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Small and large inductive dimension for ideal topological spaces

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Abstract

Undoubtedly, the small inductive dimension, ind*, and the large inductive dimension,* Ind*, for topological spaces have been studied extensively, developing an important field in Topology. Many of their properties have been studied in details (see for example [1,4,5,9,10,18]). However, researches for dimensions in the field of ideal topological spaces are in an initial stage. The covering dimension,* dim*, is an exception of this fact, since it is a meaning of dimension, which has been studied for such spaces in [17]. In this paper, based on the notions of the small and large inductive dimension, new types of dimensions for ideal topological spaces are studied. They are called $*$-small and $*$-large inductive dimension, ideal small and ideal large inductive dimension. Basic properties of these dimensions are studied and relations between these dimensions are investigated.*

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1. Introduction and preliminaries

The Dimension Theory is a developing branch of Topology, which attracts the interest of many researches (see for example [1, 2, 4–13, 18]). Especially, the covering dimension, dim, the small inductive dimension, ind, and the large inductive dimension, Ind, are three main topological dimensions, which have

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been studied extensively, and many results in various classes of topological spaces have been proved.

Simultaneously, the notion of ideal leads to an important chapter in Topology (see for example [3, 14, 15, 19]). The main notion of ideal topological space was studied in Kuratowski's monograph [16]. However, the notion of topological dimension has not been investigated under the prism of ideals. The covering dimension is an exception. In the paper [17], the meaning of the ideal covering dimension is inserted and studied in details. Thus, in this paper new notions of inductive dimensions for ideal topological spaces are introduced and studied. They are called $*$-small and $*$-large inductive dimension, ideal small and ideal large inductive dimension.

Especially, in Section 2, we insert the meanings of the so-called $*$-small inductive dimension, ind^*, and $*$-large inductive dimension, Ind^*, for an arbitrary ideal topological space and study basic results. In Sections 3, we insert and study the meanings of the ideal small inductive dimension, $\mathcal{I}$ - ind, and the ideal large inductive dimension, $\mathcal{I}$ - Ind, and finally, in Section 4, we study additional properties of the ideal topological dimensions.

It is considered to be necessary, to recall the main notions and notations that will be used in the rest of this study. Especially, the notion of the ideal topological space and the known meanings of the small inductive dimension and the large inductive dimension are presented. The standard notation of Dimension Theory is referred to [1, 5, 18].

A nonempty family $\mathcal{I}$ of subsets of a set X is called an *ideal* on X if it satisfies the following properties:

(1) If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$.
(2) If $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$.

A topological space (X, τ) with an ideal $\mathcal{I}$ is called an *ideal topological space* and is denoted by $(X, \tau, \mathcal{I})$. In [15] the authors defined a new topology τ^* on X in terms of the Kuratowski closure operator cl^*. It is known that the family

$$\beta^* = \{U \setminus I : U \in \tau, I \in \mathcal{I}\}$$

is a basis for τ^* and the topology τ^* is finer than τ. Especially, if $\mathcal{I} = \{\varnothing\}$, then $\tau^* = \tau$ and if $\mathcal{I} = P(X)$, then τ^* is the discrete topology.

In what follows, by an *open set* (resp. *closed set*), we mean an open set (resp. closed set) in the topology τ. If a set U is open in the topology τ^*, then we say that U is a $*$-open set. Similarly, we define $*$-closed sets. If $A \subseteq X$, then $\mathrm{Bd}_X(A)$ and $\mathrm{Bd}^*_X(A)$ will denote the boundary of A in (X, τ) and (X, τ^*), respectively. Similarly, $\mathrm{Cl}_X(A)$ and $\mathrm{Cl}^*_X(A)$ will denote the closure of A in (X, τ) and (X, τ^*), respectively. Also, $\mathcal{I}_f$ denotes the ideal of all finite subsets of X.

Definition 1.1. The *small inductive dimension* of a topological space X, denoted by $\mathrm{ind}(X)$, is defined as follows:

(i) $\mathrm{ind}(X) = -1$, if $X = \varnothing$.

(ii) $\text{ind}(X) \leqslant k$, where $k \in \{0, 1, \ldots\}$, if for every element $x \in X$ and for every open subset V of X with $x \in V$, there exists an open subset U of X such that

$$x \in U \subseteq V \text{ and } \text{ind}(\text{Bd}_X(U)) \leqslant k - 1.$$

(iii) $\text{ind}(X) = k$, where $k \in \{0, 1, \ldots\}$, if $\text{ind}(X) \leqslant k$ and $\text{ind}(X) \nleqslant k - 1$.
(iv) $\text{ind}(X) = \infty$, if there does not exist any $k \in \{-1, 0, 1, 2, \ldots\}$ for which $\text{ind}(X) \leqslant k$ is true.

Definition 1.2. The *large inductive dimension* of a topological space X, denoted by $\text{Ind}(X)$, is defined as follows:

(i) $\text{Ind}(X) = -1$, if $X = \varnothing$.
(ii) $\text{Ind}(X) \leqslant k$, where $k \in \{0, 1, \ldots\}$, if for every pair (F, V) of subsets of X, where F is closed, V is open and $F \subseteq V$, there exists an open set U of X such that

$$F \subseteq U \subseteq V \text{ and } \text{Ind}(\text{Bd}_X(U)) \leqslant k - 1.$$

(iii) $\text{Ind}(X) = k$, where $k \in \{0, 1, \ldots\}$, if $\text{Ind}(X) \leqslant k$ and $\text{Ind}(X) \nleqslant k - 1$.
(iv) $\text{Ind}(X) = \infty$, if there does not exist any $k \in \{-1, 0, 1, 2, \ldots\}$ for which $\text{Ind}(X) \leqslant k$ is true.

2. The small inductive dimension and the large inductive dimension for ideal topological spaces

In this section, based on the notion of the topology τ^*, the $*$-small and $*$-large inductive dimension are defined for an ideal topological space $(X, \tau, \mathcal{I})$ and basic properties are studied.

Definition 2.1. The *$*$-small inductive dimension* of an ideal topological space $(X, \tau, \mathcal{I})$, denoted by $\text{ind}^*(X)$, is defined as follows:

(i) $\text{ind}^*(X) = -1$, if $X = \varnothing$.
(ii) $\text{ind}^*(X) \leqslant k$, where $k \in \{0, 1, \ldots\}$, if for every element $x \in X$ and for every $*$-open subset V of X with $x \in V$, there exists a $*$-open subset U of X such that

$$x \in U \subseteq V \text{ and } \text{ind}^*(\text{Bd}^*_X(U)) \leqslant k - 1.$$

(iii) $\text{ind}^*(X) = k$, where $k \in \{0, 1, \ldots\}$, if $\text{ind}^*(X) \leqslant k$ and $\text{ind}^*(X) \nleqslant k - 1$.
(iv) $\text{ind}^*(X) = \infty$, if there does not exist any $k \in \{-1, 0, 1, 2, \ldots\}$ for which $\text{ind}^*(X) \leqslant k$ is true.

Definition 2.2. The *$*$-large inductive dimension* of an ideal topological space $(X, \tau, \mathcal{I})$, denoted by $\text{Ind}^*(X)$, is defined as follows:

(i) $\text{Ind}^*(X) = -1$, if $X = \varnothing$.
(ii) $\text{Ind}^*(X) \leqslant k$, where $k \in \{0, 1, \ldots\}$, if for every pair (F, V) of subsets of X, where F is $*$-closed, V is $*$-open and $F \subseteq V$, there exists a $*$-open set U of X such that

$$F \subseteq U \subseteq V \text{ and } \text{Ind}^*(\text{Bd}^*_X(U)) \leqslant k - 1.$$

(iii) $\mathrm{Ind}^*(X) = k$, where $k \in \{0, 1, \ldots\}$, if $\mathrm{Ind}^*(X) \leqslant k$ and $\mathrm{Ind}^*(X) \nleqslant k-1$.
(iv) $\mathrm{Ind}^*(X) = \infty$, if there does not exist any $k \in \{-1, 0, 1, 2, \ldots\}$ for which $\mathrm{Ind}^*(X) \leqslant k$ is true.

