'_n = \{U_V : V \in \mathcal{V}_n\}$.

Let $x \in X$.Then there exists $n \in \mathbb{N}$ such that $x \in \mathrm{Cl}_i(\mathrm{St}(\cup\mathcal{V}_n,\mathcal{U}'_n))$. For each p-open set V, we have $V \cap \mathrm{St}(\cup\mathcal{V}_n,\mathcal{U}'_n) \neq \varnothing$. Then there exists $U \in \mathcal{U}_n$ such that $(V \cap \mathrm{Cl}_i(U) \neq \varnothing$ and $\cup\mathcal{V}_n \cap \mathrm{Cl}_i(U) \neq \varnothing)$ imply that $(V \cap U \neq \varnothing$ and $\cup\mathcal{V}_n \cap \mathrm{Cl}_i(U) \neq \varnothing)$. We have that $\cup\mathcal{V}'_n \cap U \neq \varnothing$, so $x \in \mathrm{Cl}_i(\mathrm{St}(\cup\mathcal{V}'_n,\mathcal{U}_n))$. Hence, $\{\mathrm{Cl}_i(\mathrm{St}(\cup\mathcal{V}'_n,\mathcal{U}_n)) : n \in \mathbb{N}\}$ is a cover of X. □

4. p-Hurewicz and related bispaces

Definition 4.1. Call a bitopological space (X,τ_1,τ_2):

(1) weakly p-Hurewicz if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of p-open covers of X, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and each non-empty set $U \in \tau_1 \cup \tau_2$, $U \cap (\cup\mathcal{V}_n) \neq \phi$ for all but finitely many n.

(2) almost p-Hurewicz if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of p-open covers of X, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and each $x \in X$ belongs to $\cup\{Cl_i(V) : V \in \mathcal{V}_n; \mathrm{i} = \left\{ \begin{array}{l} 1 \text{ if } V \in \tau_1 \\ 2 \text{ if } V \in \tau_2 \end{array} \right\}\}$ for all but finitely many n.

Theorem 4.2. *Let X be a pairwise regular bispace. If X is an almost p-Hurewicz, then X is p-Hurewicz bispace.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of p-open covers of X. Since X is a pairwise regular bispace, using the definition, there exists for each n a p-open cover $\mathcal{V}_n$ of X such that $\mathcal{V}'_n = \{Cl_i(V) : V \in \mathcal{V}_n, V \in \tau_i; i = 1 \text{ or } i = 2\}$ forms a refinement of $\mathcal{U}_n$. By assumption, there exists a sequence $(\mathcal{W}_n : n \in \mathbb{N})$ such that for each n, $\mathcal{W}_n$ is a finite subset of $\mathcal{V}_n$ and each $x \in X$ belongs to $\cup\mathcal{W}'_n$ for all but finitely many n, where $\mathcal{W}'_n = \{Cl_i(W) : W \in \mathcal{W}_n, W \in \tau_i; i = 1, 2\}$. For every $n \in \mathbb{N}$ and every $W \in \mathcal{W}_n$ we can choose $U_W \in \mathcal{U}_n$ such that $Cl_i(W) \subset U_W$. Let $\mathcal{U}'_n = \{U_W : W \in \mathcal{W}_n\}$. We shall prove that each $x \in \cup\mathcal{U}'_n$ for all but finitely many n. Let $x \in X$. There exists $n_0 \in \mathbb{N}$ and $Cl_i(W) \in \mathcal{W}'_n$ such that $x \in Cl_i(W)$ for all $n > n_0$. By construction, there exists $U_W \in \mathcal{U}'_n$ such that $Cl_i(W) \subset U_W$. Hence, $x \in U_W$ for all $n > n_0$. □

Theorem 4.3. *A bispace X is almost p-Hurewicz if and only if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of covers of X by τ_i-regular closed sets, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and each $x \in X$ belongs to $\cup\mathcal{V}_n$ for all but finitely many $n \in \mathbb{N}$.*

Proof. Let X be an almost p-Hurewicz bispace. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of covers of X by τ_i-regular closed sets; i =1 or i = 2. This implies that $(\mathcal{U}_n : n \in \mathbb{N})$ is a sequence of p-open covers of X. By assumption, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and each $x \in X$ belongs to $\cup\mathcal{V}_n$ for all but finitely many n, where $Cl_i(V) = V$ for all $V \in \mathcal{V}_n$; $\text{i} = \left\{ \begin{array}{l} 1 \text{ if } V \in \tau_1 \\ 2 \text{ if } V \in \tau_2 \end{array} \right\}$.

Conversely, let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of p-open covers of X. Let $(\mathcal{U}'_n : n \in \mathbb{N})$ be a sequence defined by $\mathcal{U}'_n = \{Cl_i(U) : U \in \mathcal{U}_n\}$. Then each $x \in X$ belongs to $\cup\mathcal{U}'_n$ for all but finitely many n and elements of $\mathcal{U}'_n$ are τ_i-regular closed sets.Then there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}'_n$ and each $x \in X$ belongs to $\cup\mathcal{V}_n$ for all but finitely many n. By construction, for each $n \in \mathbb{N}$ and $V \in \mathcal{V}_n$ there exists $U_V \in \mathcal{U}_n$ such that $V = Cl_i(U_V)$. Hence, $x \in Cl_i(U_V) : V \in \mathcal{V}_n$ for all but finitely many n. So X is almost p-Hurewicz bispace. □

Theorem 4.4. *If a bispace X is weakly p-Hurewicz and d-paracompact, then X is almost p-Hurewicz.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of p-open covers of a bispace X. Since X is weakly p-Hurewicz, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and every non-empty set $U \in \tau_1 \cup \tau_2$, $U \cap (\cup\mathcal{V}_n) \neq \phi$ for all but finitely many n. Let $x \in X$. By the assumption $\{\mathcal{V}_n : n \in \mathbb{N}\}$ has a locally finite refinement say $\mathcal{W}$. Then $\cup\mathcal{W} = \cup_{n\in\mathbb{N}} \cup\mathcal{V}_n$ and therefore $Cl_i(\cup\mathcal{W}) = Cl_i(\cup_{n\in\mathbb{N}} \cup \mathcal{V}_n)$. As $\mathcal{W}$ is locally finite family, $Cl_i(\cup\mathcal{W}) = \cup_{W\in\mathcal{W}} Cl_i(W)$. Since for every $W \in \mathcal{W}$ there exists $V_W \in \mathcal{V}_n$, so that $W \subset V_W$, we have that each $x \in Cl_i(V)$ where $V \in \mathcal{V}_n$, for all but finitely many n. Hence, it is shown that X is almost p-Hurewicz. □

Theorem 4.5. *If a p-space X is weakly p-Hurewicz, then X is almost p-Hurewicz.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of p-open covers of X. Since X is weakly p-Hurewicz, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and every non-empty set $U \in \tau_1 \cup \tau_2$, $U \cap (\cup \mathcal{V}_n) \neq \phi$ for all but finitely many n. Let $x \in X$ and U contains x. By the condition X is p-space, the intersection of every countable family of open subsets of X is open and hence, every countable union of closed sets is closed. So, $Cl_i(\cup_{n\in\mathbb{N}} \cup \mathcal{V}_n) = \cup_{n\in\mathbb{N}}\{Cl_i(V) : V \in \mathcal{V}_n\}$ implies that $x \in Cl_i(V)$ for all but finitely many n where $V \in \mathcal{V}_n$, which shows that X is an almost p-Hurewicz space. □

Theorem 4.6. *Every $i-$clopen subset of an almost p-Hurewicz bispace is almost p-Hurewicz; i =1 or i = 2.*

Proof. Let F be an $i-$clopen subset of an almost p-Hurewicz bispace X and let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of p-open covers of F. Then $\mathcal{V}_n = \mathcal{U}_n \cup \{X - F\}$ is a p-open cover for X for every $n \in \mathbb{N}$. Since X is an almost p-Hurewicz bispace, there exist finite subsets $\mathcal{W}_n$ of $\mathcal{V}_n$ for which $x \in X$ belongs to $Cl_i(W) : W \in \mathcal{W}_n$ for all but finitely many $n \in \mathbb{N}$. Since, $Cl_i(X - F) = X - F$ and each $a \in F$ belongs to $Cl_i(W) : W \in \mathcal{W}_n, W \neq X - F$ for all but finitely many n. □

Theorem 4.7. *Every $i-$closed subset of a weakly p-Hurewicz bispace is weakly p-Hurewicz. i =1 or i = 2.*

Proof. Let F be an $i-$closed subset of a weakly p-Hurewicz space and let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of p-open covers of F. Then $\mathcal{V}_n = \mathcal{U}_n \cup \{X - F\}$ is a p-open cover of X for every $n \in \mathbb{N}$. Since X is a weakly p-Hurewicz space, there exists finite subsets $\mathcal{W}_n$ of $\mathcal{V}_n$ for each $n \in \mathbb{N}$ such that every non-empty i-open set $U \subset X$ and $U \cap (\cup \mathcal{V}_n) \neq \phi$ for all but finitely many n. Put $\mathcal{W} = \cup_{n\in\mathbb{N}}\{W : W \in \mathcal{W}_n, W \neq X - F\}$. Then every non-empty i-open set $U \subset X$, $U \cap (\mathcal{W} \cup (X - F)) \neq \phi$ for all but finitely many n. Since $F = Cl_i(Int_i(F))$ we have $Int_i(F) \cap Cl_i(X - F) = \phi$. So, $Int_i(F) \subset Cl_i(\cup \mathcal{W})$ and $F = Cl_i(Int_i(F)) \subset Cl_i(\cup \mathcal{W})$. Every non-empty i-open set $A \subset F$, $A \cap (\cup \mathcal{W}) \neq \phi$ for all but finitely many n. □

4.1. **Star p-Hurewicz bispaces.** The method of stars is one of classical popular topological methods. It has been used, for example, to study the problem of metrization of topological spaces, and for definitions and investigations of several important classical topological notions [1],[10].

Definition 4.8. A bitopological space (X,τ_1,τ_2) is said to have:

- star p-Hurewicz property, if it satisfies: For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of p-$\mathcal{O}$ there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$, and each $x \in X$ belongs to $St(\cup \mathcal{V}_n, \mathcal{U}_n)$ for all but finitely many n.

- strongly star p-Hurewicz property, if it satisfies: For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of p-$\mathcal{O}$ there is a sequence $(A_n : n \in \mathbb{N})$ of finite subsets of X, and each $x \in X$ belongs to $St(A_n,\mathcal{U}_n)$ for all but finitely many n.

Every strongly star p-Hurewicz bispace is star p-Hurewicz. The implications among the mentioned covering properties are as follows:

$$\begin{array}{ccc} p - \textit{Hurewicz} & \Rightarrow & \textit{star}\ \mathrm{p} - \textit{Hurewicz} \\ \Downarrow & & \Downarrow \\ p - \textit{Menger} & \Rightarrow & \textit{star}\ p - \textit{Menger} \end{array}$$

A bitopological space X is called strongly star pairwise-compact if for each p-open cover $\mathcal{U}$ of X there is a finite set $F \subset X$ such that $St(F,\mathcal{U}) = X$. Call a space X strongly star pairwise σ-compact if it is union of countably many strongly star pairwise-compact bispaces. Clearly, every strongly star pairwise-compact bispace is stongly star p-Hurewicz. A bitopological space X is called star-p-Lindelöf if for every p-open cover $\mathcal{U}$ of X there is a countable set $F \subset \mathcal{U}$ such that $St(F,\mathcal{U}) = X$.

Theorem 4.9. *Every star p-Hurewicz bispace is star-p-Lindelöf.*

Proof. Let X be a star p-Hurewicz bispace. Let $\mathcal{U}$ be a p-open cover of X. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence such that each $\mathcal{U}_n = \mathcal{U}$. Then, by definition, there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$, and each $x \in X$ belongs to $\cup_{n\in\mathbb{N}}(St(\cup\mathcal{V}_n,\mathcal{U}))$ for all but finitely many n. Let $\mathcal{V} = \cup_{n\in\mathbb{N}} \mathcal{V}_n$. Now $\cup_{n\in\mathbb{N}}(St(\cup\mathcal{V}_n,\mathcal{U}_n)) = St(\cup\mathcal{V},\mathcal{U})$. Then $\mathcal{V} = \cup_{n\in\mathbb{N}} \mathcal{V}_n$ is a countable subfamily of $\mathcal{U}$ satisfying $St(\cup\mathcal{V},\mathcal{U}) = \cup_{n\in\mathbb{N}} (St(\cup\mathcal{V}_n,\mathcal{U}_n)) = X$, that is X is star-p-Lindelöf. □

Theorem 4.10. *Every strongly star p-Hurewicz bispace is strongly star p-Lindelöf.*

Proof. Let X be a strongly star p-Hurewicz bispace. Let $\mathcal{U}$ be a p-open cover of X. Let $\mathcal{F}$ be the collection of all finite subsets of X. Then, by definition, there is a sequence $(F_n : n \in \mathbb{N})$ of elements of $\mathcal{F}$ such that each $x \in X$ belongs to $St(F_n,\mathcal{U}_n)$ for all but finitely many n. Let $A = \cup_{n\in\mathbb{N}}F_n$; then A is a countable set being countable union of finite sets. Also, $\cup_{n\in\mathbb{N}}St(F_n,\mathcal{U}_n)=\cup_{n\in\mathbb{N}}$ $(St(\cup_{n\in\mathbb{N}}F_n,\mathcal{U}_n)) = St(A,\mathcal{U}_n) = X$. Hence, X is strongly star-p-Lindelöf bispace. □

Theorem 4.11. *Every strongly star p-Hurewicz, p-metacompact bispace is p-Hurewicz bispace.*

Proof. Let X be a strongly star p-Hurewicz metacompact bispace. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of p-open covers of X. For each $n \in \mathbb{N}$, let $\mathcal{V}_n$ be a point-finite p-open refinement of $\mathcal{U}_n$. Since X is strongly star p-Hurewicz, there is a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of X such that each $x \in X$ belongs to $St(F_n, \mathcal{V}_n)$ for all but finitely many n.

