APPLIED GENERAL TOPOLOGY

Appl. Gen. Topol. 21, no. 2 (2020), 235-246
doi:10.4995/agt.2020.12967

On $\mathcal{I}$-quotient mappings and $\mathcal{I}$-cs'-networks under a maximal ideal

Xiangeng Zhou*

Department of Mathematics, Ningde Normal University, Fujian, 352100, P.R. China (56667400@qq.com)

Communicated by P. Das

Abstract

Let $\mathcal{I}$ be an ideal on $\mathbb{N}$ and $f : X \to Y$ be a mapping. f is said to be an $\mathcal{I}$-quotient mapping provided $f^{-1}(U)$ is $\mathcal{I}$-open in X, then U is $\mathcal{I}$-open in Y. $\mathcal{P}$ is called an $\mathcal{I}$-cs'-network of X if whenever $\{x_n\}_{n\in\mathbb{N}}$ is a sequence $\mathcal{I}$-converging to a point $x \in U$ with U open in X, then there is $P \in \mathcal{P}$ and some $n_0 \in \mathbb{N}$ such that $\{x, x_{n_0}\} \subseteq P \subseteq U$. In this paper, we introduce the concepts of $\mathcal{I}$-quotient mappings and $\mathcal{I}$-cs'-networks, and study some characterizations of $\mathcal{I}$-quotient mappings and $\mathcal{I}$-cs'-networks, especially $\mathcal{J}$-quotient mappings and $\mathcal{J}$-cs'-networks under a maximal ideal $\mathcal{J}$ of $\mathbb{N}$. With those concepts, we obtain that if X is an $\mathcal{J}$-FU space with a point-countable $\mathcal{J}$-cs'-network, then X is a meta-Lindelöf space.

2010 MSC: *54A20; 54B15; 54C08; 54D55; 40A05; 26A03.*

Keywords: *ideal convergence; maximal ideal; $\mathcal{I}$-sequential neighborhood; $\mathcal{I}$-quotient mappings; $\mathcal{I}$-cs'-networks; $\mathcal{I}$-FU spaces.*

1. Introduction

Statistical convergence was introduced by H. Fast [9] and H. Steinhaus [16], which is a generalization of the usual notion of convergence. It is doubtless that the study of statistical convergence and its various generalizations has become an active research area [2, 3, 7, 17, 18]. In particular, P. Kostyrko, T. Šalát

*This research is supported by NSFC (No. 11801254) and Ningde Normal University (No. 2017T01; 2018ZDK11; 2019ZDK11).

Received 10 January 2020 – Accepted 11 April 2020

and W. Wilczynski [11] introduced two interesting generalizations of statistical convergence by using the notion of ideals of subsets of positive integers, which were named as $\mathcal{I}$ and $\mathcal{I}^*$-convergence, and studied some properties of $\mathcal{I}$ and $\mathcal{I}^*$-convergence in metric spaces. Later, B.K. Lahiri and P. Das [12] discussed $\mathcal{I}$ and $\mathcal{I}^*$-convergence in topological spaces. Some further results connected with $\mathcal{I}$ and $\mathcal{I}^*$-convergence can be found in [4, 5, 6].

As we know, mappings and networks are important tools of investigating topological spaces. Continuous mappings, quotient mappings , pseudo-open mappings, cs-networks, sn-networks, k-networks and so on are the most important tools for studying convergence, sequential spaces, Fréchet-Urysohn spaces [14] and generalized metric spaces. For this reason, this paper draws into $\mathcal{I}$-quotient mappings and $\mathcal{I}$-cs'-networks for an ideal $\mathcal{I}$ on $\mathbb{N}$ and discusses some basic properties of them.

Recently, the researches on $\mathcal{I}$-convergence are mainly focused on aspects of $\mathcal{I}^*$-convergence [12], $\mathcal{I}$-limit points [11], $\mathcal{I}$-Cauchy sequence [5], ideal-convergence classes [4], selection principles [6], ideal sequence covering mappings [15, 19] and so on. It is expected that $\mathcal{I}$-quotient mappings and $\mathcal{I}$-cs'-networks will also play active roles in the topological spaces.

In this paper, the letter X always denote a topological space. The cardinality of a set B is denoted by $|B|$. The set of all positive integers, the first infinite ordinal, and the first uncountable ordinal are denoted by $\mathbb{N}$, ω and ω_1, respectively. The reader may refer to [8, 14] for notation and terminology not explicitly given here.

2. Preliminaries

Recall the notion of statistical convergence in topological spaces. For each subset A of $\mathbb{N}$ the *asymptotic density* of A, denoted $\delta(A)$, is given by

$$\delta(A) = \lim_{n\to\infty} \frac{1}{n} |\{k \in A : k \leq n\}|,$$

if this limit exists. Let X be a topological space. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to *converge statistically* to a point $x \in X$ [7], if

$$\delta(\{n \in \mathbb{N} : x_n \notin U\}) = 0, \text{ i.e., } \delta(\{n \in \mathbb{N} : x_n \in U\}) = 1$$

for each neighborhood U of x in X, which is denoted by $s\text{-}\lim_{n\to\infty} x_n = x$ or $x_n \xrightarrow{s} x$.

The concept of $\mathcal{I}$-convergence of sequences in a topological space is a generalization of statistical convergence which is based on the ideal of subsets of the set $\mathbb{N}$ of all positive integers. Let $\mathcal{A} = 2^{\mathbb{N}}$ be the family of all subsets of $\mathbb{N}$. An *ideal* $\mathcal{I} \subseteq \mathcal{A}$ is a hereditary family of subsets of $\mathbb{N}$ which is stable under finite unions [11], i.e., the following are satisfied: if $B \subseteq A \in \mathcal{I}$, then $B \in \mathcal{I}$; if $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$. An ideal $\mathcal{I}$ is said to be *non-trivial*, if $\mathcal{I} \neq \varnothing$ and $\mathbb{N} \notin \mathcal{I}$. A non-trivial ideal $\mathcal{I} \subseteq \mathcal{A}$ is called *admissible* if $\mathcal{I} \supseteq \{\{n\} : n \in \mathbb{N}\}$. Clearly, every non-trivial ideal $\mathcal{I}$ defines a *dual filter* $\mathcal{F}_{\mathcal{I}} = \{A \subseteq \mathbb{N} : \mathbb{N} \setminus A \in \mathcal{I}\}$ on $\mathbb{N}$.

Let $\mathcal{I}_f$ be the family of all finite subsets of $\mathbb{N}$. Then $\mathcal{I}_f$ is an admissible ideal. Let $\mathcal{I}_\delta$ [11] be the family of subsets $A \subseteq \mathbb{N}$ with $\delta(A) = 0$. Then $\mathcal{I}_\delta$ is an admissible ideal, and the dual filter $\mathcal{F}_{\mathcal{I}_\delta} = \{A \subseteq \mathbb{N} : \delta(A) = 1\}$.

Definition 2.1 ([11])**.** A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a topological space X is said to be $\mathcal{I}$-*convergent* to a point $x \in X$ provided for any neighborhood U of x, we have $\{n \in \mathbb{N} : x_n \notin U\} \in \mathcal{I}$, which is denoted by $\mathcal{I}\text{-}\lim_{n\to\infty} x_n = x$ or $x_n \xrightarrow{\mathcal{I}} x$, and the point x is called the $\mathcal{I}$-*limit* of the sequence $\{x_n\}_{n\in\mathbb{N}}$.

Definition 2.2 ([20])**.** Let $\mathcal{I}$ be an ideal on $\mathbb{N}$ and X be a topological space.

(1) A subset $F \subseteq X$ is said to be $\mathcal{I}$-*closed* if for each sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq F$ with $x_n \xrightarrow{\mathcal{I}} x \in X$, we have $x \in F$.
(2) A subset $U \subseteq X$ is said to be $\mathcal{I}$-*open* if $X \setminus U$ is $\mathcal{I}$-closed.
(3) X is called an $\mathcal{I}$-*sequential space* if each $\mathcal{I}$-closed subset of X is closed.

Obviously, each sequential space is an $\mathcal{I}$-sequential space [20].

Definition 2.3 ([20])**.** Let $\mathcal{I}$ be an ideal on $\mathbb{N}$, X, Y be topological spaces and $f : X \to Y$ be a mapping.

(1) f is called *preserving* $\mathcal{I}$-*convergence* provided for each sequence $\{x_n\}_{n\in\mathbb{N}}$ in X with $x_n \xrightarrow{\mathcal{I}} x$, the sequence $\{f(x_n)\}_{n\in\mathbb{N}}$ $\mathcal{I}$-converges to $f(x)$ [12].
(2) f is called $\mathcal{I}$-*continuous* provided U is $\mathcal{I}$-open in Y, then $f^{-1}(U)$ is $\mathcal{I}$-open in X.

It is easy to verify that a mapping $f : X \to Y$ is $\mathcal{I}$-continuous if and only if, whenever F is $\mathcal{I}$-closed in Y, then $f^{-1}(F)$ is $\mathcal{I}$-closed in X.

Lemma 2.4 ([20])**.** *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$ and X be a topological space. If a sequence $\{x_n\}_{n\in\mathbb{N}}$ $\mathcal{I}$-converges to a point $x \in X$, and $\{y_n\}_{n\in\mathbb{N}}$ is a sequence in X with $\{n \in \mathbb{N} : x_n \neq y_n\} \in \mathcal{I}$, then the sequence $\{y_n\}_{n\in\mathbb{N}}$ $\mathcal{I}$-converges to $x \in X$.*

Lemma 2.5 ([20])**.** *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. The following are equivalent for a topological space X and a subset $A \subseteq X$.*

(1) *A is $\mathcal{I}$-open.*
(2) *$\{n \in \mathbb{N} : x_n \in A\} \notin \mathcal{I}$ for each sequence $\{x_n\}_{n\in\mathbb{N}}$ in X with $x_n \xrightarrow{\mathcal{I}} x \in A$.*
(3) *$|\{n \in \mathbb{N} : x_n \in A\}| = \omega$ for each sequence $\{x_n\}_{n\in\mathbb{N}}$ in X with $x_n \xrightarrow{\mathcal{I}} x \in A$.*

Lemma 2.6 ([20])**.** *Let X, Y be topological spaces and $f : X \to Y$ be a mapping.*

(1) *If f is continuous, then f preserves $\mathcal{I}$-convergence* [12].
(2) *If f preserves $\mathcal{I}$-convergence, then f is $\mathcal{I}$-continuous.*

Definition 2.7 ([20])**.** Let $A \subseteq X$ and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X. If $\mathcal{I}$ is an ideal on $\mathbb{N}$, then $\{x_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-*eventually* in A if there is $E \in \mathcal{I}$ such that for all $n \in \mathbb{N} \setminus E, x_n \in A$.

If A is a subset of X with the property that every sequence $\mathcal{I}$-converging to a point in A is $\mathcal{I}$-eventually in A, then A is $\mathcal{I}$-open. When we assume $\mathcal{J}$ to be a maximal ideal, the following proposition shows that such sets must coincide with $\mathcal{J}$-open sets.

Proposition 2.8 ([20]). *If $\mathcal{J}$ is a maximal ideal of $\mathbb{N}$, then $A \subseteq X$ is $\mathcal{J}$-open if and only if for each $\mathcal{J}$-converging sequence $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \xrightarrow{\mathcal{J}} x \in A$, then $\{x_n\}_{n\in\mathbb{N}}$ is $\mathcal{J}$-eventually in A.*

By Definition 2.2, the union of a family of $\mathcal{I}$-open sets in a topological space is $\mathcal{I}$-open. Whenever $\mathcal{J}$ is a maximal ideal, the intersection of two $\mathcal{J}$-open sets is an $\mathcal{J}$-open set.

Proposition 2.9 ([20]). *If $\mathcal{J}$ is a maximal ideal of $\mathbb{N}$ and U, V are two $\mathcal{J}$-open subsets of X, then $U \cap V$ is $\mathcal{J}$-open in X.*

It is well known that the sequential coreflection sX of a space X is the set X endowed with the topology consisting of sequentially open subsets of X. Let $\mathcal{J}$ be a maximal ideal of $\mathbb{N}$ and X be a topological space. By Definition 2.2 and Proposition 2.9, the family of all $\mathcal{J}$-open subsets of X forms a topology of the set X. The $\mathcal{J}$-sequential coreflection of a space X is the set X endowed with the topology consisting of $\mathcal{J}$-open subsets of X, which is denoted by $\mathcal{J}$-sX. The spaces X and $\mathcal{J}$-sX have the same $\mathcal{J}$-convergent sequences; $\mathcal{J}$-sX is an $\mathcal{J}$-sequential space; a space X is an $\mathcal{J}$-sequential space if and only if $\mathcal{J}$-$sX = X$ [20].

If no otherwise specified, we consider ideal $\mathcal{I}$ is always an admissible ideal on $\mathbb{N}$, all mappings are continuous and surjection, all spaces are Hausdorff.

3. $\mathcal{I}$-quotient mappings

In this section, we introduce the concept of $\mathcal{I}$-quotient mappings, and obtain some characterizations of $\mathcal{I}$-quotient mappings, especially $\mathcal{J}$-quotient mappings under a maximal ideal of $\mathbb{N}$. Let X, Y be arbitrary topological spaces, and $f : X \to Y$ be a mapping. f is said to be *quotient* provided $f^{-1}(U)$ is open in X, then U is open in Y; f is said to be *sequentially quotient* provided $f^{-1}(U)$ is sequentially open in X, then U is sequentially open in Y [1].