We observe that the dimensions $\mathrm{ind}^*(X)$ and $\mathrm{Ind}^*(X)$ are the small and the large inductive dimension, respectively, of the topological space (X, τ^*). The following result will be useful in the rest of this study.

Theorem 2.3.
(1) *For any subset* A *of* X, $\mathrm{ind}^*(A) \leqslant \mathrm{ind}^*(X)$.
(2) *For any* $*$*-closed subset* A *of* X, $\mathrm{Ind}^*(A) \leqslant \mathrm{Ind}^*(X)$.

Proof. Based on Definition 2.1 and Definition 2.2, (1) and (2) can be proved by induction on the dimension $\mathrm{ind}^*(X)$ and $\mathrm{Ind}^*(X)$, respectively. □

However, the following examples prove that these dimensions are different to each other and different from the small inductive dimension, ind, and the large inductive dimension, Ind.

Example 2.4. Let $X = \{a, b, c\}$ with the topology $\tau = \{\varnothing, \{b, c\}, X\}$. If we consider the ideal $\mathcal{I} = \{\varnothing, \{a\}\}$, then $\mathrm{ind}^*(X) = 1$ and $\mathrm{Ind}^*(X) = 0$.

Example 2.5.
(1) We consider the set $X = \{a, b, c, d\}$ with the topology
$$\tau = \{\varnothing, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, X\}.$$
Then $\mathrm{ind}(X) = 1$. However, if we consider the ideal $\mathcal{I} = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$, then the space (X, τ^*) is the discrete space and thus, $\mathrm{ind}^*(X) = 0$.
(2) We consider the set $X = \{a, b, c\}$ with the topology $\tau = \{\varnothing, X\}$. Then $\mathrm{ind}(X) = 0$. However, if we consider the ideal $\mathcal{I} = \{\varnothing, \{b\}\}$, then $\mathrm{ind}^*(X) = 1$.

Example 2.6.
(1) We consider the set $X = \{a, b, c, d\}$ with the topology
$$\tau = \{\varnothing, \{a\}, \{a, b\}, \{a, d\}, \{a, b, d\}, \{a, c, d\}, X\}.$$
Then $\mathrm{Ind}(X) = 1$. However, if we consider the ideal $\mathcal{I} = \{\varnothing, \{a\}, \{d\}, \{a, d\}\}$, then the space (X, τ^*) is the discrete space and thus, $\mathrm{Ind}^*(X) = 0$.
(2) We consider the set $X = \{a, b, c, d\}$ with the topology
$$\tau = \{\varnothing, \{a\}, \{b\}, \{a, b\}, \{a, b, d\}, X\}.$$
Then $\mathrm{Ind}(X) = 0$. However, if we consider the ideal $\mathcal{I} = \{\varnothing, \{b\}, \{d\}, \{b, d\}\}$, then $\mathrm{Ind}^*(X) = 1$.

Remark 2.7. For any ideal topological space $(X, \tau, \mathcal{I})$ for which $\tau = \tau^*$ we have
(1) $\mathrm{ind}(X) = \mathrm{ind}^*(X)$ and
(2) $\mathrm{Ind}(X) = \mathrm{Ind}^*(X)$.

However, the converse of Remark 2.7 does not always hold and the following examples prove this claim.

Example 2.8.

(1) Let $X = \{a, b, c\}$ with the topology $\tau = \{\varnothing, \{b, c\}, X\}$. Then $\operatorname{ind}(X) = 1$. If we consider the ideal $\mathcal{I} = \{\varnothing, \{b\}\}$, then $\operatorname{ind}^*(X) = 1$ but $\tau \neq \tau^*$.

(2) Let $X = \{a, b, c, d\}$ with the topology

$$\tau = \{\varnothing, \{a\}, \{b\}, \{a, b\}, \{a, b, d\}, X\}.$$

Then $\operatorname{Ind}(X) = 0$. However, if we consider the ideal $\mathcal{I} = \{\varnothing, \{b\}\}$, then $\operatorname{Ind}^*(X) = 0$ but $\tau \neq \tau^*$.

Moreover, in the following propositions we can prove further relations between these dimensions.

Proposition 2.9. *For every ideal topological space $(X, \tau, \mathcal{I})$, where $\mathcal{I} \subseteq \tau^c$, the following are satisfied:*

(1) $\operatorname{ind}(X) = \operatorname{ind}^*(X)$ *and*

(2) $\operatorname{Ind}(X) = \operatorname{Ind}^*(X)$.

Proof. Since $\mathcal{I} \subseteq \tau^c$, we have that $\beta^* \subseteq \tau$. Therefore, $\tau = \tau^*$ and by Remark 2.7 we have that

$$\operatorname{ind}(X) = \operatorname{ind}^*(X) \text{ and } \operatorname{Ind}(X) = \operatorname{Ind}^*(X).$$

□

Proposition 2.10. *For every ideal topological* T_1*-space $(X, \tau, \mathcal{I})$, where $\mathcal{I} \subseteq \mathcal{I}_f$, the following are satisfied:*

(1) $\operatorname{ind}(X) = \operatorname{ind}^*(X)$ *and*

(2) $\operatorname{Ind}(X) = \operatorname{Ind}^*(X)$.

Proof. Since the ideal topological space $(X, \tau, \mathcal{I})$ is T_1, every $I \in \mathcal{I}$ is closed in (X, τ). Therefore, $\tau = \tau^*$ and by Proposition 2.9 we have that

$$\operatorname{ind}(X) = \operatorname{ind}^*(X) \text{ and } \operatorname{Ind}(X) = \operatorname{Ind}^*(X). \ \square$$

Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If β^* is a topology on X (and hence $\tau^* = \beta^*$), then the ideal $\mathcal{I}$ is called *τ-simple* [14]. Also, the ideal $\mathcal{I}$ is called *τ-codense* if $\mathcal{I} \cap \tau = \{\varnothing\}$, that is each member of $\mathcal{I}$ has empty interior with respect to the topology τ [3]. Moreover, we state that if $A \subseteq X$, then the family

$$\mathcal{I}_A = \{A \cap I : I \in \mathcal{I}\}$$

is an ideal on A. So, we can consider the ideal topological space $(A, \tau_A, \mathcal{I}_A)$, where τ_A is the subspace topology on A. The topology $(\tau_A)^*$ is equal to the subspace topology $(\tau^*)_A$ on A [15]. □

Proposition 2.11. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If $\mathcal{I}$ is τ-simple and τ-codense and for each closed subset F of X the ideal $\mathcal{I}_F$ is τ_F-codense, then*

$$\operatorname{ind}(X) \leqslant \operatorname{ind}^*(X).$$

Proof. Clearly, if $\operatorname{ind}^*(X) = -1$ or $\operatorname{ind}^*(X) = \infty$, then the inequality holds. We suppose that the inequality holds for all integers $m < k$ and we shall prove it for k. Let $\operatorname{ind}^*(X) = k$, $x \in X$ and V be an open set in X with $x \in V$. Then V is a $*$-open set in X. Since $\operatorname{ind}^*(X) = k$, there exists a $*$-open set U in X with $x \in U \subseteq V$ and $\operatorname{ind}^*(\operatorname{Bd}^*_X(U)) \leqslant k - 1$. Also, since $\mathcal{I}$ is τ-simple, $U = W \setminus I$, where $W \in \tau$ and $I \in \mathcal{I}$.

