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On a probabilistic version of Meir-Keeler type fixed point theorem for a family of discontinuous operators

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Abstract

A Meir-Keeler type fixed point theorem for a family of mappings is proved in Menger probabilistic metric space (Menger PM-space). We establish that completeness of the space is equivalent to fixed point property for a larger class of mappings that includes continuous as well as discontinuous mappings. In addition to it, a probabilistic fixed point theorem satisfying $(\epsilon - \delta)$ type non-expansive mappings is established.

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1. Introduction and preliminaries

The idea of statistical metric space or probabilistic Menger space can be traced back to Menger [10], who extended the concept of metric space (X, d), by replacing the notion of distance $d(x, y)$ $(x, y \in X)$ by a distributive function $F_{x,y} : X \times X \to \mathbb{R}$, where $F_{x,y}(t)$ represents the probability that the distance between x and y is less than t. Schweizer and Sklar [22, 23] studied various properties, e.g., topology, convergence of sequences, continuity of mappings,

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completeness, etc., of these spaces. In 1972, Sehgal and Bharucha–Reid [24] showed the role of distributive functions in metric fixed point theory and established the probabilistic metric version of the classical Banach contraction mapping principle. Since then the study of fixed point theorems in PM-space has emerged as an active area of research.

Let g be a selfmapping which satisfy some contractive condition on a complete Menger PM-space $(X, \mathcal{F}, T)$. Then there exists a Cauchy sequence of successive iterates $\{g^n x\}_{n\in\mathbb{N}}$ for each x in X which converges to some point, say $z \in X$, and the limiting point z of the sequence of iterates is nothing but a fixed point of g. However, there exist various contractive definitions which ensure the existence of the Cauchy sequence of iterates converging to some limit point, but the limit point may not be a fixed point.

Pant et al. [17] (see also Bisht [2]) proved the following theorem where the Meir-Keeler [9] type operator ensures the convergence of sequence of iterates but does not ensure the existence of a fixed point.

Lemma 1.1. *Let $(X, \mathcal{F}, T)$ be a complete Menger PM-space, and let f be self-mapping of X satisfying one of the following conditions*

(i) for every $\epsilon \in (0,1)$ there exists $\delta \in (0, \epsilon]$ such that

$$\epsilon - \delta < \min\left\{F_{x,gx}(t), F_{y,gy}(t)\right\} < \epsilon \;\Rightarrow\; F_{gx,gy}(t) \geq \epsilon,$$

(ii) $F_{gx,gy}(t) > \min\left\{F_{x,gx}(t), F_{y,gy}(t)\right\}$,

or

(i') for every $\epsilon \in (0,1)$ there exists $\delta \in (0, \epsilon]$ such that

$$\epsilon - \delta \leq \min\left\{F_{x,gx}(t), F_{y,gy}(t)\right\} < \epsilon \;\Rightarrow\; F_{gx,gy}(t) > \epsilon,$$

for all $x, y \in X$. Then for any x in X the sequence of iterates $\{g^n x\}_{n\in\mathbb{N}}$ is a Cauchy sequence and there exists a point z in X such that $\lim\limits_{n\to\infty} g^n x = z$ for each x in X.

The triple $(X, \mathcal{F}, T_{min})$ is a complete Menger PM-space, for $X \subseteq \mathbb{R}$ (see Remark 2.3). The following example illustrates Lemma 1.1, but does not possess a fixed point.

Example 1.2. Let $X = [1,2] \cup \left\{1 - \dfrac{1}{3^n} : n = 0, 1, 2, \cdots\right\}$ and d be the usual metric. Define $g : X \to X$ by

$$gx = \begin{cases} 0 & \text{if } 1 \leq x \leq 2. \\ 1 - \dfrac{1}{3^{n+1}} & \text{if } x = 1 - \dfrac{1}{3^n}, \qquad n = 1, 2, \cdots. \end{cases}$$

Then

$$g(X) = \left\{1 - \frac{1}{3^n} : n = 1, 2, \cdots\right\}$$

and g is fixed point free. The mapping g satisfies the contractive condition (i') of Lemma 1.1 with

$$\delta(\varepsilon) = \begin{cases} \frac{1}{3^n} - \varepsilon & if \quad \frac{1}{3^{n+1}} \leq \varepsilon < \frac{1}{3^n}, n = 1, 2, \cdots \\ \varepsilon & if \quad \varepsilon \geq 1. \end{cases}$$

Therefore, to ensure the existence of a fixed point under such contractive definitions, one needs to assume some additional hypotheses on the mappings.

Ćirić [5] introduced the notion of orbital continuity. If g is a self-mapping of a metric space (X, d) then the set $O_g(x) = \{g^n x \mid n = 0, 1, 2, \ldots\}$ is called the orbit of g at x and g is called orbitally continuous if $u = \lim_i g^{m_i} x$ implies $gu = \lim_i gg^{m_i} x$. Every continuous self-mapping is orbitally continuous but not conversely. In 1977, Jaggi [7] introduced the concept of x_0-orbital continuity which is weaker than orbital continuity of the mapping. A self-mapping g of a metric space (X, d) is called x_0-orbitally continuous for some $x_0 \in X$ if its restriction to the set $\overline{O(g, x_0)}$, is continuous, i.e., $g : \overline{O(g, x_0)} \to X$, is continuous, here $\overline{O(g, x)}$ represents closure of the orbit of g at x_0. The mapping g is said to be orbitally continuous if it is x_0-orbitally continuous for all $x_0 \in X$. In 2011, Jungck [8] gave a generalized notion of orbital continuity, namely, almost orbital continuity. A self-mapping g of a metric space (X, d) is called almost orbitally continuous at $x_0 \in X$ if whenever $\lim_n g^{i_n}(x) = x_0$ for some $x \in X$ and subsequence $\{g^{i_n}(x)\}$ of $g^n(x)$, there exists a subsequence $\{g^{j_n}(x)\}$ of $g^n(x)$ such that $\lim_n g^{j_n}(x) = g(x_0)$. Orbital continuity implies almost orbital continuity, but the implication is not reversible. In 2017, Pant and Pant [11] introduced the notion of $k-$continuity. A self-mapping g of a metric space X is called k-continuous, $k = 1, 2, 3, \ldots$, if $g^k x_n \to gt$, whenever $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in X such that $g^{k-1}x_n \to t$. It may be observed that 1-continuity is equivalent to continuity and continuity implies 2-continuity, 2-continuity implies 3-continuity and so on but not conversely. It is important to note that $k-$continuity of the mapping implies orbital continuity but not conversely. More recently, Pant et al. [12] introduced the notion of weak orbital continuity, which is weaker than orbital continuity of the mapping. A self-mapping g of a metric space (X, d) is called weakly orbitally continuous [12] if the set $\{y \in X : \lim_i g^{m_i} y = u \text{ implies } \lim_i gg^{m_i} y = gu\}$ is nonempty, whenever the set $\{x \in X : \lim_i g^{m_i} x = u\}$ is nonempty.

Example 1.3. Let $X = [0, 2]$ and d be the usual metric. Define $g : X \to X$ by

$$gx = \frac{(1 + x)}{2} \quad \text{if } \ 0 \leq x < 1, \quad gx = 0 \quad \text{if } \ 1 \leq x < 2, \quad g2 = 2.$$

Then [12]:

(i) g is not orbitally continuous. Since $g^n 0 \to 1$ and $g(g^n 0) \to 1 \neq g1$.

(ii) g is weakly orbitally continuous. If we take $x = 2$ then $g^n 2 \to 2$ and $g(g^n 2) \to 2 = g2$.

(iii) g is not $k-$continuous. If we consider the sequence $\{g^n 0\}$, then for any positive integer k, we have $g^{k-1}(g^n 0) \to 1$ and $g^k(g^n 0) \to 1 \neq g1$.

Example 1.4. Let $X = [0, +\infty)$ and d be the usual metric. Define $g : X \to X$ by

$$gx = 1 \text{ if } 0 \leq x \leq 1, \quad gx = \frac{x}{5} \text{ if } x > 1.$$

Then g is orbitally continuous. Let $k \geq 1$ be any integer. Consider the sequence $\{x_n\}$ given by $x_n = 5^{k-1} + \frac{1}{n}$. Then $g^{k-1}x_n = 1 + \frac{1}{n5^{k-1}}, g^k x_n = \frac{1}{5} + \frac{1}{n5^k}$. This implies $g^{k-1}x_n \to 1, g^k x_n \to \frac{1}{5} \neq g1$ as $n \to +\infty$. Hence g is not $k-$continuous.

The above examples show that orbital continuity implies weak orbital continuity but the converse need not be true. Also, every k-continuous mapping is orbitally continuous, but the converse is not true.

The question of continuity of contractive definitions at their fixed point in metric space was studied by Rhoades [20] (see also, Hicks and Rhoades [6]). All the contractive definitions studied by them forced the mappings to be continuous at the fixed point. Rhoades [20] also listed the question of the existence of a contractive condition that intromits discontinuity at the fixed point as an open problem. Pant [15] gave the first affirmative answer to this problem in the setting of metric space. Various other distinct answers to this problem and their possible applications to neural networks having discontinuities in activation functions can be found in Bisht and Pant [1], Bisht and Rakočević [3], Pant and Pant [11], Pant et al. [12, 16, 17, 18], Taş and Özgür [27].

Bisht and Rakočević [4] presented some new solutions to Rhoades' open problem on the existence of contractive mappings that admit discontinuity at the fixed point. This was done via new fixed point theorems for a generalized class of Meir-Keeler type mappings which were proved by the authors. Rhoade's question was related, in part, to the important problem of characterizing metric completeness in terms of fixed point results; in this direction solutions to that problem were deduced. In 2020. Romaguera [21] introduced and studied the notion of w-distance for fuzzy metric spaces and he obtained a characterization of complete fuzzy metric spaces via a suitable fixed point theorem.

In this paper, we prove a Meir-Keeler type fixed point theorem for a family of mappings in Menger PM- space. A probabilistic fixed point theorem satisfying $(\epsilon - \delta)$ type non-expansive mappings is also established. We assume the notions of weak continuity which may imply discontinuity at the fixed point but characterize completeness of the space.

2. Preliminaries

We start with some standard definitions and notations of a probabilistic metric space.

Let D^+ be the set of all distribution functions $F : \mathbb{R} \to [0, 1]$ such that F is a non-decreasing, left-continuous mapping satisfying $F(0) = 0$ and $\sup_{x \in \mathbb{R}} F(x) =$

1. The space D^+ is partially ordered by the usual point-wise ordering of functions, i.e., $F \le G$ if and only if $F(t) \le G(t)$ for all $t \in \mathbb{R}$. The maximal element for D^+ in this order is the distribution function given by

$$\varepsilon_0(t) = \begin{cases} 0, & t \le 0, \\ 1, & t > 0. \end{cases}$$

Definition 2.1 ([23])**.** A binary operation $T : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if T satisfies the following conditions:

(a) T is commutative and associative;
(b) T is continuous;
(c) $T(a,1) = a$ for all $a \in [0,1]$;
(d) $T(a,b) \le T(c,d)$, whenever $a \le c$ and $b \le d$, and $a,b,c,d \in [0,1]$.

Some of the simple examples of t-norm are $T(a,b) = \max\{a+b-1, 0\}$, $T(a,b) = \min\{a,b\}$, $T(a,b) = ab$ and

$$T(a,b) = \begin{cases} \frac{ab}{a+b-ab}, & ab \ne 0, \\ 0, & ab = 0. \end{cases}$$

The t-norms are defined recursively by $T^1 = T$ and

$$T^n(x_1, \dots, x_{n+1}) = T(T^{n-1}(x_1, \dots, x_n), x_{n+1}),$$

for $n \ge 2$ and $x_i \in [0,1]$ for all $i \in \{1, \dots, n+1\}$.

Definition 2.2. *A Menger probabilistic metric space (briefly, Menger PM-space) is a triple $(X, \mathcal{F}, T)$ where X is a non-void set, T is a continuous t-norm, and $\mathcal{F}$ is a mapping from $X \times X$ into D^+ such that, if $F_{x,y}$ denotes the value of $\mathcal{F}$ at the pair (x,y), then the following conditions hold:*

(PM1) $F_{x,y}(t) = \varepsilon_0(t)$ *if and only if* $x = y$;
(PM2) $F_{x,y}(t) = F_{y,x}(t)$;
(PM3) $F_{x,z}(t+s) \ge T(F_{x,y}(t), F_{y,z}(s))$ *for all* $x, y, z \in X$ *and* $s, t \ge 0$.

Remark 2.3 ([24]). Every metric space is a PM-space. Let (X, d) be a metric space and $T(a,b) = \min\{a,b\}$ is a continuous t-norm. Define $F_{x,y}(t) = \varepsilon_0(t - d(x,y))$ for all $x, y \in X$ and $t > 0$. The triple $(X, \mathcal{F}, T)$ is a PM-space induced by the metric d.

Definition 2.4. Let $(X, \mathcal{F}, T)$ be a Menger PM-space.

(1) A sequence $\{x_n\}_{n=1,2,\dots}$ in X is said to be convergent to x in X if, for every $\varepsilon > 0$ and $\lambda > 0$ there exists positive integer N such that $F_{x_n,x}(\varepsilon) > 1 - \lambda$ whenever $n \ge N$.
(2) A sequence $\{x_n\}_{n=1,2,\dots}$ in X is called Cauchy sequence if, for every $\varepsilon > 0$ and $\lambda > 0$ there exists positive integer N such that $F_{x_n,x_m}(\varepsilon) > 1 - \lambda$ whenever $n, m \ge N$.
(3) A Menger PM-space is said to be complete if every Cauchy sequence in X is convergent to a point in X.

The following lemma was given in [22, 23].

Lemma 2.5 ([23]). *Let $(X, \mathcal{F}, T)$ be a Menger PM-space. Then the function $\mathcal{F}$ is lower semi-continuous for every fixed $t > 0$, i.e., for every fixed $t > 0$ and every two convergent sequences $\{x_n\}, \{y_n\} \subseteq X$ such that $x_n \to x, y_n \to y$ it follows that*

$$\liminf_{n\to+\infty} F_{x_n,y_n}(t) = F_{x,y}(t).$$

3. Main results

3.1. **Fixed points of a family of Meir-Keeler type mappings in Menger PM-space.** The Meir-Keeler type contractive condition employed in the next theorem for a family of self-mappings ensures the convergence of sequence of iterates as well as the existence of fixed points under some weaker notion of continuity assumption.

Theorem 3.1. *Let $(X, \mathcal{F}, T)$ be a complete Menger PM-space, and let $\{f_j : 0 \leq j \leq 1\}$ be a family of self-mappings of X such that for any given f_j the following conditions are satisfied:*

(i) $F_{f_j x, f_j y}(t) \geq \min\{F_{x,f_j x}(t), F_{y,f_j y}(t)\}$ *for all $x, y \in X$;*

(ii) given $\epsilon \in (0, 1)$ there exists $\delta \in (0, \epsilon]$ such that $x, y \in X$ with

$$\epsilon - \delta \leq \min\{F_{x,f_j x}(t), F_{y,f_j y}(t)\} < \epsilon \text{ implies } F_{f_j x, f_j y}(t) > \epsilon.$$

If f_j is weakly orbitally continuous, then f_j has a unique fixed point, say z, and $\lim\limits_{n\to+\infty} f_j^n x_0 = z$ for each x in X. Moreover, if every pair of mappings (f_r, f_s) satisfies the condition

(iii) $F_{f_j x, f_s y}(t) \geq \min\{F_{x,f_j x}(t), F_{y,f_s y}(t)\}$;

then the mappings $\{f_j\}$ have a unique common fixed point which is also the unique fixed point of each f_r.

Proof. Consider any mapping f_j. By virtue of (ii), it is obvious that f_j satisfies the following condition:

$$F_{f_j x, f_j y}(t) > \min\{F_{x,f_j x}(t), F_{y,f_j y}(t)\}. \tag{3.1}$$

Let x_0 be any point in X. Define a sequence $\{x_n\}$ in X recursively by $x_n = f_j x_{n-1}, n = 1, 2, \ldots$. If $x_p = x_{p+1}$ for some $p \in \mathbb{N}$, then x_p is a fixed point of f_j. