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Fixed point property of amenable planar vortexes

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Abstract

This article, dedicated to Mahlon M. Day, introduces free group presentations of planar vortexes in a CW space that are a natural outcome of results for amenable groups and fixed points found by M.M. Day during the 1960s and a fundamental result for fixed points given by L.E.J. Brouwer.

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1. Introduction

This article introduces consequences of results for amenable groups and fixed points found by M.M. Day during the 1960s in terms of free group presentations [17] of planar vortexes in a CW (Closure-finite Weak) space. Results given here spring from a fundamental result for fixed points given by L.E.J. Brouwer [3].

Theorem 1.1 (Brouwer Fixed Point Theorem [18, §4.7, p. 194])**.**
Every continuous map from $\mathbb{R}^n$ to itself has a fixed point.

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Briefly, let Σ be a finite group and let $m(\Sigma)$ be a set of bounded, real-valued functions on Σ. Then Σ is amenable, provided there is a mean μ on $m(\Sigma)$ which is both left and right invariant.

Theorem 1.2 (Day Abelian Group [Semigroup] Theorem ([6, 7])**.**
Every finite group is amenable.

The study of amenable groups led to the following extension of the Kakutani-Markov Theorem by Day.

Theorem 1.3 (Day Fixed Point Theorem ([7])**.**
Let K be a compact convex subset of a locally convex linear topological space X, and let Σ be a semigroup (under functional composition) of continuous affine transformations of K into itself. If Σ, when regarded as an abstract semigroup, is amenable, or if it has a left-invariant mean, then there is in K a common fixed point of the family Σ.

A direct consequence of Theorem 1.3 is that each amenable group of a planar vortex has a fixed point in a CW space.

Definition 1.4. A planar vortex $\mathrm{vor}E$ is a finite cell complex, which is a collection of path-connected vertices in nested, filled 1-cycles in a CW complex K. A 1-cycle in $\mathrm{vor}E$ (denoted by $\mathrm{cyc}A$) is a sequence of edges with no end vertex and with a nonempty interior. A geometric realization of $\mathrm{vor}E$ is denoted by $|\mathrm{vor}E|$ on $|K|$ in the Euclidean plane.

A nonvoid collection of cell complexes K is a *Closure finite Weak* CW space, provided K is Hausdorff (every pair of distinct cells is contained in disjoint neighbourhoods [11, §5.1, p. 94]) and the collection of cell complexes in K satisfy the Whitehead [19, pp. 315-317], [20, §5, p. 223] CW conditions, namely, the closure of each cell complex is in K and the nonempty intersection of cell complexes is in K.

A number of important results concerning fixed points in this paper spring from Čech proximities, leading to descriptive proximally continuous maps. A descriptive proximally continuous map is defined over descriptive Čech proximity spaces [5, §4.1] in which the description of a nonempty set is in the form of a feature vector derived from probe functions, one for each feature of the set. For the details, see App. C.

2. Conjugacy between proximal descriptively continuous maps

This section introduces proximal conjugacy between two dynamical systems, which is an easy extension of topological conjugacy [1, §8.1,p. 243]. Proximal conjugacy is akin to strongly amenable groups in which each of its proximal topological actions has a fixed point [8]. Let $\sum$ denote either a semigroup or a group. Also, let lub, glb denote least upper bound and greatest lower bound, respectively, and let $m(\sum)$ be the set of bounded, real-valued functions θ on $\sum$ for which

$$\|\theta\| = lub_{x\in\sum} |\theta(x)|\,.$$

A **mean** μ on $m(\sum)$ is an element of the $m(\sum)^*$ (in the conjugate space B^* of a Banach space B [6, p.510]) such that, for each $x \in m(\sum)$, we have

$$glb_{x\in\sum}\theta(x) \leq \mu(x) \leq lub_{x\in\sum}\theta(x).$$

An element of μ of $m(\sum)^*$ is left[right] invariant, provided

$$\mu(\ell_\sigma x) = \mu(x)\ [\mu(r_\sigma x) = \mu(x)],\ \text{for all } x \in m(\sum), \sigma \in \sum,$$

where $(\ell_\sigma x)\sigma' = x(\sigma\sigma')$ and $(r_\sigma x)\sigma' = x(\sigma'\sigma)$ for all $\sigma' \in \Sigma$.

Definition 2.1 ([6])**.** A semigroup (also group) $\sum$ is amenable, provided there is a mean μ on $m(\sum)$, which is both left and right invariant.