Definition 3.1. Let $\mathcal{I}$ be an ideal on $\mathbb{N}$ and $f : X \to Y$ be a mapping.

(1) f is said to be an $\mathcal{I}$-*quotient mapping* (or shortly, $\mathcal{I}$-quotient) provided $f^{-1}(U)$ is $\mathcal{I}$-open in X, then U is $\mathcal{I}$-open in Y.
(2) f is said to be an $\mathcal{I}$-*covering mapping* (or shortly, $\mathcal{I}$-covering) if, whenever $\{y_n\}_{n\in\mathbb{N}}$ is a sequence in Y $\mathcal{I}$-converging to y in Y, there exist a sequence $\{x_n\}_{n\in\mathbb{N}}$ of points $x_n \in f^{-1}(y_n)$ for all $n \in \mathbb{N}$ and $x \in f^{-1}(y)$ such that $x_n \xrightarrow{\mathcal{I}} x$.

In [20], it was showed that each $\mathcal{I}$-covering mapping is $\mathcal{I}$-quotient.

Definition 3.2. Let $\mathcal{I}$ be an ideal on $\mathbb{N}$, X be a topological space and $P \subset X$. P is called an *$\mathcal{I}$-sequential neighborhood* of x, if each sequence $\{x_n\}_{n\in\mathbb{N}}$ $\mathcal{I}$-converges to a point $x \in P$, then $\{x_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-eventually in P, i.e., there is $I \in \mathcal{I}$ such that $\{n \in \mathbb{N} : x_n \notin P\} = I$.

Remark 3.3. Let $\mathcal{J}$ be a maximal ideal of $\mathbb{N}$ and $A \subseteq X$. By Proposition 2.8, A is $\mathcal{J}$-open in X if and only if A is an $\mathcal{J}$-sequential neighborhood of x for each $x \in A$.

Proposition 3.4. *Let $\mathcal{J}$ be a maximal ideal of $\mathbb{N}$ and $A \subseteq X$. If A is not an $\mathcal{J}$-sequential neighborhood of x, then there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $X \setminus A$ such that $x_n \xrightarrow{\mathcal{J}} x$.*

Proof. If A is not an $\mathcal{J}$-sequential neighborhood of x, then there is a sequence $\{y_n\}_{n\in\mathbb{N}}$ in X such that $y_n \xrightarrow{\mathcal{J}} x$, but $\{n \in \mathbb{N} : y_n \notin A\} \notin \mathcal{J}$. Since $\mathcal{J}$ is a maximal ideal of $\mathbb{N}$, this means that $\{n \in \mathbb{N} : y_n \in A\} \in \mathcal{J}$. Let $\{n \in \mathbb{N} : y_n \in A\} = J \in \mathcal{J}$. And since $\mathcal{J}$ is a non-trivial ideal, it follows that $A \neq X$. Take a point $a \in X \setminus A$. Define a sequence $\{x_n\}_{n\in\mathbb{N}}$ by $x_n = a$ if $n \in J$; $x_n = y_n$ if $n \in \mathbb{N} \setminus J$. Then the sequence $\{x_n\}_{n\in\mathbb{N}}$ in $X \setminus A$ and $x_n \xrightarrow{\mathcal{J}} x$ from Lemma 2.5. □

Theorem 3.5. *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. If $f : X \to Y$ is an $\mathcal{I}$-quotient mapping, then for each $\mathcal{I}$-convergent sequence $\{y_n\}_{n\in\mathbb{N}}$ in Y with $y_n \xrightarrow{\mathcal{I}} y$, there is a sequence $\{x_i\}_{i\in\mathbb{N}}$ in X such that $\{x_i : i \in \mathbb{N}\} \subseteq f^{-1}(\{y_n : n \in \mathbb{N}\})$ and $x_i \xrightarrow{\mathcal{I}} x \notin f^{-1}(\{y_n : n \in \mathbb{N}\})$.*

Proof. Suppose that $f : X \to Y$ is an $\mathcal{I}$-quotient mapping and $\{y_n\}_{n\in\mathbb{N}}$ is a sequence in Y with $y_n \xrightarrow{\mathcal{I}} y$. Without loss of generality, we can assume that $y_n \neq y$ for each $n \in \mathbb{N}$. Let $U = Y \setminus \{y_n : n \in \mathbb{N}\}$. Then U is not $\mathcal{I}$-open in Y. Since f is an $\mathcal{I}$-quotient mapping, $f^{-1}(U) = f^{-1}(Y \setminus \{y_n : n \in \mathbb{N}\}) = X \setminus f^{-1}(\{y_n : n \in \mathbb{N}\})$ is not $\mathcal{I}$-open in X. Thus there is a sequence $\{x_i\}_{i\in\mathbb{N}}$ in $X \setminus f^{-1}(U) = f^{-1}(\{y_n : n \in \mathbb{N}\})$ such that $x_i \xrightarrow{\mathcal{I}} x \notin f^{-1}(\{y_n : n \in \mathbb{N}\})$. □

In [20], it was discussed that quotient mappings, sequentially quotient mappings and $\mathcal{I}$-quotient mappings are mutually independent; and the following two theorems are useful and can be seen in it.

Theorem 3.6. *Let $f : X \to Y$ be a mapping.*

(1) *If X is an $\mathcal{I}$-sequential space and f is quotient, then Y is an $\mathcal{I}$-sequential space and f is $\mathcal{I}$-quotient.*
(2) *If Y is an $\mathcal{I}$-sequential space and f is $\mathcal{I}$-quotient, then f is quotient.*
(3) *X is an $\mathcal{I}$-sequential space if and only if for an arbitrary topological space Y, if f is quotient, then f is $\mathcal{I}$-quotient.*

Theorem 3.7. *Let $\mathcal{J}$ be a maximal ideal of $\mathbb{N}$ and X be a topological space. Then X is an $\mathcal{J}$-sequential space if and only if each $\mathcal{J}$-quotient mapping onto X is quotient.*

Let $\mathcal{J}$ be a maximal ideal of $\mathbb{N}$ and $A \subseteq X$. Denote

$[A]_{\mathcal{J}\text{-}s} = \{x \in X :$ there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ in A such that $x_n \xrightarrow{\mathcal{J}} x\}$;

$(A)_{\mathcal{J}\text{-}s} = \{x \in X : A$ is an $\mathcal{J}$ -sequential neighborhood of $x\}$.

A subset $U \subseteq X$ is said to be an *$\mathcal{J}$-sequential neighborhood* of A if $A \subseteq (U)_{\mathcal{J}\text{-}s}$.