We consider the open set $V \cap W$ in X. Then $x \in U \subseteq V \cap W \subseteq V$. It suffices to prove that $\operatorname{ind}(\operatorname{Bd}_X(V \cap W)) \leqslant k - 1$. Let $Y = \operatorname{Bd}_X(V \cap W)$ with the subspace topology τ_Y and the ideal $\mathcal{I}_Y$. We have that $\mathcal{I}_Y$ is τ_Y-simple, as β^*_Y is a topology on Y, and τ_Y-codense, as Y is a closed subset of X. Also, we should prove that for every closed set K in Y, the ideal $\mathcal{I}_K$ is τ_K-codense. Let K be a closed set in Y. Since Y is closed in X, K is also closed in X and thus, $\mathcal{I}_K$ is τ_K-codense. Moreover, we observe that

$$\operatorname{Bd}_X(V \cap W) \subseteq \operatorname{Bd}^*_X(U).$$

Indeed, we suppose that there exists $y \in \operatorname{Bd}_X(V \cap W)$ such that $y \notin \operatorname{Bd}^*_X(U)$. We have that $y \in X \setminus (V \cap W)$ and thus, $y \in X \setminus U$. Therefore, $y \notin \operatorname{Cl}^*_X(U)$. That is, there exists a $*$-open set O in X with $y \in O$ and $O \cap U = \varnothing$. Since $\mathcal{I}$ is a τ-simple ideal, we have that $O = P \setminus J$, where $P \in \tau$ and $J \in \mathcal{I}$. Thus, P is an open set in X with $y \in P$. Since $y \in \operatorname{Cl}_X(V \cap W)$, we have that $P \cap (V \cap W) \neq \varnothing$ or equivalently, $(P \cap W) \cap V \neq \varnothing$. Since $(P \cap W) \cap V \subseteq P \cap W$, we have that $P \cap W \neq \varnothing$. Also, the relation $O \cap U = \varnothing$ implies the relation $(P \setminus J) \cap (W \setminus I) = \varnothing$ and thus, $P \cap W \subseteq I \cup J$. Since $\mathcal{I}$ is an ideal and $I, J \in \mathcal{I}$, we have that $I \cup J \in \mathcal{I}$. Thus, the member $I \cup J$ of the ideal $\mathcal{I}$ has non empty interior with respect to the topology τ, which is a contradiction as $\mathcal{I}$ is τ-codense. Thus, $\operatorname{Bd}_X(V \cap W) \subseteq \operatorname{Bd}^*_X(U)$.

By the Subspace Theorem for the dimension ind^* (see Theorem 2.3), we have that

$$\operatorname{ind}^*(\operatorname{Bd}_X(V \cap W)) \leqslant \operatorname{ind}^*(\operatorname{Bd}^*_X(U)) \leqslant k - 1$$

and by inductive hypothesis, we have that

$$\operatorname{ind}(\operatorname{Bd}_X(V \cap W)) \leqslant \operatorname{ind}^*(\operatorname{Bd}_X(V \cap W)).$$

Therefore, $\operatorname{ind}(\operatorname{Bd}_X(V \cap W)) \leqslant k - 1$. Thus, $\operatorname{ind}(X) \leqslant k$. □

Proposition 2.12. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If $\mathcal{I}$ is τ-simple and τ-codense and for each closed subset F of X the ideal $\mathcal{I}_F$ is τ_F-codense, then*

$$\operatorname{Ind}(X) \leqslant \operatorname{Ind}^*(X).$$

Proof. It is similar to the proof of Proposition 2.11. □

Proposition 2.13. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and U be a $*$-open set in X. Then*

$$\mathcal{I}_{\operatorname{Bd}^*_X(U)} \subseteq (\tau^*_{\operatorname{Bd}^*_X(U)})^c.$$

Proof. Let $A \in \mathcal{I}_{\mathrm{Bd}^*_X(U)}$. Then there exists $I \in \mathcal{I}$ such that $A = \mathrm{Bd}^*_X(U) \cap I$. Since I is closed in the space (X, τ^*) [15], that is $*$-closed in X, A is $*$-closed in $\mathrm{Bd}^*_X(U)$, proving the relation of the proposition. □

3. The ideal small inductive dimension $\mathcal{I}$-ind and the ideal large inductive dimension $\mathcal{I}$-Ind for ideal topological spaces

In this section, the notions of the ideal small inductive dimension, $\mathcal{I}$-ind, and the ideal large inductive dimension, $\mathcal{I}$-Ind, of an ideal topological space $(X, \tau, \mathcal{I})$, are defined, combining the topologies τ and τ^*, and basic properties of these dimensions are investigated.

Definition 3.1. The *ideal small inductive dimension* of an ideal topological space $(X, \tau, \mathcal{I})$, denoted by $\mathcal{I}\text{-}\mathrm{ind}(X)$, is defined as follows:

(i) $\mathcal{I}\text{-}\mathrm{ind}(X) = -1$, if $X = \varnothing$.

(ii) $\mathcal{I}\text{-}\mathrm{ind}(X) \leqslant k$, where $k \in \{0, 1, \ldots\}$, if for every element $x \in X$ and for every open subset V of X with $x \in V$, there exists a $*$-open subset U of X with

$$x \in U \subseteq V \text{ and } \mathcal{I}_{\mathrm{Bd}^*_X(U)}\text{-ind}(\mathrm{Bd}^*_X(U)) \leqslant k - 1.$$

(iii) $\mathcal{I}\text{-}\mathrm{ind}(X) = k$, where $k \in \{0, 1, \ldots\}$, if $\mathcal{I}\text{-}\mathrm{ind}(X) \leqslant k$ and $\mathcal{I}\text{-}\mathrm{ind}(X) \nleqslant k - 1$.

(iv) $\mathcal{I}\text{-}\mathrm{ind}(X) = \infty$, if there does not exist any $k \in \{-1, 0, 1, 2, \ldots\}$ for which $\mathcal{I}\text{-}\mathrm{ind}(X) \leqslant k$ is true.

It is observed that the ideal small inductive dimension $\mathcal{I}$-ind is different from the dimensions ind and ind* and the following examples prove this assertion.

Example 3.2.

(1) Let $k \geqslant 1$. We consider the set $X_k = \{0, 1, 2, \ldots, k\}$ and the topology generated by the family $\{\varnothing, \{0\}, \{0, 1\}, \ldots, \{0, 1, \ldots, k\}\}$. Then $\mathrm{ind}(X_k) = k$. If we consider the ideal $\mathcal{I}$, consisting of all subsets of X, then τ^* is the discrete topology and therefore, $\mathcal{I}\text{-}\mathrm{ind}(X) = 0$.

(2) We consider the indiscrete space $(\mathbb{R}, \tau)$ and the ideal $\mathcal{I} = \{I \subseteq \mathbb{R} : 0 \notin I\}$. Then $\tau^* = \{\varnothing\} \cup \{U \subseteq \mathbb{R} : 0 \in U\}$, $\mathrm{ind}^*(\mathbb{R}) = 1$ and $\mathcal{I}\text{-}\mathrm{ind}(\mathbb{R}) = 0$.

Lemma 3.3. *Let X be a non-empty set, A, B subsets of X such that $A \subseteq B$ and $\mathcal{I}$ an ideal on X. Then $(\mathcal{I}_B)_A = \mathcal{I}_A$.*

Theorem 3.4. *If $(X, \tau, \mathcal{I})$ is an ideal topological space and A is a subset of X, then*

$$\mathcal{I}_A\text{-ind}(A) \leqslant \mathcal{I}\text{-}\mathrm{ind}(X).$$

Proof. Obviously, if $\mathcal{I}\text{-}\mathrm{ind}(X) = -1$ or $\mathcal{I}\text{-}\mathrm{ind}(X) = \infty$, then the inequality holds. We suppose that the inequality holds for all integers $m < k$ and we shall prove it for k. Let $\mathcal{I}\text{-}\mathrm{ind}(X) = k$, $x \in A$ and V_A be an open set in A with $x \in V_A$. Then there exists an open set V in X such that $V_A = V \cap A$. Clearly, $x \in V$. Since $\mathcal{I}\text{-}\mathrm{ind}(X) = k$, there exists a $*$-open set U in X with $x \in U \subseteq V$ and $\mathcal{I}_{\mathrm{Bd}^*_X(U)}\text{-ind}(\mathrm{Bd}^*_X(U)) \leqslant k - 1$.