Theorem 2.2. *A free group presentation of a planar vortex in a CW space is amenable.*

Proof. This result is a direct consequence of Theorem 1.2 from Day [7] and (I) [6, p.516], since, by construction, every free group G presentation of a planar vortex in a CW space is finite and, by definition, an Abelian semigroup. □

Theorem 2.2 stems from Day's extension of Theorem 2.3 (restated by Day [7, p. 585]) to cover the case when the family in question is a semigroup.

Theorem 2.3 (Kakutani-Markov Theorem [9, 10])**.** *Let K be a compact convex set in a locally convex linear topological space, and let F be a commuting family of continuous, affine transformations, f, of K into itself. Then there is a common fixed point of the functions in F; that is, there is an x in K such that $f(x) = x$ for every f in F.*

If a planar vortex vor$E$ has no hole inside, then we no longer need to require it be convex. It is automatically convex and we have the following.

Theorem 2.4. *For a CW complex K, let (K, δ) be a proximity space that contains a planar vortex vor$E$ without a planar hole and let $f : \text{vor}E \to \text{vor}E$ be proximal continuous. Then vor$E$ has a fixed point of f.*

Proof. Since vor$E$ is finite, the topology on the geometric realization $|\text{vor}E|$ of vor$E$ can be regarded as the subspace topology inherited from the Euclidean space $\mathbb{R}^2$. Then if we have the geometric realization of f, denoted $|f|$, we see that $|f|$ is a continuous affine transformation. This is true since, f maps two near subsets to the two near subsets. Also notice that $|\text{vor}E|$ is convex compact subset of $\mathbb{R}^2$ and the collection of maps $\{|f^n| \, : \, n = 1, 2 \ldots\}$ is amenable, since it is a semigroup under composition. Then by Theorem 2.3, for the family of continuous affine transformations $\{|f^n| \, : \, n = 1, 2, \ldots\}$, there is an x in $|\text{vor}E|$ such that $f(x) = x$ and so $|\text{vor}E|$ has a fixed point of $|f|$. This also allows us to conclude that vor$E$ has a fixed point of f without considering the geometric realization. □

If a planar vortex vor$E$ has a (planar) hole inside, then we could consider a subset vor$E$ such that its geometric realization is convex compact.

Theorem 2.5. *For a CW complex K, let (K, δ) be a proximity space that contains a planar vortex vor$E$ with a planar hole and let $X \subset$ vor$E$ such that its geometric realization $|X|$ is convex compact. If $f : X \to X$ is proximal continuous, then X has a fixed point under f.*

Proof. By the construction of a planar vortex, there is subset X of vor$E$ such that its geometric realization $|X|$ is a convex compact subset of $|\text{vor}E|$. Then a proximal continuous map $f : X \to X$ has a corresponding continuous affine transformation $|f| : |X| \to |X|$. Again by Theorem 2.3, for the family of continuous affine transformations $\{|f^n| \ : \ n = 1, 2, \dots\}$, there is an x in $|X|$ such that $f(x) = x$ and so $|X|$ has a fixed point of $|f|$. Hence, vor$E$ has a fixed point of f without considering the geometric realization. □

Remark 2.6. From what have observed, notice that any vortex has a locally compact abelian group presentation, since it is a locally compact Hausdorff space and its underlying group structure is abelian. In that case, any proximal continuous map from a vortex to itself can be also considered as a group action. Hence, by a direct consequence of Theorem 2.2 and a result of a generalization of the Kakutani-Markov Theorem 2.3, each amenable vortex has a fixed point.

Corollary 2.7. *If f is a proximal continuous map from a vortex to itself, then f has a fixed point.*

Next, we introduce the (descriptive) proximal conjugate between two proximal (descriptive) continuous maps. Note that a (descriptive) Čech proximity space X together with a (descriptive) proximal continuous self map on X can be considered as a (descriptive) proximal dynamical system. Now we introduce a (descriptive) proximal conjugacy between two (descriptive) dynamical systems, so that the existence of it guarantees the (descriptive) dynamical systems having equivalent flows and related (descriptive) fixed points.

Definition 2.8. Two proximal continuous maps $f : (X, \delta_1) \to (X, \delta_1)$ and $g : (Y, \delta_2) \to (Y, \delta_2)$ are said to be proximal conjugates, provided there exists a proximal isomorphism $h : (X, \delta_1) \to (Y, \delta_2)$ such that $g \circ h = h \circ f$. The function h is called a proximal conjugacy between f and g.