Proposition 3.8. *Let $\mathcal{J}$ be a maximal ideal of $\mathbb{N}$ and $A \subseteq X$. Then $[A]_{\mathcal{J}\text{-}s} = X \setminus (X \setminus A)_{\mathcal{J}\text{-}s}$.*

Proof. Suppose that $x \in [A]_{\mathcal{J}\text{-}s}$, then there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ in A such that $x_n \xrightarrow{\mathcal{J}} x$. Thus $X \setminus A$ is not an $\mathcal{J}$-sequential neighborhood of x in X. In fact, if $X \setminus A$ is an $\mathcal{J}$-sequential neighborhood of x in X, then $\{x_n\}_{n\in\mathbb{N}}$ is $\mathcal{J}$-eventually in $X \setminus A$, i.e., there is $E \in \mathcal{J}$ such that for all $n \in \mathbb{N} \setminus E, x_n \in X \setminus A$. Since $\mathcal{J}$ is an admissible ideal, this contradicts to $\{x_n\}_{n\in\mathbb{N}}$ in A. Therefore $x \notin (X \setminus A)_{\mathcal{J}\text{-}s}$, and further $x \in X \setminus (X \setminus A)_{\mathcal{J}\text{-}s}$.

On the other hand, assume that $x \in X \setminus (X \setminus A)_{\mathcal{J}\text{-}s}$, then $x \notin (X \setminus A)_{\mathcal{J}\text{-}s}$, and hence $X \setminus A$ is not an $\mathcal{J}$-sequential neighborhood of x in X. By Proposition 3.4, there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ in A such that $x_n \xrightarrow{\mathcal{J}} x$. Thus $x \in [A]_{\mathcal{J}\text{-}s}$. □

By Definition 2.2 and Proposition 3.8, the following proposition is correct.

Proposition 3.9. *Let $\mathcal{J}$ be a maximal ideal of $\mathbb{N}$ and $A, B \subseteq X$. Then*

(1) $[\varnothing]_{\mathcal{J}\text{-}s} = \varnothing$, $A^\circ \subseteq (A)_{\mathcal{J}\text{-}s} \subseteq A \subseteq [A]_{\mathcal{J}\text{-}s} \subseteq \overline{A}$.
(2) *A is $\mathcal{J}$-open in X if and only if $A = (A)_{\mathcal{J}\text{-}s}$.*
(3) *A is $\mathcal{J}$-closed in X if and only if $A = [A]_{\mathcal{J}\text{-}s}$.*
(4) *If $B \subseteq A$, then $(B)_{\mathcal{J}\text{-}s} \subseteq (A)_{\mathcal{J}\text{-}s}$ and $[B]_{\mathcal{J}\text{-}s} \subseteq [A]_{\mathcal{J}\text{-}s}$.*
(5) $(A \cap B)_{\mathcal{J}\text{-}s} = (A)_{\mathcal{J}\text{-}s} \cap (B)_{\mathcal{J}\text{-}s}$ *and* $[A \cup B]_{\mathcal{J}\text{-}s} = [A]_{\mathcal{J}\text{-}s} \cup [B]_{\mathcal{J}\text{-}s}$.

Proof. We only prove that (5) is true. Since $A \cap B \subseteq A$, $A \cap B \subseteq B$, it follows that $(A \cap B)_{\mathcal{J}\text{-}s} \subseteq (A)_{\mathcal{J}\text{-}s}$, $(A \cap B)_{\mathcal{J}\text{-}s} \subseteq (B)_{\mathcal{J}\text{-}s}$. Hence $(A \cap B)_{\mathcal{J}\text{-}s} \subseteq (A)_{\mathcal{J}\text{-}s} \cap (B)_{\mathcal{J}\text{-}s}$. On the other hand, assume that $x \in (A)_{\mathcal{J}\text{-}s} \cap (B)_{\mathcal{J}\text{-}s}$. Then for each sequence $\{x_n\}_{n\in\mathbb{N}}$ in X with $x_n \xrightarrow{\mathcal{J}} x$, there is $E, F \in \mathcal{J}$, such that for each $n \in \mathbb{N} \setminus E$, $x_n \in A$ and for each $n \in \mathbb{N} \setminus F$, $x_n \in B$. Since $E \cup F \in \mathcal{J}$ and for each $n \in \mathbb{N} \setminus (E \cup F)$, $x_n \in A \cap B$. This means that $A \cap B$ is an $\mathcal{J}$-sequential neighborhood of x in X. Thus $x \in (A \cap B)_{\mathcal{J}\text{-}s}$.

Now replace $X \setminus A$ with A and $X \setminus B$ with B, it follows that $((X \setminus A) \cap (X \setminus B))_{\mathcal{J}\text{-}s} = (X \setminus A)_{\mathcal{J}\text{-}s} \cap (X \setminus B)_{\mathcal{J}\text{-}s}$. Hence $[A \cup B]_{\mathcal{J}\text{-}s} = X \setminus (X \setminus (A \cup B))_{\mathcal{J}\text{-}s} = X \setminus ((X \setminus A) \cap (X \setminus B))_{\mathcal{J}\text{-}s} = X \setminus ((X \setminus A))_{\mathcal{J}\text{-}s} \cap (X \setminus B))_{\mathcal{J}\text{-}s}) = (X \setminus (X \setminus A)_{\mathcal{J}\text{-}s}) \cup (X \setminus (X \setminus B)_{\mathcal{J}\text{-}s}) = [A]_{\mathcal{J}\text{-}s} \cup [B]_{\mathcal{J}\text{-}s}$. □

Theorem 3.10. *Let $\mathcal{J}$ be a maximal ideal of $\mathbb{N}$ and $f : X \to Y$ be a mapping. Then the following conditions are equivalent.*

(1) *For each $\mathcal{J}$-convergent sequence $\{y_n\}_{n\in\mathbb{N}}$ in Y with $y_n \xrightarrow{\mathcal{J}} y$, there is a sequence $\{x_i\}_{i\in\mathbb{N}}$ in X with $x_i \xrightarrow{\mathcal{J}} x \in f^{-1}(y)$ and $\{x_i : i \in \mathbb{N}\} \subseteq f^{-1}(\{y_n : n \in \mathbb{N}\})$.*
(2) *For each $A \subseteq Y$, it has $f([f^{-1}(A)]_{\mathcal{J}\text{-}s}) = [A]_{\mathcal{J}\text{-}s}$.*

(3) *If* $y \in [A]_{\mathcal{J}\text{-}s} \subseteq Y$, *then* $f^{-1}(y) \cap [f^{-1}(A)]_{\mathcal{J}\text{-}s} \neq \varnothing$.
(4) *If* $y \in [A]_{\mathcal{J}\text{-}s} \subseteq Y$, *then there is a point* $x \in f^{-1}(y)$ *such that whenever* V *is an* $\mathcal{J}$*-sequential neighborhood of* x, $y \in [f(V) \cap A]_{\mathcal{J}\text{-}s}$.
(5) *If* $y \in [A]_{\mathcal{J}\text{-}s} \subseteq Y$, *then there is a point* $x \in f^{-1}(y)$ *such that whenever* V *is an* $\mathcal{J}$*-sequential neighborhood of* x, $f(V) \cap A \neq \varnothing$.
(6) *For each* $y \in Y$, *if* U *is an* $\mathcal{J}$*-sequential neighborhood of* $f^{-1}(y)$, *then* $f(U)$ *is an* $\mathcal{J}$*-sequential neighborhood of* y.