We consider the $*$-open set $U_A = U \cap A$ in A. Then $x \in U_A \subseteq V_A$. We shall prove that

$$(\mathcal{I}_A)_{\mathrm{Bd}^*_A(U_A)}\text{-ind}(\mathrm{Bd}^*_A(U_A)) \leqslant k-1.$$

By Lemma 3.3, it suffices to prove that

$$\mathcal{I}_{\mathrm{Bd}^*_A(U_A)}\text{-ind}(\mathrm{Bd}^*_A(U_A)) \leqslant k-1.$$

Since

$$\mathrm{Bd}^*_A(U_A) = \mathrm{Bd}^*_A(U \cap A) \subseteq A \cap \mathrm{Bd}^*_X(U) \subseteq \mathrm{Bd}^*_X(U),$$

$\mathrm{Bd}^*_A(U_A)$ is a subset of $\mathrm{Bd}^*_X(U)$ and by inductive hypothesis, we have that

$$(\mathcal{I}_{\mathrm{Bd}^*_X(U)})_{\mathrm{Bd}^*_A(U_A)}\text{-ind}(\mathrm{Bd}^*_A(U_A)) \leqslant k-1.$$

Moreover, applying Lemma 3.3 we have that

$$(\mathcal{I}_{\mathrm{Bd}^*_X(U)})_{\mathrm{Bd}^*_A(U_A)} = \mathcal{I}_{\mathrm{Bd}^*_A(U_A)}.$$

Thus, $\mathcal{I}_{\mathrm{Bd}^*_A(U_A)}$-ind$(\mathrm{Bd}^*_A(U_A)) \leqslant k-1$ and so $\mathcal{I}_A$-ind$(A) \leqslant k$. □

Proposition 3.5. *For any ideal topological space* $(X, \tau, \mathcal{I})$ *we have that*

$$\mathcal{I}\text{-ind}(X) \leqslant \min\{\mathrm{ind}(X), \mathrm{ind}^*(X)\}.$$

Proof. Firstly, we shall prove that $\mathcal{I}$-ind$(X) \leqslant$ ind(X). Clearly, if ind$(X) = -1$ or ind$(X) = \infty$, then the inequality of the proposition holds. We suppose that the inequality holds for all integers $m < k$ and we shall prove it for k. Let ind$(X) = k$, $x \in X$ and V be an open set in X with $x \in V$. Since ind$(X) = k$, there exists an open set U in X with $x \in U \subseteq V$ and ind$(\mathrm{Bd}_X(U)) \leqslant k-1$. Then U is a $*$-open set and by inductive hypothesis, we have that

$$\mathcal{I}_{\mathrm{Bd}_X(U)}\text{-ind}(\mathrm{Bd}_X(U)) \leqslant k-1.$$

Since $\mathrm{Bd}^*_X(U) \subseteq \mathrm{Bd}_X(U)$, by Theorem 3.4 we have

$$(\mathcal{I}_{\mathrm{Bd}_X(U)})_{\mathrm{Bd}^*_X(U)}\text{-ind}(\mathrm{Bd}^*_X(U)) \leqslant k-1.$$

By Lemma 3.3 we have that

$$\mathcal{I}_{\mathrm{Bd}^*_X(U)}\text{-ind}(\mathrm{Bd}^*_X(U)) \leqslant k-1.$$

and therefore, $\mathcal{I}$-ind$(X) \leqslant k$.

We shall prove that $\mathcal{I}$-ind$(X) \leqslant$ ind$^*(X)$. Clearly, if ind$^*(X) = -1$ or ind$^*(X) = \infty$, then the inequality of the proposition holds. We suppose that the inequality holds for all integers $m < k$ and we shall prove it for k. Let ind$^*(X) = k$, $x \in X$ and V be an open set in X with $x \in V$. Then V is also a $*$-open set in X. Since ind$^*(X) = k$, there exists a $*$-open set U in X with $x \in U \subseteq V$ and ind$^*(\mathrm{Bd}^*_X(U)) \leqslant k-1$. Then by inductive hypothesis we have that

$$\mathcal{I}_{\mathrm{Bd}^*_X(U)}\text{-ind}(\mathrm{Bd}^*_X(U)) \leqslant k-1$$

and thus, $\mathcal{I}$-ind$(X) \leqslant k$. □

Proposition 3.6. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $k \in \{0, 1, \dots\}$. If there exists a base B of (X, τ^*) such that $\mathcal{I}_{\mathrm{Bd}^*_X(U)}$-$\mathrm{ind}(\mathrm{Bd}^*_X(U)) \leqslant k-1$, for every $U \in B$, then $\mathcal{I}$-$\mathrm{ind}(X) \leqslant k$.*

In what follows, we show characterizations for the ideal small inductive dimension, assuming that (X, τ) is a regular space. We state firstly that a subset L of the space (X, τ^*) is called a *$*$-partition* between two subsets A and B of X if there exist $*$-open sets U, V with $A \subseteq U$, $B \subseteq V$, $U \cap V = \varnothing$ and $X \setminus L = U \cup V$.

Proposition 3.7. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space, where (X, τ) is a regular space. Then $\mathcal{I}$-$\mathrm{ind}(X) \leqslant k$, where $k \in \{0, 1, \dots\}$, if and only if for every point $x \in X$ and for every closed subset F of X with $x \notin F$, there exists a $*$-partition L between $\{x\}$ and F with $\mathcal{I}_L$-$\mathrm{ind}(L) \leqslant k-1$.*

Proof. Let $k \in \{0, 1, \dots\}$. Firstly, we suppose that $\mathcal{I}$-$\mathrm{ind}(X) \leqslant k$. Let $x \in X$ and F be a closed subset of X with $x \notin F$. Then we consider the set $V = X \setminus F$. Then V is an open set in X with $x \in V$. Since (X, τ) is a regular space, there exists an open set V_1 (and thus a $*$-open set) in X such that

$$x \in V_1 \subseteq \mathrm{Cl}_X(V_1) \subseteq V.$$

Since $\mathcal{I}$-$\mathrm{ind}(X) \leqslant k$ and $x \in V_1$, there exists a $*$-open set U in X with $x \in U \subseteq V_1$ and $\mathcal{I}_{\mathrm{Bd}^*_X(U)}$-$\mathrm{ind}(\mathrm{Bd}^*_X(U)) \leqslant k-1$. Then

$$x \in \mathrm{Cl}^*_X(U) \subseteq \mathrm{Cl}^*_X(V_1) \subseteq \mathrm{Cl}_X(V_1) \subseteq V$$

and the set $L = \mathrm{Bd}^*_X(U)$ is a $*$-partition between the sets $\{x\}$ and F such that $\mathcal{I}_L$-$\mathrm{ind}(L) \leqslant k-1$.

Conversely, we shall prove that $\mathcal{I}$-$\mathrm{ind}(X) \leqslant k$. Let $x \in X$ and V_x be an open set in X with $x \in V_x$. We set $F = X \setminus V_x$. Then F is a closed subset of X with $x \notin F$. By assumption there exists a $*$-partition L between $\{x\}$ and F with $\mathcal{I}_L$-$\mathrm{ind}(L) \leqslant k-1$. That is, there exist $*$-open sets U and V in X with $x \in U$, $F \subseteq V$, $U \cap V = \varnothing$ and $X \setminus L = U \cup V$. Since $\mathrm{Bd}^*_X(U) \subseteq L$, by Theorem 3.4, we have that

$$(\mathcal{I}_L)_{\mathrm{Bd}^*_X(U)}\text{-}\mathrm{ind}(\mathrm{Bd}^*_X(U)) \leqslant k-1$$

and by Lemma 3.3,

$$\mathcal{I}_{\mathrm{Bd}^*_X(U)}\text{-}\mathrm{ind}(\mathrm{Bd}^*_X(U)) \leqslant k-1.$$

Therefore, $\mathcal{I}$-$\mathrm{ind}(X) \leqslant k$. □

Corollary 3.8. *A non-empty ideal topological space X is ideal zero dimensional (with respect to the ideal small inductive dimension), where (X, τ) is a regular space, if and only if for every point $x \in X$ and each closed subset F of X such that $x \notin F$ the empty set is a $*$-partition between $\{x\}$ and F.*

Proof. It follows directly by Proposition 3.7. □

Proposition 3.9. *If $(X, \tau, \mathcal{I})$ is an ideal topological space, where (X, τ) is a regular space, and $k \in \{0, 1, \dots\}$, then the following statements are equivalent:*

(1) $\mathcal{I}\text{-ind}(X) \leqslant k$,
(2) *for every open neighborhood V of a point x in X, there exists a $*$-open set U and a $*$-closed set E in X such that*
$$x \in U \subseteq E \subseteq V \text{ and } \mathcal{I}_{E\setminus U}\text{-ind}(E \setminus U) \leqslant k-1,$$
(3) *for every open neighborhood V of a point x in X, there exists a $*$-open set U in X such that*
$$x \in U \subseteq \mathrm{Cl}^*_X(U) \subseteq V \text{ and } \mathcal{I}_{\mathrm{Bd}^*_X(U)}\text{-ind}(\mathrm{Bd}^*_X(U)) \leqslant k-1.$$

Proof. We prove the implication (1) $\Rightarrow$ (2). Let $\mathcal{I}\text{-ind}(X) \leqslant k$ and V be an open neighborhood of a point x in X. Then by Proposition 3.7, there exists a $*$-partition L between $\{x\}$ and $X \setminus V$ with $\mathcal{I}_L\text{-ind}(L) \leqslant k-1$. We consider disjoint $*$-open sets U and G in X with $x \in U$, $X \setminus V \subseteq G$ and $X \setminus L = U \cup G$. We set $E = X \setminus G$. Then E is a $*$-closed set in X with $x \in U \subseteq E \subseteq V$ and $E \setminus U = L$, completing the proof of this assertion.