The following theorem states that if two proximal continuous maps are proximal conjugate, then their corresponding iterated functions are also proximal conjugate.

Theorem 2.9. *Let h be a proximal conjugacy between $f : (X, \delta_1) \to (X, \delta_1)$ and $g : (Y, \delta_2) \to (Y, \delta_2)$. Then for each $A \subseteq X$ and $n \in \mathbb{Z}_+$, we have $h(f^n(A)) = g^n(h(A))$.*

Proof. The proof follows from the induction on n. □

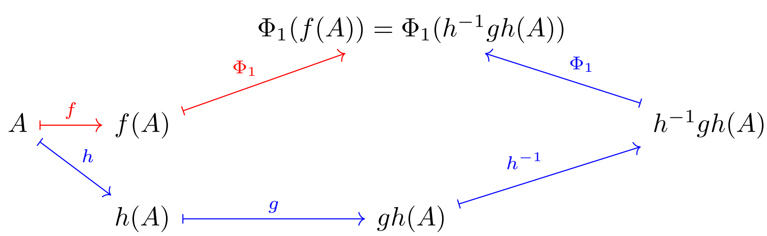
Definition 2.10. Let (X, δ_Φ) be a descriptive proximity space with a probe function $\Phi : X \to \mathbb{R}^n$ and $A, B \in 2^X$. Then A and B are said to be descriptively equal, provided $\Phi(A) = \Phi(B)$. In that case, we write $A \underset{\text{des}}{=} B$.

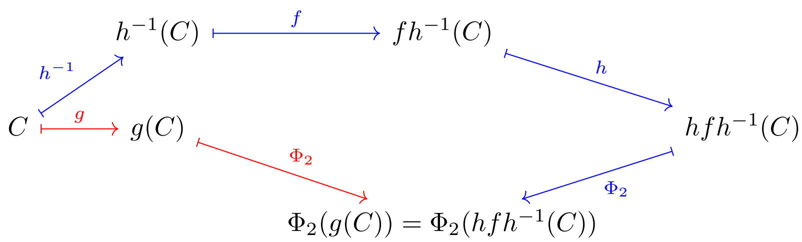
Definition 2.11. Let $\Phi(E) \in \mathbb{R}^n$ be a vector of n real-values that describe a nonempty set E. Two proximal descriptive continuous maps $f : (X, \delta_{\Phi_1}) \to (X, \delta_{\Phi_1})$ and $g : (Y, \delta_{\Phi_2}) \to (Y, \delta_{\Phi_2})$ are said to be proximal descriptive conjugates, provided there exists a proximal descriptive isomorphism $h : (X, \delta_{\Phi_1}) \to (Y, \delta_{\Phi_2})$ such that $g \circ h(A) \underset{\text{des}}{=} h \circ f(A)$ for any $A \in 2^X$. The function h is called a proximal descriptive conjugacy between f and g.

Remark 2.12. We see from the definition of a proximal descriptive conjugacy that $g \circ h(A)$ and $h \circ f(A)$ may not be equal but we have

$$\Phi_2(g \circ h(A)) = \Phi_2(h \circ f(A))$$

for $A \in 2^X$. Moreover $g \circ h(A) \underset{\text{des}}{=} h \circ f(A)$ implies $g(A) \underset{\text{des}}{=} h \circ f \circ h^{-1}(A)$ and $f(A) \underset{\text{des}}{=} h^{-1} \circ g \circ h(A)$, so that we have the following commutative diagrams.





Remark 2.13. For proximal descriptive conjugates $f : (X, \delta_{\Phi_1}) \to (X, \delta_{\Phi_1})$ and $g : (Y, \delta_{\Phi_2}) \to (Y, \delta_{\Phi_2})$, Def. 2.11 tells us that for $A \subseteq X$ and $C \subseteq Y$, we have

$$\Phi_2(g \circ h(A)) = \Phi_2(h \circ f(A)),$$
$$\Phi_1(f \circ h^{-1}(C)) = \Phi_1(h^{-1} \circ g(C)).$$

Note that if h is a proximal descriptive conjugacy between $f : (X, \delta_{\Phi_1}) \to (X, \delta_{\Phi_1})$ and $g : (Y, \delta_{\Phi_2}) \to (Y, \delta_{\Phi_2})$, then $A \underset{\text{des}}{=} B$ implies $h(A) \underset{\text{des}}{=} h(B)$ for $A, B \in 2^X$.

Theorem 2.14. *Let h be a proximal descriptive conjugacy between $f : (X, \delta_{\Phi_1}) \to (X, \delta_{\Phi_1})$ and $g : (Y, \delta_{\Phi_2}) \to (Y, \delta_{\Phi_2})$. Then for each $A \in 2^X$ and $n \in \mathbb{Z}_+$, we have $h(f^n(A)) \underset{des}{=} g^n(h(A))$.*

Proof. The proof follows from the induction on n. □

Definition 2.15. Let (X, δ_Φ) be a descriptive proximity space with a probe function $\Phi : X \to \mathbb{R}^n$, $A \in 2^X$, and $f : (X, \delta_\Phi) \to (X, \delta_\Phi)$ a descriptive proximally continuous map.

(i) A is a descriptive fixed subset of f, provided $\Phi(f(A)) = \Phi(A)$.

(ii) A is an eventual descriptive fixed subset of f, provided A is not a descriptive fixed subset while $f^t(A)$ is a descriptive fixed subset for some $t \neq 1$.

(iii) A is an almost descriptive fixed subset of f, provided $\Phi(f(A)) = \Phi(A)$ or $\Phi(f(A))\ \delta_\Phi\ \Phi(A)$ so that $f(A) \underset{\Phi}{\cap} A \neq \varnothing$ by Lemma C.1. (For the analogy of a point being almost fixed in digital topology, we refer to [2, 16].)

Corollary 2.16. *Let h be a proximal descriptive conjugacy between $f : (X, \delta_{\Phi_1}) \to (X, \delta_{\Phi_1})$ and $g : (Y, \delta_{\Phi_2}) \to (Y, \delta_{\Phi_2})$.*

a) If A is a descriptively fixed subset of f, then $h(A)$ is a descriptively fixed subset of g.

b) If A is an eventual descriptively fixed subset of f, then $h(A)$ is an eventual descriptively fixed subset of g.

c) If A is an almost descriptively fixed subset of f, then $h(A)$ is an almost descriptively fixed subset of g.

Proof. a) Let A be a descriptively fixed subset of f. That is, $\Phi_1(f(A)) = \Phi_1(A)$. In other words, we have $f(A) \underset{\text{des}}{=} A$. Since h is a proximal isomorphism, h preserves desciptive proximity $h(f(A)) \underset{\text{des}}{=} h(A)$. By Theorem 2.14, $g(h(A)) \underset{\text{des}}{=} h(A)$ so that $h(A)$ is a descriptively fixed subset of g.

b) Let A be an eventual descriptively fixed subset of f. That is, A is not a descriptively fixed subset of f but $\Phi_1(f^n(A)) = \Phi_1(A)$ for some positive integer $n > 1$. In other words, we have $f^n(A) \underset{\text{des}}{=} A$. Since h is a proximal isomorphism, h preserves being equal in a descriptive sense: $h(f^n(A)) \underset{\text{des}}{=} h(A)$. By Theorem 2.14, $g^n(h(A)) \underset{\text{des}}{=} h(A)$. Note that $h(A)$ is not a descriptively fixed subset of g since A is not a descriptively fixed subset of f and h is an isomorphism. So, $h(A)$ is an eventual descriptively fixed subset of g.

c) Let A be an almost descriptively fixed subset of f. That is, $f(A) \underset{\text{des}}{=} A$ or $A\ \delta_{\Phi_1}\ f(A)$. If $f(A) \underset{\text{des}}{=} A$, then we are done. Let $A\ \delta_{\Phi_1}\ f(A)$. Since h is a proximal isomorphism, we have $h(A)\ \delta_{\Phi_2}\ h(f(A))$. By Theorem 2.14, $h(A)\ \delta_{\Phi_2}\ g(h(A))$ so that $h(A)$ is a descriptively fixed subset of g.

□

Further, the existence of proximal conjugacy between two dynamical systems of cell complexes such as vortexes also guarantees isomorphic amenable group structures and hence related fixed points, which is another consequence of Theorem 2.2.