Since $\mathcal{V}_n$ is a point-finite refinement and each F_n is finite, elements of each F_n belongs to finitely many members of $\mathcal{V}_n$ say $V_{n_1}, V_{n_2}, V_{n_3}, ..., V_{n_k}$. Let $\mathcal{V}'_n = \{V_{n_1}, V_{n_2}, V_{n_3}, ..., V_{n_k}\}$. Then $St(F_n, \mathcal{V}_n) = \cup\mathcal{V}'_n$ for each $n \in \mathbb{N}$. We have that each $x \in X$ belongs to $\cup\mathcal{V}'_n$ for all but finitely many n. For every $V \in \mathcal{V}'_n$ choose $U_V \in \mathcal{U}_n$ such that $V \subset U_V$. Then, for every n, $\{U_V : V \in \mathcal{V}'_n\} = \mathcal{W}_n$ is a finite subfamily of $\mathcal{U}_n$ and each $x \in X$ belongs to $\cup\mathcal{W}_n$ for all but finitely many n, that is X is p-Hurewicz bispace. □

Theorem 4.12. *Every strongly star p-Hurewicz, p-metaLindelöf bispace is p-Lindelöf.*

Proof. Let X be a strongly star p-Hurewicz, p- metaLindelof bispace. Let $\mathcal{U}$ be a p-open cover of X then there exists $\mathcal{V}$, a point-countable p-open refinement of $\mathcal{U}$. Since X is strongly star p-Hurewicz, there exists a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of X such that for each $x \in X, x \in St(F_n, \mathcal{V}_n)$ for all but finitely many n.

For every $n \in \mathbb{N}$ denote by $\mathcal{W}_n$ the collection of all members of $\mathcal{V}$ which intersects with F_n. Since $\mathcal{V}$ is point-countable and F_n is finite, $\mathcal{W}_n$ is countable. So, the collection $\mathcal{W} = \cup_{n\in\mathbb{N}}\mathcal{W}_n$ is countable subfamily of $\mathcal{V}$ and is a cover of X. For every $W \in \mathcal{W}$ pick a member $U_W \in \mathcal{U}$ such that $W \subset U_W$. Then $\{U_W : W \in \mathcal{W}\}$ is a countable subcover of $\mathcal{U}$. Hence, X is a p-Lindelof bispace. □

References

[1] O. T. Alas, L. R. Junqueira and R. G. Wilson, Countability and star covering properties, Topology Appl. 158 (2011), 620–626.

[2] B. P. Dvalishvili, Bitopological spaces: theory, relations with generalized algebraic structures, and applications, North-Holand Math. Stud. (2005).

[3] W. Hurewicz, Über die verallgemeinerung des borelschen theorems, Math. Z. 24 (1925), 401–425.

[4] J. C. Kelly Bitopological spaces, Proc. London Math. Soc. 13, no. 3 (1963), 71–89.

[5] D. Kocev, Almost Menger and related spaces, Mat. Vesnik 61 (2009), 173–180.

[6] Lj. D. R. Kočinac , Star-Menger and related spaces, Publ. Math. Debrecen 55 (1999), 421–431.

[7] Lj. D. R. Kočinac, Star-Menger and related spaces II, Filomat 13 (1999), 129–140.

[8] Lj. D. R. Kočinac and S. Özçağ, Versions of separability in bitopological spaces, Topology Appl. 158 (2011), 1471–1477.

[9] Lj. D. R. Kočinac and S. Özçağ, Bitopological spaces and selection principles, Proceedings ICTA2011, Cambridge Scientific Publishers, (2012), 243–255.

[10] M. V. Matveev, A survey on star covering properties, Topology Atlas, preprint No. 330 (1998).

[11] S. Özçağ and A. E. Eysen, Almost Menger property in bitopological spaces, Ukranian Math. J. 68, no. 6 (2016), 950–958.

[12] F. Rothberger, Eine Ver schörfung der Eigenschaftc, Fundamenta Mathematicae 30 (1938), 50–55.