Next, we prove the implication (2) $\Rightarrow$ (3). Let V be an open neighborhood of a point x in X. By (2), there exists a $*$-open set U and a $*$-closed set E in X with $x \in U \subseteq E \subseteq V$ and $\mathcal{I}_{E\setminus U}\text{-ind}(E \setminus U) \leqslant k-1$. We observe that $\mathrm{Cl}^*_X(U) \subseteq E$ and $\mathrm{Bd}^*_X(U) \subseteq E \setminus U$. Thus, by Theorem 3.4, we have that
$$(\mathcal{I}_{E\setminus U})_{\mathrm{Bd}^*_X(U)}\text{-ind}(\mathrm{Bd}^*_X(U)) \leqslant k-1$$
and by Lemma 3.3 we have that
$$\mathcal{I}_{\mathrm{Bd}^*_X(U)}\text{-ind}(\mathrm{Bd}^*_X(U)) \leqslant k-1.$$

Finally, we prove the implication (3) $\Rightarrow$ (1). Let F be a closed set in X and $x \in X$ with $x \notin F$. We put $V = X \setminus F$. Then V is an open neighborhood of x. By (3), there exists a $*$-open set U in X with $x \in U \subseteq \mathrm{Cl}^*_X(U) \subseteq V$ and $\mathcal{I}_{\mathrm{Bd}^*_X(U)}\text{-ind}(\mathrm{Bd}^*_X(U)) \leqslant k-1$. Then U and $X \setminus \mathrm{Cl}^*_X(U)$ are disjoint $*$-open sets in X, containing x and F, respectively, and
$$X \setminus (U \cup (X \setminus \mathrm{Cl}^*_X(U))) = (X \setminus U) \cap \mathrm{Cl}^*_X(U) = \mathrm{Bd}^*_X(U).$$
Thus, $\mathrm{Bd}^*_X(U)$ is a $*$-partition between $\{x\}$ and F and therefore, by Proposition 3.7, we have that $\mathcal{I}\text{-ind}(X) \leqslant k$. □

In the next results we study the ideal small inductive dimension of the topological sum.

Lemma 3.10. *Let $\{X_s : s \in S\}$ be a set of pairwise disjoint topological spaces and $\bigoplus_{s\in S} X_s$ be their topological sum. Let also for each $s \in S$, $\mathcal{I}_s$ be an ideal of X_s. Then*
$$\mathcal{I} = \{\bigcup E_s : E_s \in \mathcal{I}_s \text{ for each } s \in S\}$$
is an ideal of $\bigoplus_{s\in S} X_s$.

Theorem 3.11. *Let $\{X_s : s \in S\}$ be a set of pairwise disjoint topological spaces and $\bigoplus_{s\in S} X_s$ be their topological sum. Let also for each $s \in S$, $\mathcal{I}_s$ be an ideal of X_s. Then $\mathcal{I}\text{-ind}(\bigoplus_{s\in S} X_s) \leqslant k$, where $k \in \{0, 1, \dots\}$ and $\mathcal{I}$ is the ideal of Lemma 3.10, if and only if $\mathcal{I}_s\text{-ind}(X_s) \leqslant k$ for each $s \in S$.*

Proof. Let $k \in \{0, 1, \ldots\}$. We suppose that $\mathcal{I}\text{-}\operatorname{ind}(\bigoplus_{s\in S} X_s) \leqslant k$ and we shall prove that $\mathcal{I}_s\text{-ind}(X_s) \leqslant k$ for each $s \in S$. Let $s \in S$. In the proof we shall write $Y = \bigoplus_{s\in S} X_s$. Let $x \in X_s$ and V be an open subset of X_s with $x \in V$. Then V is an open subset of Y. Since $\mathcal{I}\text{-}\operatorname{ind}(Y) \leqslant k$, there exists a $*$-open subset U of Y such that $x \in U \subseteq V$ and

$$\mathcal{I}_{\mathrm{Bd}^*_Y(U)}\text{-ind}(\mathrm{Bd}^*_Y(U)) \leqslant k-1.$$

We observe that $\mathcal{I}_{X_s} = \mathcal{I}_s$ and the set $U \cap X_s$ is a $*$-open subset of X_s with $x \in U \cap X_s \subseteq U \subseteq V$. It suffices to prove that

$$(\mathcal{I}_{X_s})_{\mathrm{Bd}^*_{X_s}(U\cap X_s)}\text{-ind}(\mathrm{Bd}^*_{X_s}(U \cap X_s)) \leqslant k-1$$

or by Lemma 3.3,

$$\mathcal{I}_{\mathrm{Bd}^*_{X_s}(U\cap X_s)}\text{-ind}(\mathrm{Bd}^*_{X_s}(U \cap X_s)) \leqslant k-1.$$

We have that $\mathrm{Bd}^*_{X_s}(U \cap X_s) \subseteq \mathrm{Bd}^*_Y(U)$. Therefore, by Theorem 3.4 we have that

$$(\mathcal{I}_{\mathrm{Bd}^*_Y(U)})_{\mathrm{Bd}^*_{X_s}(U\cap X_s)}\text{-ind}(\mathrm{Bd}^*_{X_s}(U \cap X_s)) \leqslant \mathcal{I}_{\mathrm{Bd}^*_Y(U)}\text{-ind}(\mathrm{Bd}^*_Y(U))$$

and by Lemma 3.3 we have that

$$\mathcal{I}_{\mathrm{Bd}^*_{X_s}(U\cap X_s)}\text{-ind}(\mathrm{Bd}^*_{X_s}(U \cap X_s)) \leqslant \mathcal{I}_{\mathrm{Bd}^*_Y(U)}\text{-ind}(\mathrm{Bd}^*_Y(U))$$

and therefore,

$$\mathcal{I}_{\mathrm{Bd}^*_{X_s}(U\cap X_s)}\text{-ind}(\mathrm{Bd}^*_{X_s}(U \cap X_s)) \leqslant k-1.$$

Conversely, we suppose that $\mathcal{I}_s\text{-ind}(X_s) \leqslant k$ for each $s \in S$ and we shall prove that $\mathcal{I}\text{-}\operatorname{Ind}(Y) \leqslant k$. Let $x \in Y$ and V be an open subset of Y with $x \in V$. Then there exists $s \in S$ such that $x \in X_s$ and the set $V \cap X_s$ is an open subset of X_s with $x \in V \cap X_s$. Since $\mathcal{I}_s\text{-ind}(X_s) \leqslant k$, there exists a $*$-open subset U_s of X_s such that $x \in U_s \subseteq V \cap X_s \subseteq V$ and

$$(\mathcal{I}_s)_{\mathrm{Bd}^*_{X_s}(U_s)}\text{-ind}(\mathrm{Bd}^*_{X_s}(U_s)) \leqslant k-1$$

or equivalently,

$$(\mathcal{I}_{X_s})_{\mathrm{Bd}^*_{X_s}(U_s)}\text{-ind}(\mathrm{Bd}^*_{X_s}(U_s)) \leqslant k-1$$

and by Lemma 3.3,

$$\mathcal{I}_{\mathrm{Bd}^*_{X_s}(U_s)}\text{-ind}(\mathrm{Bd}^*_{X_s}(U_s)) \leqslant k-1.$$

The set U_s is $*$-open subset of Y and it suffices to prove that

$$\mathcal{I}_{\mathrm{Bd}^*_Y(U_s)}\text{-ind}(\mathrm{Bd}^*_Y(U_s)) \leqslant k-1.$$

We have that $\mathrm{Bd}^*_{X_s}(U_s) = \mathrm{Bd}^*_Y(U_s)$ and thus,

$$\mathcal{I}_{\mathrm{Bd}^*_Y(U_s)}\text{-ind}(\mathrm{Bd}^*_Y(U_s)) \leqslant k-1.$$

□

In what follows, we shall define and study the ideal large inductive dimension $\mathcal{I}\text{-}\operatorname{Ind}$ of an ideal topological space $(X, \tau, \mathcal{I})$.