Corollary 2.17. *If there exists a descriptive proximal conjugacy between two descriptive dynamical systems, then they have isomorphic descriptive fixed subsets.*

Proof. Let $f : (X, \delta_{\Phi_1}) \to (X, \delta_{\Phi_1})$ and $g : (Y, \delta_{\Phi_2}) \to (Y, \delta_{\Phi_2})$ be proximal descriptive conjugates and $h : (X, \delta_{\Phi_1}) \to (Y, \delta_{\Phi_2})$ be the proximal descriptive conjugacy between them. If $A \in 2^X$ is a descriptive fixed subset of f, then $h(A)$ is a descriptive fixed subset of g by Corollary 2.16 so that A and $h(A)$ are descriptively isomorphic. Similarly if $B \in 2^Y$ is a descriptive fixed subset of g, then $h^{-1}(B)$ is a descriptive fixed subset of f so that B and $h^{-1}(B)$ are descriptively isomorphic. Hence there is a one-to-one correspondence between the set of the descriptive fixed subsets of f and the set of the descriptive fixed subsets of g. □

3. Weak conjugacy between descriptive proximally continuous maps

This section introduces weak conjugacy between descriptive proximally continuous maps.

Definition 3.1. Two proximally continuous maps $f : (X, \delta_1) \to (X, \delta_1)$ and $g : (Y, \delta_2) \to (Y, \delta_2)$ are said to be weakly proximal conjugates, provided there exists a proximal isomorphism $h : (X, \delta_1) \to (Y, \delta_2)$ such that for any $A \in 2^X$, $g \circ h(A)\ \delta_2\ h \circ f(A)$. Note that this also implies that $f \circ h^{-1}(C)\ \delta_1\ h^{-1} \circ g(C)$ for any $C \in 2^Y$. The function h is called a weakly proximal conjugacy between f and g.

Theorem 3.2. *Let h be a weakly proximal conjugacy between $f : (X, \delta_1) \to (X, \delta_1)$ and $g : (Y, \delta_2) \to (Y, \delta_2)$. Then for each $A \in 2^X$ and $n \in \mathbb{Z}_+$, we have $h(f^n(A))\ \delta_2\ g^n(h(A))$.*

Proof. The proof follows from the induction on n. □

Definition 3.3. Two descriptive proximally continuous maps $f : (X, \delta_{\Phi_1}) \to (X, \delta_{\Phi_1})$ and $g : (Y, \delta_{\Phi_2}) \to (Y, \delta_{\Phi_2})$ are said to be weakly proximal descriptive conjugates, provided there exists a proximal descriptive isomorphism $h : (X, \delta_{\Phi_1}) \to (Y, \delta_{\Phi_2})$ such that $g \circ h(A)\ \delta_{\Phi_2}\ h \circ f(A)$ for any $A \in 2^X$. Note that this also implies $f \circ h^{-1}(C)\ \delta_{\Phi_2}\ h^{-1} \circ g(C)$ for any $C \in 2^Y$. The function h is called a weakly proximal descriptive conjugacy between f and g.

Remark 3.4. For weakly proximal descriptive conjugates $f : (X, \delta_{\Phi_1}) \to (X, \delta_{\Phi_1})$ and $g : (Y, \delta_{\Phi_2}) \to (Y, \delta_{\Phi_2})$, Def. 3.3 and Lemma C.1 tell us that for $A \in 2^X$ and $C \in 2^Y$, we have

$$g \circ h(A) \underset{\Phi}{\cap} f \circ h(A) \neq \varnothing,$$

$$f \circ h^{-1}(C) \underset{\Phi}{\cap} h^{-1} \circ g(C) \neq \varnothing.$$

Appendix A. Planar Vortexes

This section briefly looks at planar vortex structures in planar CW spaces. For simplicity, we consider only 2 cycle vortexes containing a pair of nested 1-cycles that intersect or attached to each other via at least one bridge edge.

Definition A.1. ([15]) Let cyc$A$, cyc$B$ be a collection of path-connected vertexes on nested filled 1-cycles (with cyc$B$ in the interior of cyc$A$) defined on a finite, bounded, planar region in a CW space K. A *planar 2 cycle vortex* vor$E$ is defined by