Proof. (1) $\Rightarrow$ (2) Suppose that $x \in [f^{-1}(A)]_{\mathcal{J}\text{-}s}$. Then there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $f^{-1}(A)$ such that $x_n \xrightarrow{\mathcal{J}} x$. Hence $\{f(x_n) : n \in \mathbb{N}\} \subseteq A$ and $f(x_n) \xrightarrow{\mathcal{J}} f(x)$. This means that $f(x) \in [A]_{\mathcal{J}\text{-}s}$. Hence $f([f^{-1}(A)]_{\mathcal{J}\text{-}s}) \subseteq [A]_{\mathcal{J}\text{-}s}$.

On the other hand, assume that $y \in [A]_{\mathcal{J}\text{-}s}$. Then there is a sequence $\{y_n\}_{n\in\mathbb{N}}$ in A such that $y_n \xrightarrow{\mathcal{J}} y$. By the condition (1), there is a sequence $\{x_i\}_{i\in\mathbb{N}}$ in X with $\{x_i : i \in \mathbb{N}\} \subseteq f^{-1}(\{y_n : n \in \mathbb{N}\}) \subseteq f^{-1}(A)$ and $x_i \xrightarrow{\mathcal{J}} x \in f^{-1}(y)$. Thus $x \in [f^{-1}(A)]_{\mathcal{J}\text{-}s}$, hence $y = f(x) \in f([f^{-1}(A)]_{\mathcal{J}\text{-}s})$, and further $[A]_{\mathcal{J}\text{-}s} \subseteq f([f^{-1}(A)]_{\mathcal{J}\text{-}s})$.

(2) $\Rightarrow$ (3) Let $y \in [A]_{\mathcal{J}\text{-}s}$ for each $A \subseteq Y$. By the condition (2), it follows that $y \in f([f^{-1}(A)]_{\mathcal{J}\text{-}s})$. Thus $f^{-1}(y) \cap [f^{-1}(A)]_{\mathcal{J}\text{-}s} \neq \varnothing$.

(3) $\Rightarrow$ (4) Let $y \in [A]_{\mathcal{J}\text{-}s} \subseteq Y$. By the condition (3), assume that $x \in f^{-1}(y) \cap [f^{-1}(A)]_{\mathcal{J}\text{-}s}$. Then there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $f^{-1}(A)$ such that $x_n \xrightarrow{\mathcal{J}} x$. If V is an $\mathcal{J}$-sequential neighborhood of x, then there is $E \in \mathcal{J}$ such that $x_n \in V$ for all $n \in \mathbb{N} \setminus E$. Hence $f(x_n) \in f(V) \cap A$ for all $n \in \mathbb{N} \setminus E$ and $f(x_n) \xrightarrow{\mathcal{J}} f(x)$. Take a point $a \in f(V) \cap A$. Define a sequence $\{y_n\}_{n\in\mathbb{N}}$ by $y_n = f(x_n)$ if $n \in \mathbb{N} \setminus E$; $y_n = a$ if $n \in E$. Then $\{y_n : n \in \mathbb{N}\} \subseteq f(V) \cap A$ and $y_n \xrightarrow{\mathcal{J}} f(x) = y$ from Lemma 2.4. Thus $y \in [f(V) \cap A]_{\mathcal{J}\text{-}s}$.

(4) $\Rightarrow$ (5) It is clear.

(5) $\Rightarrow$ (6) Let $y \in Y$ and U be an $\mathcal{J}$-sequential neighborhood of $f^{-1}(y)$. If $f(U)$ is not an $\mathcal{J}$-sequential neighborhood of y, then $y \in Y \setminus (f(U))_{\mathcal{J}\text{-}s} = [Y \setminus f(U)]_{\mathcal{J}\text{-}s}$. By the condition (5), it follows that $f(U) \cap (Y \setminus f(U)) = \varnothing$, a contradiction.

(6) $\Rightarrow$ (3) Let $y \in [A]_{\mathcal{J}\text{-}s} \subseteq Y$. Suppose that $f^{-1}(y) \cap [f^{-1}(A)]_{\mathcal{J}\text{-}s} = \varnothing$. Then $f^{-1}(y) \subseteq X \setminus [f^{-1}(A)]_{\mathcal{J}\text{-}s} = (X \setminus f^{-1}(A))_{\mathcal{J}\text{-}s}$. This means that $X \setminus f^{-1}(A)$ is an $\mathcal{J}$-sequential neighborhood of $f^{-1}(y)$. By the condition (6), $y \in (f(X \setminus f^{-1}(A)))_{\mathcal{J}\text{-}s} = (Y \setminus A)_{\mathcal{J}\text{-}s} = Y \setminus [A]_{\mathcal{J}\text{-}s}$, a contradiction.

(3) $\Rightarrow$ (1) Let $\{y_n\}_{n\in\mathbb{N}}$ be an $\mathcal{J}$-convergent sequence in Y with $y_n \xrightarrow{\mathcal{J}} y$. Put $A = \{y_n : n \in \mathbb{N}\}$, then $y \in [A]_{\mathcal{J}\text{-}s}$. By the condition (3), there is $x \in f^{-1}(y) \cap [f^{-1}(A)]_{\mathcal{J}\text{-}s}$. Hence there is a sequence $\{x_i\}_{i\in\mathbb{N}}$ in X with $\{x_i : i \in \mathbb{N}\} \subseteq f^{-1}(A) \subseteq f^{-1}(\{y_n : n \in \mathbb{N}\})$ and $x_i \xrightarrow{\mathcal{J}} x \in f^{-1}(y)$. □

Remark 3.11.

(1) Theorem 3.5 is different from Lemma 3.10 (1). In Lemma 3.10 (1), $x_i \xrightarrow{\mathcal{J}} x \in f^{-1}(y)$. But we don't know whether the $\mathcal{I}$-limit point x in $f^{-1}(y)$ or not in Theorem 3.5.