Definition 3.12. The *ideal large inductive dimension* of an ideal topological space $(X, \tau, \mathcal{I})$, denoted by $\mathcal{I}\text{-}\mathrm{Ind}(X)$, is defined as follows:

(i) $\mathcal{I}\text{-}\mathrm{Ind}(X) = -1$, if $X = \varnothing$.

(ii) $\mathcal{I}\text{-}\mathrm{Ind}(X) \leqslant k$, where $k \in \{0, 1, \ldots\}$, if for every pair (F, V) of subsets of X, where F is closed, V is open and $F \subseteq V$, there exists a $*$-open subset U of X such that

$$F \subseteq U \subseteq V \text{ and } \mathcal{I}_{\mathrm{Bd}^*_X(U)}\text{-}\mathrm{Ind}(\mathrm{Bd}^*_X(U)) \leqslant k - 1.$$

(iii) $\mathcal{I}\text{-}\mathrm{Ind}(X) = k$, where $k \in \{0, 1, \ldots\}$, if $\mathcal{I}\text{-}\mathrm{Ind}(X) \leqslant k$ and $\mathcal{I}\text{-}\mathrm{Ind}(X) \nleqslant k - 1$.

(iv) $\mathcal{I}\text{-}\mathrm{Ind}(X) = \infty$, if there does not exist any $k \in \{-1, 0, 1, 2, \ldots\}$ for which $\mathcal{I}\text{-}\mathrm{Ind}(X) \leqslant k$ is true.

We observe that the ideal large inductive dimension is different from the dimensions Ind and Ind^* and the following examples prove this assertion.

Example 3.13.

(1) We consider the space $X = \{a, b, c, d\}$ with the topology

$$\tau = \{\varnothing, \{a\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, X\}$$

and the ideal $\mathcal{I} = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$. Then $\mathrm{Ind}(X) = 1$ and $\mathcal{I}\text{-}\mathrm{Ind}(X) = 0$.

(2) We consider the space $X = \{a, b, c, d, e\}$ with the topology

$$\tau = \{\varnothing, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, b, c, e\}, X\}$$

and the ideal $\mathcal{I} = \{\varnothing, \{c\}, \{e\}, \{c, e\}\}$. Then $\mathrm{Ind}^*(X) = 2$ and $\mathcal{I}\text{-}\mathrm{Ind}(X) = 0$.

Theorem 3.14. *If $(X, \tau, \mathcal{I})$ is an ideal topological space and A is a closed subset of X, then*

$$\mathcal{I}_A\text{-}\mathrm{Ind}(A) \leqslant \mathcal{I}\text{-}\mathrm{Ind}(X).$$

Proof. Obviously, if $\mathcal{I}\text{-}\mathrm{Ind}(X) = -1$ or $\mathcal{I}\text{-}\mathrm{Ind}(X) = \infty$, then the inequality holds. We suppose that the inequality holds for all integers $m < k$ and we shall prove it for k. Let $\mathcal{I}\text{-}\mathrm{Ind}(X) = k$, F_A be a closed subset of A and V_A be an open subset of A such that $F_A \subseteq V_A$. Since A is a closed subset of X, F_A is also a closed subset of X. Also, there exists an open set V in X such that $V_A = V \cap A$. Since $\mathcal{I}\text{-}\mathrm{Ind}(X) = k$, there exists a $*$-open set U in X with $F_A \subseteq U \subseteq V$ and $\mathcal{I}_{\mathrm{Bd}^*_X(U)}\text{-}\mathrm{Ind}(\mathrm{Bd}^*_X(U)) \leqslant k - 1$.

We consider the $*$-open set $U_A = U \cap A$ in A. Then $F_A \subseteq U \cap A \subseteq V \cap A$ or equivalently, $F_A \subseteq U_A \subseteq V_A$. We shall prove that

$$(\mathcal{I}_A)_{\mathrm{Bd}^*_A(U_A)}\text{-}\mathrm{Ind}(\mathrm{Bd}^*_A(U_A)) \leqslant k - 1.$$

By Lemma 3.3 it suffices to prove that

$$\mathcal{I}_{\mathrm{Bd}^*_A(U_A)}\text{-}\mathrm{Ind}(\mathrm{Bd}^*_A(U_A)) \leqslant k - 1.$$

Since $\mathrm{Bd}^*_A(U_A)$ is a closed subset of $\mathrm{Bd}^*_X(U)$, by inductive hypothesis we have that

$$(\mathcal{I}_{\mathrm{Bd}^*_X(U)})_{\mathrm{Bd}^*_A(U_A)}\text{-}\mathrm{Ind}(\mathrm{Bd}^*_A(U_A)) \leqslant k-1.$$

Moreover, applying Lemma 3.3 we have that

$$(\mathcal{I}_{\mathrm{Bd}^*_X(U)})_{\mathrm{Bd}^*_A(U_A)} = \mathcal{I}_{\mathrm{Bd}^*_A(U_A)}.$$

Thus, $\mathcal{I}_{\mathrm{Bd}^*_A(U_A)}$-Ind$(\mathrm{Bd}^*_A(U_A)) \leqslant k-1$ and so $\mathcal{I}_A$-Ind$(A) \leqslant k$. □

The assumption that A is a closed subset of X cannot be dropped and the following example justifies this.

Example 3.15. Let $X = \{a,b,c,d,e,f\}$ with the topology

$$\tau = \{\varnothing, \{f\}, \{a,f\}, \{b,f\}, \{a,b,f\}, \{a,b,c,f\}, \{a,b,d,f\}, \{a,b,c,d,f\}, X\}.$$

If we consider the ideal $\mathcal{I} = \{\varnothing, \{e\}\}$, then $\mathcal{I}$-Ind$(X) = 0$. However, if we consider the set $A = \{a,b,c,d\}$, which is not closed in X, then $\mathcal{I}_A$-Ind$(A) = 1$.

Proposition 3.16. *For any ideal topological space $(X,\tau,\mathcal{I})$ we have that*

$$\mathcal{I}\text{-}\mathrm{Ind}(X) \leqslant \min\{\mathrm{Ind}(X), \mathrm{Ind}^*(X)\}.$$

Proof. It is similar to Proposition 3.5. □

The following results characterize the ideal large inductive dimension in various classes of spaces.

Proposition 3.17. *An ideal topological space $(X,\tau,\mathcal{I})$, where (X,τ) is a normal space, satisfies the inequality $\mathcal{I}$-Ind$(X) \leqslant k$, where $k \in \{0,1,\ldots\}$, if and only if for every pair A, B of disjoint closed subsets of X, there exists a $*$-partition L between A and B such that $\mathcal{I}_L$-Ind$(L) \leqslant k-1$.*

Proof. It is similar to the proof of Proposition 3.7. □

Corollary 3.18. *A non-empty ideal topological space X is ideal zero dimensional (with respect to the ideal large inductive dimension), where (X,τ) is a normal space, if and only if for every disjoint closed subsets A and B of X, the empty set is a $*$-partition between A and B.*

Proof. It follows by Proposition 3.17. □

Proposition 3.19. *If $(X,\tau,\mathcal{I})$ is an ideal topological space, where (X,τ) is a normal space, and $k \in \{0,1,\ldots\}$, then the following statements are equivalent:*

(1) $\mathcal{I}$-Ind$(X) \leqslant k$,

(2) *for every open neighborhood V of a closed set E of X, there exists a $*$-open set U and a $*$-closed set F in X such that*

$$E \subseteq U \subseteq F \subseteq V \quad \textit{and} \quad \mathcal{I}_{F\setminus U}\text{-}\mathrm{Ind}(F \setminus U) \leqslant k-1,$$

(3) *for every open neighborhood V of a closed set E of X, there exists a $*$-open set U in X with*

$$E \subseteq U \subseteq \mathrm{Cl}^*_X(U) \subseteq V \quad \textit{and} \quad \mathcal{I}_{\mathrm{Bd}^*_X(U)}\text{-}\mathrm{Ind}(\mathrm{Bd}^*_X(U)) \leqslant k-1.$$

Proof. It is similar to the proof of Proposition 3.9. □

The following result presents the behavior of the ideal large inductive dimension of the topological sum.