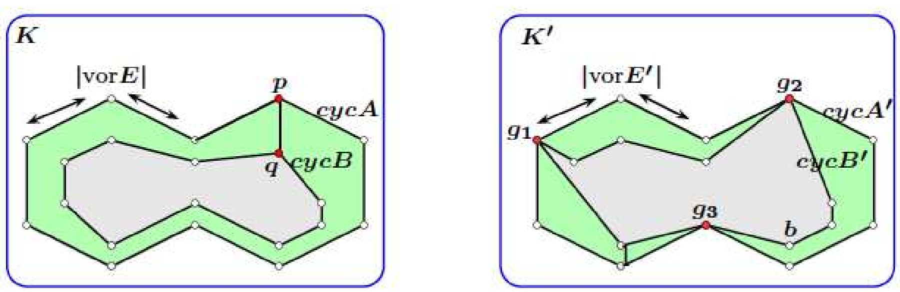
$$\mathrm{vor}E = \overbrace{\left\{\mathrm{cl}(\mathrm{cyc}A) : \mathrm{cl}(\mathrm{cyc}B) \subset \mathrm{int}(\mathrm{cl}(\mathrm{cyc}A))\right\}}^{\textbf{cl(cyc}B\textbf{) is contained (nested) in the interior of cl(cyc}A\textbf{).}}.$$

A vortex containing adjacent non-intersecting cycles has a bridge edge attached to vertexes on the cycles.

Definition A.2. A vortex bridge edge is an edge attached to vertexes on a pair of non-intersecting, filled 1-cycles.

Remark A.3. From Def. A.1, the cycles in a 2 cycle vortex can either have nonempty intersection (see, *e.g.*, cyc$A' \cap$ cyc$B' \neq \varnothing$ in $|\mathrm{vor}E'|$ in Fig. 1b) or there is a bridge edge between the cycles (see, *e.g.*, $\widehat{pq}$ $|\mathrm{vor}E|$ in Fig. 1a). In effect, every pair of vertexes in a 2 cycle vortex is path-connected.

Remark A.4. The structure of a 2 cycle vortex extends to a vortex with $k > 2$ nested filled 1-cycles, provided adjacent pairs of cycles cyc$A$, cycA' in a k-cycle vortex either intersect or there is a bridge edge attached between cyc$A$, cycA'.



(a) Vortex $|\mathrm{vor}E|$ with non-intersecting 1-cycles cyc$A$, cyc$B$

(b) Vortex $|\mathrm{vor}E|$ with intersecting 1-cycles cycA', cycB'

Figure 1. Sample planar 2-cycle vortexes

Appendix B. Free Group Presentation of a Vortex

A finite group G is free, provided every element $x \in G$ is a linear combination of its basis elements (called generators). We write $\mathcal{B}$ to denote a nonempty set of generators $\left\{g_1, \ldots, g_{|\mathcal{B}|}\right\}$ and $G(\mathcal{B}, +)$ to denote the free group with binary operation $+$.

Example B.1. The basis $\{g_1, g_2, g_3\}$ generates a group G whose geometric realization is $|\mathrm{vor}E'|$ in Fig. 1b. The $+$ operation on G corresponds to a move from a generator to a neighbouring vertex. For example,

$$b = \overbrace{3g_1 + 3g_2}^{\textbf{traversing 3 cyc}\boldsymbol{A'}\textbf{ \& 3 cyc}\boldsymbol{B'}\textbf{ edges to reach }\boldsymbol{b}\textbf{ via }\boldsymbol{g_1, g_2}}$$

$$b = \overbrace{4g_1 + 3g_2}^{\textbf{traversing 7 cyc}\boldsymbol{B'}\textbf{ edges to reach }\boldsymbol{b}\textbf{ via }\boldsymbol{g_1, g_2}}$$

$$b = \overbrace{0g_1 + 1g_3.}^{\textbf{traversing 1 cyc}\boldsymbol{B'}\textbf{ edge to reach }\boldsymbol{b}\textbf{ via }\boldsymbol{g_3}}$$

The identity element 0 in G is represented by a zero move from a generator g to another vertex (denoted by $0g$) and an inverse in G is represented by a reverse move $-g$.

Definition B.2. Let 2^K be the collection of cell complexes in a CW space K, vortex $|\mathrm{vor}E| \in 2^K$, basis $\mathcal{B} \in |\mathrm{vor}E|$, k_i the i^{th} integer coefficient in a linear combination $\sum\limits_{i,j} k_i g_j$ of generating elements $g_j \in \mathcal{B}$. A free group G presentation of $|\mathrm{vor}E|$ is a continuous self-map $f : 2^K \to 2^K$ defined by

$$f(|\mathrm{vor}E|) = \left\{ v := \sum_{i,j} k_i g_j : v \in |\mathrm{vor}E|, g_j \in \mathcal{B} \right\}$$

$$= \overbrace{\boldsymbol{G}(\{\boldsymbol{g_1, \ldots, g_{|\mathcal{B}|}}\}, +).}^{|\mathbf{vor}\boldsymbol{E}| \mapsto \textbf{ free group }\boldsymbol{G}}$$

Appendix C. Descriptive Proximity Spaces

This section briefly introduces descriptive Čech proximity spaces, paving the way for descriptive proximally continuous maps. The simplest form of proximity relation (denoted by δ) on a nonempty set was introduced by E. Čech [4]. A nonempty set X equipped with the relation δ is a Čech proximity space (denoted by (X, δ)), provided the following axioms are satisfied.