(2) One of the above six conditions can deduce that f is an $\mathcal{J}$-quotient mapping.

In fact, let U be non-$\mathcal{I}$-closed in Y. Then there is a sequence $\{y_n\}_{n\in\mathbb{N}}$ in U $\mathcal{J}$-converging to $y \in Y\backslash U$. Thus $y \neq y_n$ for each $n \in \mathbb{N}$. By the assumption of the condition (1), there is a sequence $\{x_i\}_{i\in\mathbb{N}}$ in X such that $\{x_i : i \in \mathbb{N}\} \subseteq f^{-1}(\{y_n : n \in \mathbb{N}\}) \subseteq f^{-1}(U)$ and $x_i \xrightarrow{\mathcal{J}} x \in f^{-1}(y) \notin f^{-1}(U)$. This implies that $f^{-1}(U)$ is non-$\mathcal{J}$-closed in X. Hence, f is an $\mathcal{J}$-quotient mapping.

(3) If the maximal ideal $\mathcal{J}$ is replaced by $\mathcal{I}_f$ in Theorem3.10, then (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) $\Leftrightarrow$ f is an $\mathcal{I}_f$-quotient mapping. But the following example shows that there exist a T_1 space X, an ideal $\mathcal{I}$ of $\mathbb{N}$ and an $\mathcal{I}$-quotient mapping f such that f does not satisfy the condition (6) of Theorem 3.10.

Example 3.12. There exist a T_1 space X, an ideal $\mathcal{I}$ of $\mathbb{N}$ and an $\mathcal{I}$-quotient mapping f, but f does not satisfy the condition (6) of Theorem 3.10.

Proof. Let $\mathcal{I} = \{A \subseteq \mathbb{N} : A$ contains at most only finite odd positive integers$\}$. Then $\mathcal{I}$ is an admissible ideal of $\mathbb{N}$. Let Y be the set ω which is endowed with the finite complement topology. Then Y is a first-countable T_1-space. Put $X_0 = Y \setminus \{0\}$ and $X_1 = \{2k : k \in \omega\}$ as the subspaces of the space Y, and $X = X_0 \bigoplus X_1$. A mapping $f : X \to Y$ is defined by the natural mapping. It is easy to see that the mapping f is a continuous quotient mapping. Since X_0 and X_1 are first-countable space, X is a first-countable space. Thus, X is an $\mathcal{I}$-sequential space. By Theorem 3.6, it follows that f is an $\mathcal{I}$-quotient mapping.

Note that the set X_1 is open in X and $f^{-1}(0) \subseteq X_1$, and hence X_1 is an $\mathcal{I}$-sequential neighborhood of $f^{-1}(0)$. For each open neighborhood U of 0 in Y, $\{n \in \mathbb{N} : n \notin U\}$ is a finite subset, hence $\{n \in \mathbb{N} : n \notin U\} \in \mathcal{I}$. This means that the sequence $\{n\}_{n\in\mathbb{N}}$ in Y satisfies $n \xrightarrow{\mathcal{I}} 0$. But $\{n \in \mathbb{N} : n \notin f(X_1)\} = \{2k+1, k \in \omega\} \notin \mathcal{I}$. Thus $f(X_1)$ is not an $\mathcal{I}$-sequential neighborhood of 0 in Y. □

Problem 3.13. *For some maximal ideal $\mathcal{J}$ of $\mathbb{N}$ and an $\mathcal{J}$-quotient mapping f, does it satisfy the condition (6) of Theorem 3.10?*

4. On Spaces with $\mathcal{I}$-cs'-Networks

In this section, we introduce the concepts of $\mathcal{I}$-cs-networks, $\mathcal{I}$-cs'-networks and $\mathcal{I}$-wcs'-networks for a space X; and obtain that if X is an $\mathcal{J}$-FU space with a point-countable $\mathcal{J}$-cs'-network, then X is a meta-Lindelöf space, for a maximal ideal $\mathcal{J}$ of $\mathbb{N}$.

Definition 4.1 ([13])**.** Let $\mathcal{I}$ be an ideal on $\mathbb{N}$, X be a topological space and $\mathcal{P}$ be a cover of X.

(1) $\mathcal{P}$ is a *network* of X if whenever $x \in U$ with U open in X, then $x \in P \subseteq U$ for some $P \in \mathcal{P}$.

(2) $\mathcal{P}$ is called an *$\mathcal{I}$-cs-network* of X if whenever $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in X $\mathcal{I}$-converging to a point $x \in U$ with U open in X, then $\{x_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-eventually in P and $x \in P \subseteq U$ for some $P \in \mathcal{P}$.

(3) $\mathcal{P}$ is called an *$\mathcal{I}$-cs'-network* of X if whenever $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in X $\mathcal{I}$-converging to a point $x \in U$ with U open in X, then there is $P \in \mathcal{P}$ and some $n_0 \in \mathbb{N}$ such that $\{x, x_{n_0}\} \subseteq P \subseteq U$.

(4) $\mathcal{P}$ is called an *$\mathcal{I}$-wcs'-network* of X if whenever $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in X $\mathcal{I}$-converging to a point $x \in U$ with U open in X, then there is $P \in \mathcal{P}$ and some $n_0 \in \mathbb{N}$ such that $\{x_{n_0}\} \subseteq P \subseteq U$.

Obviously, $\mathcal{I}$-cs-networks $\Rightarrow$ $\mathcal{I}$-cs'-networks $\Rightarrow$ $\mathcal{I}$-wcs'-networks $\Rightarrow$ networks.

Definition 4.2. Let $\mathcal{J}$ be a maximal ideal of $\mathbb{N}$ and X be a topological space. $\mathcal{U}$ is said to be *$\mathcal{J}$-sn-cover* of X, if $\{(U)_{\mathcal{J}\text{-}s} : U \in \mathcal{U}\}$ is a cover of X.

Theorem 4.3. *Each $\mathcal{I}$-cs-network is preserved by an $\mathcal{I}$-covering mapping.*

Proof. Let $f : X \to Y$ be an $\mathcal{I}$-covering mapping and $\mathcal{P}$ be an $\mathcal{I}$-cs-network of X. Suppose that $\{y_n\}_{n\in\mathbb{N}}$ is a sequence $\mathcal{I}$-converging to a point $y \in U$ with U open in Y. Since f is an $\mathcal{I}$-covering mapping, there exist a sequence $\{x_n\}_{n\in\mathbb{N}}$ of points $x_n \in f^{-1}(y_n)$ for all $n \in \mathbb{N}$ and $x \in f^{-1}(y)$ such that $x_n \xrightarrow{\mathcal{I}} x$. Since $\mathcal{P}$ is an $\mathcal{I}$-cs-network of X, there is some $P \in \mathcal{P}$ such that $\{x_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-eventually in P and $x \in P \subseteq f^{-1}(U)$. Thus there is $E \in \mathcal{I}$ such that $\{n \in \mathbb{N} : x_n \notin P\} \subseteq E$. Note that $\{n \in \mathbb{N} : y_n \notin f(P)\} \subseteq \{n \in \mathbb{N} : x_n \notin P\} \subseteq E$, hence $y_n \in f(P)$ for all $n \in \mathbb{N} \setminus E$, i.e. $\{y_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-eventually in $f(P)$ and $y \in f(P) \subseteq U$. This means that $f(\mathcal{P}) = \{f(P) : P \in \mathcal{P}\}$ is an $\mathcal{I}$-cs-network of Y. □

Corollary 4.4. *Each $\mathcal{I}$-cs'-network is preserved by an $\mathcal{I}$-covering mapping.*

Theorem 4.5. *Each $\mathcal{I}$-wcs'-network is preserved by an $\mathcal{I}$-quotient mapping.*

Proof. Let $f : X \to Y$ be an $\mathcal{I}$-quotient mapping and $\mathcal{P}$ be an $\mathcal{I}$-wcs'-network of X. Suppose that $\{y_n\}_{n\in\mathbb{N}}$ is a sequence $\mathcal{I}$-converging to a point $y \in U$ with U open in Y. Since f is an $\mathcal{I}$-quotient mapping, there is a sequence $\{x_i\}_{i\in\mathbb{N}}$ in X such that $\{x_i : i \in \mathbb{N}\} \subseteq f^{-1}(\{y_n : n \in \mathbb{N}\})$ and $x_i \xrightarrow{\mathcal{I}} x \notin f^{-1}(\{y_n : n \in \mathbb{N}\})$. And because $\mathcal{P}$ is an $\mathcal{I}$-wcs'-network of X, there is some $P_0 \in \mathcal{P}$ and $i_0 \in \mathbb{N}$ such that $\{x_{i_0}\} \subseteq P_0 \subseteq f^{-1}(U)$. And hence $\{f(x_{i_0})\} = \{y_{n_0}\} \subseteq f(P_0) \subseteq U$ for some $n_0 \in \mathbb{N}$. This implies that $f(\mathcal{P}) = \{f(P) : P \in \mathcal{P}\}$ is an $\mathcal{I}$-wcs'-network of Y. □

Lemma 4.6. *Let $\mathcal{J}$ be a maximal ideal of $\mathbb{N}$ and $\mathcal{P}$ be a family of subsets of X. Then $\mathcal{P}$ is an $\mathcal{J}$-cs'-network of X if and only if, whenever U is an open neighborhood of x, $\bigcup\{P \in \mathcal{P} : x \in P \subseteq U\}$ is an $\mathcal{J}$-sequential neighborhood of x.*

Proof. Necessity: Let U be an open neighborhood of x. If $\bigcup\{P \in \mathcal{P} : x \in P \subseteq U\}$ is not an $\mathcal{J}$-sequential neighborhood of x, then there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n \xrightarrow{\mathcal{J}} x$ and $x_n \notin \bigcup\{P \in \mathcal{P} : x \in P \subseteq U\}$ for each $n \in \mathbb{N}$. Since $\mathcal{P}$ is an $\mathcal{J}$-cs'-network of X, there is $P_0 \in \mathcal{P}$ and $n_0 \in \mathbb{N}$ such that $\{x, x_{n_0}\} \subseteq P_0 \subseteq U$, a contradiction.

Sufficiency: Suppose that $x_n \xrightarrow{\mathcal{J}} x \in U \in \tau_X$ and $\bigcup\{P \in \mathcal{P} : x \in P \subseteq U\}$ is an $\mathcal{J}$-sequential neighborhood of x. Then $\{x_n\}_{n\in\mathbb{N}}$ is $\mathcal{J}$-eventually in $\bigcup\{P \in \mathcal{P} : x \in P \subseteq U\}$. Hence there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} \in \bigcup\{P \in \mathcal{P} : x \in P \subseteq U\}$. And hence there is $P_0 \in \mathcal{P}$ such that $x_{n_0} \in P_0$ and $x \in P_0 \subseteq U$. Thus $\{x, x_{n_0}\} \subseteq P_0 \subseteq U$. This means that $\mathcal{P}$ is an $\mathcal{J}$-*cs*′-network of X. □

Theorem 4.7. *Let $\mathcal{J}$ be a maximal ideal of $\mathbb{N}$ and a space X be of a point-countable $\mathcal{J}$-cs′-network. Then each open cover of X has a point-countable $\mathcal{J}$-sn refinement.*

Proof. Suppose that $\mathcal{P}$ is a point-countable $\mathcal{J}$-*cs*′-network for a space X. Let $\mathcal{U} = \{U_\alpha\}_{\alpha<\gamma}$ be an open cover of X, where γ is an ordinal. For each $\alpha < \gamma$, put

$$V_\alpha = \bigcup\{P \in \mathcal{P} : P \subseteq U_\alpha, P \not\subseteq U_\beta \text{ if } \beta < \alpha\}.$$

Clearly, $V_\alpha \subseteq U_\alpha$. Next we shall show that the family $\mathcal{V} = \{V_\alpha\}_{\alpha<\gamma}$ is a point-countable $\mathcal{J}$-*sn*-cover of X. For each $x \in X$, let $\alpha(x) = \min\{\alpha < \gamma : x \in U_\alpha\}$. Then $x \in U_{\alpha(x)}$ and

$$\bigcup\{P \in \mathcal{P} : x \in P \subseteq U_{\alpha(x)}\} \subseteq \bigcup\{P \in \mathcal{P} : P \subseteq U_{\alpha(x)}, P \not\subseteq U_\beta \text{ if } \beta < \alpha(x)\}.$$

Since $\mathcal{P}$ is an $\mathcal{J}$-*cs*′-network for a space X, it follows from Lemma 4.6 that

$$\begin{aligned} x &\in (\bigcup\{P \in \mathcal{P} : x \in P \subseteq U_{\alpha(x)}\})_{\mathcal{J}\text{-}s} \\ &\subseteq (\bigcup\{P \in \mathcal{P} : P \subseteq U_{\alpha(x)}, P \not\subseteq U_\beta \text{ if } \beta < \alpha(x)\})_{\mathcal{J}\text{-}s} \\ &= (V_{\alpha(x)})_{\mathcal{J}\text{-}s}. \end{aligned}$$

This means that $\mathcal{V} = \{V_\alpha\}_{\alpha<\gamma}$ is an $\mathcal{J}$-*sn*-cover of X.