Theorem 3.20. *Let $\{X_s : s \in S\}$ be a set of pairwise disjoint topological spaces and $\bigoplus_{s\in S} X_s$ be their topological sum. Let also for each $s \in S$, $\mathcal{I}_s$ be an ideal of X_s. Then $\mathcal{I}$-$\mathrm{Ind}(\bigoplus_{s\in S} X_s) \leqslant k$, where $\mathcal{I}$ is the ideal of Lemma 3.10, if and only if $\mathcal{I}_s$-$\mathrm{Ind}(X_s) \leqslant k$ for each $s \in S$.*

Proof. It is similar to the proof of Theorem 3.11. □

4. Additional Results on Dimensions for Ideal Topological Spaces

In what follows, properties of the ideal small and ideal large inductive dimension, using different ideals on the underlying sets, are studied. Moreover, relations between the dimensions $\mathcal{I}$-ind and $\mathcal{I}$-Ind and the ideal covering dimension $\mathcal{I}$-dim are investigated.

Proposition 4.1. *Let (X, τ) be a topological space and $\mathcal{I}_1, \mathcal{I}_2$ be two ideals on X.*

(1) *If $\mathcal{I}_1 \subseteq \mathcal{I}_2$, then $\mathcal{I}_2$-$\mathrm{Ind}(X) \leqslant \mathcal{I}_1$-$\mathrm{Ind}(X)$.*
(2) *If $\mathcal{I}_1 \subseteq \mathcal{I}_2$, then $\mathcal{I}_2$-$\mathrm{ind}(X) \leqslant \mathcal{I}_1$-$\mathrm{ind}(X)$.*

Proof. For the simplicity of the writing, we denote by $*_1$ and $*_2$ the $*$-open set referred to the ideal topological spaces $(X, \tau, \mathcal{I}_1)$ and $(X, \tau, \mathcal{I}_2)$, respectively.

(1) We suppose that the inequality holds for all integers $m < k$ and we shall prove it for k. Let $\mathcal{I}_1$-$\mathrm{Ind}(X) = k$. We shall prove that $\mathcal{I}_2$-$\mathrm{Ind}(X) \leqslant k$. Let F and V be a closed and an open set in X, respectively, such that $F \subseteq V$. Since $\mathcal{I}_1$-$\mathrm{Ind}(X) = k$, there exists a $*_1$-open set U in X such that $F \subseteq U \subseteq V$ and

$$(\mathcal{I}_1)_{\mathrm{Bd}_X^{*_1}(U)}\text{-}\mathrm{Ind}(\mathrm{Bd}_X^{*_1}(U)) \leqslant k-1.$$

Since $\mathcal{I}_1 \subseteq \mathcal{I}_2$, the set U is also a $*_2$-open set with $\mathrm{Bd}_X^{*_2}(U) \subseteq \mathrm{Bd}_X^{*_1}(U)$ and thus, applying Lemma 3.3 and Theorem 3.4, we have that

$$(\mathcal{I}_1)_{\mathrm{Bd}_X^{*_2}(U)}\text{-}\mathrm{Ind}(\mathrm{Bd}_X^{*_2}(U)) \leqslant k-1.$$

Also, $(\mathcal{I}_1)_{\mathrm{Bd}_X^{*_2}(U)} \subseteq (\mathcal{I}_2)_{\mathrm{Bd}_X^{*_2}(U)}$. Therefore, by inductive hypothesis, we have that $(\mathcal{I}_2)_{\mathrm{Bd}_X^{*_2}(U)}$-$\mathrm{Ind}(\mathrm{Bd}_X^{*_2}(U)) \leqslant k-1$ and thus, $\mathcal{I}_2$-$\mathrm{Ind}(X) \leqslant k$.

(2) It is similar to (1). □

Corollary 4.2. *Let (X, τ) be a topological space and $\mathcal{I}_1, \mathcal{I}_2$ be two ideals on X. Then*

(1) $\max\{\mathcal{I}_1\text{-}\mathrm{Ind}(X), \mathcal{I}_2\text{-}\mathrm{Ind}(X)\} \leqslant \mathcal{I}_1 \cap \mathcal{I}_2\text{-}\mathrm{Ind}(X)$.
(2) $\max\{\mathcal{I}_1\text{-}\mathrm{ind}(X), \mathcal{I}_2\text{-}\mathrm{ind}(X)\} \leqslant \mathcal{I}_1 \cap \mathcal{I}_2\text{-}\mathrm{ind}(X)$.

Corollary 4.3. *Let (X, τ) be a topological space, A, B subsets of X and $\mathcal{I}_1 = P(A)$, $\mathcal{I}_2 = P(B)$ and $\mathcal{I}_3 = P(A \cup B)$ be three ideals on X. Then*

(1) $\mathcal{I}_3$-$\mathrm{Ind}(X) \leqslant \min\{\mathcal{I}_1\text{-}\mathrm{Ind}(X),\ \mathcal{I}_2\text{-}\mathrm{Ind}(X)\}$.

(2) $\mathcal{I}_3$-ind$(X) \leqslant \min\{\mathcal{I}_1\text{-ind}(X), \mathcal{I}_2\text{-ind}(X)\}$.

Proposition 4.4. *For any ideal topological space* $(X, \tau, \mathcal{I})$*, where* (X, τ) *is a* T_1*-space, we have that*

$$\mathcal{I}\text{-}\mathrm{ind}(X) \leqslant \mathcal{I}\text{-}\mathrm{Ind}(X).$$

Proof. Let $(X, \tau, \mathcal{I})$ be an ideal topological space for which (X, τ) is a T_1-space. We suppose that the inequality holds for all integers $m < k$ and we shall prove it for k. Let $\mathcal{I}\text{-}\mathrm{Ind}(X) = k$. We shall prove that $\mathcal{I}\text{-}\mathrm{ind}(X) \leqslant k$. Let $x \in X$ and V be an open subset of X with $x \in V$. Then the set $\{x\}$ is a closed subset of X. Since $\mathcal{I}\text{-}\mathrm{Ind}(X) \leqslant k$, for the pair $(\{x\}, V)$ there exists a $*$-open subset U of X with $x \in U \subseteq V$ and

$$\mathcal{I}_{\mathrm{Bd}^*_X(U)}\text{-Ind}(\mathrm{Bd}^*_X(U)) \leqslant k-1.$$

Therefore, by inductive hypothesis, we have that $\mathcal{I}_{\mathrm{Bd}^*_X(U)}$-ind$(\mathrm{Bd}^*_X(U)) \leqslant k-1$ and thus, $\mathcal{I}\text{-}\mathrm{ind}(X) \leqslant k$. □

Example 4.5. We consider the space $X = \{a, b, c, d, e\}$ with the topology

$$\tau = \{\varnothing, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$$

and the ideal $\mathcal{I} = \{\varnothing, \{d\}\}$. Then we observe that the space X is not T_1, $\mathcal{I}\text{-}\mathrm{ind}(X) = 2$ and $\mathcal{I}\text{-}\mathrm{Ind}(X) = 0$.

Especially, for the ideal small inductive dimension of X, we have the following statements:

- for the element $x = a$ and every open subset V of X with $a \in V$, there exists the $*$-open subset $U = \{a\}$ of X such that $a \in U \subseteq V$ and $\mathcal{I}_{\mathrm{Bd}^*_X(\{a\})}$-ind$(\mathrm{Bd}^*_X(\{a\})) = 1$,
- for the element $x = b$ and every open subset V of X with $b \in V$, there exists the $*$-open subset $U = \{b\}$ of X such that $a \in U \subseteq V$ and $\mathcal{I}_{\mathrm{Bd}^*_X(\{b\})}$-ind$(\mathrm{Bd}^*_X(\{b\})) = 1$,
- for the element $x = c$ and every open subset V of X with $c \in V$, there exists the $*$-open subset $U = \{a, b, c\}$ of X such that $c \in U \subseteq V$ and $\mathcal{I}_{\mathrm{Bd}^*_X(\{a,b,c\})}$-ind$(\mathrm{Bd}^*_X(\{a, b, c\})) = 0$,
- for the element $x = d$ and every open subset V of X with $d \in V$, there exists the $*$-open subset $U = X$ of X such that $d \in U \subseteq V$ and $\mathcal{I}_{\mathrm{Bd}^*_X(X)}$-ind$(\mathrm{Bd}^*_X(X)) = -1$ and
- for the element $x = e$ and every open subset V of X with $e \in V$, there exists the $*$-open subset $U = X$ of X such that $e \in U \subseteq V$ and $\mathcal{I}_{\mathrm{Bd}^*_X(X)}$-ind$(\mathrm{Bd}^*_X(X)) = -1$.

Therefore, $\mathcal{I}\text{-}\mathrm{ind}(X) = 2$.

Moreover, for the ideal large inductive dimension of X we observe that for every pair (F, V) of subsets of X, where F is closed, V is open and $F \subseteq V$, there exists the $*$-open subset $U = X$ of X such that $F \subseteq U \subseteq V$ and $\mathcal{I}_{\mathrm{Bd}^*_X(U)}$-Ind$(\mathrm{Bd}^*_X(U)) = -1$. Therefore, $\mathcal{I}\text{-}\mathrm{Ind}(X) = 0$.

For the rest of the paper, we remind that for a topological space (X, τ) a non-empty family c of open subsets of X is called an *open cover* if the union of all elements of c is X. A family r of subsets of X is said to be a *refinement* of a family c of subsets of X if each element of r is contained in an element of c.

Especially, in what follows, for an ideal topological space $(X, \tau, \mathcal{I})$ a non empty family c of open sets (respectively, of $*$-open sets) will be said a τ-*cover* (respectively, a τ^*-*cover*) of X if the union of all elements of c is X.

The *order* of a family r of subsets of a topological space X is defined as follows:

(1) $\operatorname{ord}(r) = -1$, if r consists of the empty set only.
(2) $\operatorname{ord}(r) = k$, where $k \in \{0, 1, \ldots\}$, if the intersection of any $k+2$ distinct elements of r is empty and there exist $k+1$ distinct elements of r whose intersection is not empty.
(3) $\operatorname{ord}(r) = \infty$ if for every $k \in \{1, 2, \ldots\}$ there exist k distinct elements of r whose intersection is not empty.

Definition 4.6 ([17])**.** The ideal covering dimension, denoted by $\mathcal{I}\text{-}\dim$, is defined as follows:

(i) $\mathcal{I}\text{-}\dim(X) = -1$ if and only if $X = \varnothing$.
(ii) $\mathcal{I}\text{-}\dim(X) \leqslant k$, where $k \in \{0, 1, \ldots\}$, if for every finite τ-cover c of X there exists a finite τ^*-cover r of X, which is a refinement of c with $\operatorname{ord}(r) \leqslant k$.
(iii) $\mathcal{I}\text{-}\dim(X) = k$, where $k \in \{0, 1, \ldots\}$, if $\mathcal{I}\text{-}\dim(X) \leqslant k$ and $\mathcal{I}\text{-}\dim(X) \nleqslant k-1$.
(iv) $\mathcal{I}\text{-}\dim(X) = \infty$, if there does not exist any $k \in \{-1, 0, 1, 2, \ldots\}$ for which $\mathcal{I}\text{-}\dim(X) \leqslant k$ is true.

Proposition 4.7. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space, where (X, τ) is a normal space. If $\mathcal{I}\text{-}\dim(X) = 0$, then $\mathcal{I}\text{-}\operatorname{Ind}(X) = 0$.*

Proof. Let $\mathcal{I}\text{-}\dim(X) = 0$ and E, F disjoint closed subsets of X. Then the family

$$c = \{X \setminus E, X \setminus F\}$$

is a τ-cover of X. Since $\mathcal{I}\text{-}\dim(X) = 0$, there exists a finite τ^*-cover r of X, which refines c and has order less than or equal to 0. Thus, every point of X is contained in a member of r, a member of r is disjoint from at least one of E and F, and any two members of r are disjoint. Hence,

$$U = \bigcup\{G \in r : G \cap E \neq \varnothing\} \text{ and } V = \bigcup\{G \in r : G \cap E = \varnothing\}$$

are disjoint $*$-open sets of X with $E \subseteq U$, $F \subseteq V$ and $X = U \cup V$. Thus, $\varnothing$ is a $*$-partition between E and F with $\mathcal{I}_{\varnothing}\text{-}\operatorname{Ind}(\varnothing) = -1$. Therefore, by Corollary 3.18, $\mathcal{I}\text{-}\operatorname{Ind}(X) = 0$. ☐

Corollary 4.8. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space, where (X, τ) is a normal space. If $\mathcal{I}\text{-}\dim(X) = 0$, then $\mathcal{I}\text{-}\operatorname{ind}(X) = 0$.*

Proof. Since $\mathcal{I}\text{-}\dim(X) = 0$, by Proposition 4.7 we have that $\mathcal{I}\text{-}\mathrm{Ind}(X) = 0$ and thus, by Proposition 4.4, we have the desired result. □

Example 4.9. We consider the space $X = \{a, b, c\}$ with the topology

$$\tau = \{\varnothing, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$$

and the ideal $\mathcal{I} = \{\varnothing, \{b\}\}$. Then $\mathcal{I}\text{-}\mathrm{ind}(X) = \mathcal{I}\text{-}\mathrm{Ind}(X) = \mathcal{I}\text{-}\dim(X) = 0$.

In general, we observe that the ideal topological dimensions of the types ind, Ind and dim are different and the following examples prove this claim.

Example 4.10.

(1) We consider the space $X = \{a, b, c, d, e\}$ with the topology

$$\tau = \{\varnothing, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}, X\}$$

and the ideal $\mathcal{I} = \{\varnothing, \{c\}\}$. Then $\mathcal{I}\text{-}\mathrm{ind}(X) = 2$ and $\mathcal{I}\text{-}\dim(X) = 0$.

(2) We consider the space $X = \{a, b, c, d\}$ with the topology generated by the family

$$\beta = \{\varnothing, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}\}$$

and the ideal $\mathcal{I} = \{\varnothing, \{b\}\}$. Then $\mathcal{I}\text{-}\mathrm{Ind}(X) = 1$ and $\mathcal{I}\text{-}\dim(X) = 2$.

However, under some conditions for topological spaces and ideals we can obtain the following relation between $\mathcal{I}\text{-}\mathrm{ind}$, $\mathcal{I}\text{-}\mathrm{Ind}$ and $\mathcal{I}\text{-}\dim$.

Proposition 4.11. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If $\mathcal{I} \subseteq \tau^c$ and (X, τ) is a separable metric space, then*

$$\mathcal{I}\text{-}\mathrm{ind}(X) = \mathcal{I}\text{-}\mathrm{Ind}(X) = \mathcal{I}\text{-}\dim(X).$$

Proof. By Remark 2.7, Proposition 2.9 and [17], we have that $\mathrm{ind}(X) = \mathrm{ind}^*(X)$, $\mathrm{Ind}(X) = \mathrm{Ind}^*(X)$ and $\dim(X) = \dim^*(X)$. Also, based on the definitions of the ideal topological spaces and the fact that $\tau = \tau^*$, we have that

$$\mathcal{I}\text{-}\mathrm{ind}(X) = \mathrm{ind}(X),\ \mathcal{I}\text{-}\mathrm{Ind}(X) = \mathrm{Ind}(X) \ \text{ and } \ \mathcal{I}\text{-}\dim(X) = \dim(X).$$

Finally, the desired relation follows from the fact that $\mathrm{ind}(X) = \mathrm{Ind}(X) = \dim(X)$, whenever (X, τ) is a separable metric space. □

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