Čech Axioms

(P.0): All nonempty subsets in X are far from the empty set, *i.e.*, $A \not\delta \varnothing$ for all $A \subseteq X$.

(P.1): $A \ \delta \ B \Rightarrow B \ \delta \ A$.

(P.2): $A \cap B \neq \varnothing \Rightarrow A\ \delta\ B$.
(P.3): $A\ \delta\ (B \cup C) \Rightarrow A\ \delta\ B$ or $A\ \delta\ C$.

Given that a nonempty set E has $k \geq 1$ features such as Fermi energy E_{Fe}, cardinality E_{card}, a description $\Phi(E)$ of E is a feature vector, *i.e.*, $\Phi(E) = (E_{Fe}, E_{card})$. Nonempty sets A, B with overlapping descriptions are descriptively proximal (denoted by $A\ \delta_\Phi\ B$). The descriptive intersection of nonempty subsets in $A \cup B$ (denoted by $A \underset{\Phi}{\cap} B$) is defined by

$$A \underset{\Phi}{\cap} B = \overbrace{\{x \in A \cup B\ :\ \Phi(x) \in \Phi(A)\ \cap\ \Phi(B)\}}^{\textit{i.e.}\text{, Descriptions } \Phi(A)\ \&\ \Phi(B) \text{ overlap}}.$$

Let 2^X denote the collection of all subsets in a nonvoid set X. A nonempty set X equipped with the relation δ_Φ with non-void subsets $A, B, C \in 2^X$ is a descriptive proximity space, provided the following descriptive forms of the Čech axioms are satisfied.

Descriptive Čech Axioms

(dP.0): All nonempty subsets in 2^X are descriptively far from the empty set, *i.e.*, $A\ \not\delta_\Phi\ \varnothing$ for all $A \in 2^X$.
(dP.1): $A\ \delta_\Phi\ B \Rightarrow B\ \delta_\Phi\ A$.
(dP.2): $A \underset{\Phi}{\cap} B \neq \varnothing \Rightarrow A\ \delta_\Phi\ B$.
(dP.3): $A\ \delta_\Phi\ (B \cup C) \Rightarrow A\ \delta_\Phi\ B$ or $A\ \delta_\Phi\ C$.

The converse of Axiom (**dp**.2) also holds.

Lemma C.1 ([14]). *Let X be equipped with the relation δ_Φ, $A, B \in 2^X$. Then $A\ \delta_\Phi\ B$ implies $A \underset{\Phi}{\cap} B \neq \varnothing$.*

Proof. Let $A, B \in 2^X$. By definition, $A\ \delta_\Phi\ B$ implies that there is at least one member $x \in A$ and $y \in B$ so that $\Phi(x) = \Phi(y)$, *i.e.*, x and y have the same description. Then $x, y \in A \underset{\Phi}{\cap} B$. Hence, $A \underset{\Phi}{\cap} B \neq \varnothing$, which is the converse of (**dp**.2). □

Theorem C.2. *Let K be a cell complex, $Vor(K) \subset K$ a collection of planar vortexes equipped with the proximity δ_Φ and let $vorA, vorB \in Vor(K)$. Then $vorA\ \delta_\Phi\ vorB$ implies $vorA \underset{\Phi}{\cap} vorB \neq \varnothing$.*

Proof. Immediate from Lemma C.1. □

Let (X, δ_1) and (Y, δ_2) be two Čech proximity spaces. Then a map $f : (X, \delta_1) \to (Y, \delta_2)$ is *proximal continuous*, provided $A\ \delta_1\ B$ implies $f(A)\ \delta_2\ f(B)$, *i.e.*, $f(A)\ \delta_2\ f(B)$, provided $f(A) \cap f(B) \neq \varnothing$ for $A, B \in 2^X$ [12, §1.4]. In general, a proximal continuous function preserves the nearness of pairs of sets [11,

§1.7,p. 16]. Further, f is a *proximal isomorphism*, provided f is proximal continuous with a proximal continuous inverse f^{-1}.