We claim that $\mathcal{V}$ is point-countable. Suppose, to the contrary, that there exist a point $x \in X$ and an uncountable subset Γ of γ such that $x \in V_\alpha$ for each $\alpha \in \Gamma$. Hence there is $P_\alpha \in \mathcal{P}$ such that $x \in P_\alpha \subseteq U_\alpha$ and $P_\alpha \not\subseteq U_\beta$ for $\beta < \alpha$. Since $\mathcal{P}$ is a point-countable family and Γ is an uncountable set, there are $\alpha, \beta \in \Gamma, \alpha \neq \beta$ such that $P_\alpha = P_\beta$. Assume that $\beta < \alpha$, then $U_\beta \supseteq P_\beta = P_\alpha \not\subseteq U_\beta$, a contradiction. □

Definition 4.8.

(1) A space X is called $\mathcal{I}$-*Fréchet-Urysohn* (or shortly, $\mathcal{I}$-*FU*) space, if for each $A \subset X$ and each $x \in \overline{A}$, there exists a sequence in A $\mathcal{I}$-converging to the point x in X [20].

(2) A space X is called a *meta-Lindelöf* space if each open cover of X has a point-countable open refinement [13].

Corollary 4.9. *Let $\mathcal{J}$ be a maximal ideal of $\mathbb{N}$. If X is an $\mathcal{J}$-FU space with a point-countable $\mathcal{J}$-cs′-network, then X is a meta-Lindelöf space.*

Proof. X is an $\mathcal{J}$-FU space $\Leftrightarrow \overline{A} = [A]_{\mathcal{J}\text{-}s}$ for each $A \subseteq X \Leftrightarrow \text{int}A = (A)_{\mathcal{J}\text{-}s}$ for each $A \subseteq X$. □

Theorem 4.10. *Let $\mathcal{J}$ be a maximal ideal of $\mathbb{N}$. The following are equivalent for a space X.*

(1) *$\mathcal{J}$-sX is an $\mathcal{J}$-Fréchet-Urysohn space.*
(2) $\mathrm{cl}_{\mathcal{J}\text{-}sX}(A) = [A]_{\mathcal{J}\text{-}s}$, *for each $A \subseteq X$.*
(3) *$[A]_{\mathcal{J}\text{-}s}$ is $\mathcal{J}$-closed in X, for each $A \subseteq X$.*
(4) *$(A)_{\mathcal{J}\text{-}s}$ is $\mathcal{J}$-open in X, for each $A \subseteq X$.*

Proof. Since the spaces X and $\mathcal{J}$-sX have the same $\mathcal{J}$-convergent sequences, by the Definition 4.8 and Proposition 3.8, it follows that (1) $\Leftrightarrow$ (2) and (3) $\Leftrightarrow$ (4). Hence, it suffices to show that (2) $\Leftrightarrow$ (3). If $\mathrm{cl}_{\mathcal{J}\text{-}sX}(A) = [A]_{\mathcal{J}\text{-}s}$, then $[A]_{\mathcal{J}\text{-}s}$ is closed in $\mathcal{J}$-sX, and hence $[A]_{\mathcal{J}\text{-}s}$ is $\mathcal{J}$-closed in X, for each $A \subseteq X$. On the other hand, if $[A]_{\mathcal{J}\text{-}s}$ is $\mathcal{J}$-closed in X, then $[A]_{\mathcal{J}\text{-}s}$ is closed in $\mathcal{J}$-sX, and further $\mathrm{cl}_{\mathcal{J}\text{-}sX}(A) = [A]_{\mathcal{J}\text{-}s}$, for each $A \subseteq X$. □

Acknowledgements. *My gratitude goes to Professor Shou Lin, for his friendly encouragement and inspiring suggestions.*

References

[1] J. R. Boone and F. Siwiec, Sequentially quotient mappings, Czech. Math. J. 26 (1976), 174–182.
[2] L. X. Cheng, G. C. Lin, Y. Y. Lan and H. Liu, Measure theory of statistical convergence, Sci. China Ser. A 51 (2008), 2285–2303.
[3] L. X. Cheng, G. C. Lin and H. H. Shi, On real-valued measures of statistical type and their applications to statistical convergence, Math. Comput. Modelling 50 (2009), 116–122.
[4] P. Das, Some further results on ideal convergence in topological spaces, Topol. Appl. 159 (2012), 2621–2626.
[5] P. Das and S. Ghosal, When $\mathcal{I}$-Cauchy nets in complete uniform spaces are $\mathcal{I}$-convergent, Topol. Appl. 158 (2011), 1529–1533.
[6] P. Das, Lj.D.R. Kočinac and D. Chandra, Some remarks on open covers and selection principles using ideals, Topol. Appl. 202 (2016), 183–193.
[7] G. Di Maio and Lj. D. R. Kočinac, Statistical convergence in topology, Topol. Appl. 156 (2008), 28–45.
[8] R. Engelking, General Topology (revised and completed edition), Heldermann Verlag, Berlin, 1989.
[9] H. Fast, Sur la convergence statistique, Colloq. Math. 2 (1951), 241–244.
[10] L. Gillman and M. Jerison, Rings of Continuous Functions, Van Nostrand, Princeton, 1960.
[11] P. Kostyrko, T. Šalát and W. Wilczynski, $\mathcal{I}$-convergence, Real Anal. Exch. 26 (2000/2001), 669–686.
[12] B. K. Lahiri and P. Das, $\mathcal{I}$ and $\mathcal{I}^*$-convergence in topological spaces, Math. Bohemica 130, no. 2 (2005), 153–160.
[13] S. Lin, Point-countable covers and sequence-covering mappings, Science Press, Beijing, 2015 (in Chinese).
[14] S. Lin and Z.Q. Yun, Generalized metric spaces and mapping, Atlantis Studies in Mathematics 6, Atlantis Press, Paris, 2016.

[15] S. K. Pal, N. Adhikary and U. Samanta, On ideal sequence covering maps, Appl. Gen. Topol. 20, no. 2 (2019), 363–377.
[16] H. Steinhaus, Sur la convergence ordinaire et la convergence asymptotique, Colloq. Math. 2 (1951), 73–74.
[17] Z. Tang and F. Lin, Statistical versions of sequential and Fréchet-Urysohn spaces, Adv. Math. (China) 44 (2015), 945–954.
[18] X. G. Zhou and M. Zhang, More about the kernel convergence and the ideal convergence, Acta Math. Sinica, English Series 29 (2013), 2367–2372.
[19] X. G. Zhou and L. liu, On $\mathcal{I}$-covering mappings and 1-$\mathcal{I}$-covering mappings, J. Math. Res. Appl. (China) 40, no. 1 (2020) 47–56.
[20] X. G. Zhou, L. Liu and S. Lin, On topological spaces defined by $\mathcal{I}$-convergence, Bull. Iran. Math. Soc. 46 (2020), 675–692.