Let (X, δ_{Φ_1}) and (Y, δ_{Φ_2}) be descriptive proximity spaces with probe functions $\Phi_1 : X \to \mathbb{R}^n$, $\Phi_2 : Y \to \mathbb{R}^n$, and $A, B \in 2^X$. Then a map $f : (X, \delta_{\Phi_1}) \to (Y, \delta_{\Phi_2})$ is said to be *descriptive proximally continuous*, provided $A\ \delta_{\Phi_1}\ B$ implies $f(A)\ \delta_{\Phi_2}\ f(B)$, *i.e.*, $f(A)\ \delta_{\Phi_2}\ f(B)$, provided $f(A) \underset{\Phi}{\cap} f(B) \neq \varnothing$. Further f is a *descriptive proximal isomorphism*, provided f and its inverse f^{-1} are descriptively proximally continuous.

Definition C.3. Let (X, δ_Φ) be a descriptive Čech proximity space and $f : (X, \delta_\Phi) \to (X, \delta_\Phi)$ a descriptive proximally continuous map. A set $A \in 2^X$ is said to be descriptively invariant with respect to f, provided $\Phi(f(A)) \subseteq \Phi(A)$.

Notice that if A is a descriptively invariant set with respect to f, then $\Phi(f^n(A)) \subseteq \Phi(A)$ for all positive integer n.

Theorem C.4. *Let (X, δ_Φ) be a descriptive Čech proximity space and $f : (X, \delta_\Phi) \to (X, \delta_\Phi)$ a proximal descriptive continuous map. If $\{A_i\}_{i\in I} \subseteq 2^X$ is a collection of descriptively invariant sets with respect to f, then*

i) $\cup_{i\in I} A_i$ is descriptively invariant with respect to f, and

ii) $\cap_{i\in I} A_i$ is descriptively invariant with respect to f.

Proof. From our assumption, we have $\Phi(f(A_i)) \subseteq \Phi(A_i)$ for all $i \in I$ so that

i)

$$\begin{aligned} f(\cup_{i\in I} A_i) &= \cup_{i\in I} f(A_i) \\ \Phi(f(\cup_{i\in I} A_i)) &= \Phi(\cup_{i\in I} f(A_i)) \\ &= \cup_{i\in I} \Phi(f(A_i)) \\ &\subseteq \cup_{i\in I} \Phi(A_i) \end{aligned}$$

and

ii)

$$\begin{aligned} f(\cap_{i\in I} A_i) &\subseteq \cap_{i\in I} f(A_i) \\ \Phi(f(\cap_{i\in I} A_i)) &\subseteq \Phi(\cap_{i\in I} f(A_i)) \\ &\subseteq \cap_{i\in I} \Phi(f(A_i)) \\ &\subseteq \cap_{i\in I} \Phi(A_i) \end{aligned}$$

□

Theorem C.5. *Let (X, δ_Φ) be a descriptive Čech proximity space and $f : (X, \delta_\Phi) \to (X, \delta_\Phi)$ a descriptive proximally continuous map. If $A \in 2^X$ is descriptively invariant with respect to f then $cl_{\delta_\Phi} A$ is also descriptively invariant with respect to with respect to f.*

Proof. The descriptive closure of a subset A of X is defined in [13, §1.21.2] as follows:

$$\mathrm{cl}_\Phi A = \{x \in X \mid x\ \delta_\Phi\ A\}.$$

Take an element x in $\mathrm{cl}_\Phi A$ so that $x\ \delta_\Phi\ A$ and $\Phi(x) \in \Phi(A)$ by Lemma C.1. Since f is descriptive proximally continuous $f(x)\ \delta_\Phi\ f(A)$ and $\Phi(f(x)) \in \Phi(f(A))$ by Lemma C.1. We also have $\Phi(f(x)) \in \Phi(A)$ since A is an invariant set with respect to f. Therefore $f(x)\ \delta_\Phi\ A$ and $f(x) \in \mathrm{cl}_\Phi A$. Since this holds for all $x \in \mathrm{cl}_\Phi A$, we have $f(\mathrm{cl}_\Phi A) \subseteq \mathrm{cl}_\Phi A$ so that $\Phi(f(\mathrm{cl}_\Phi A)) \subseteq \Phi(\mathrm{cl}_\Phi A)$. □

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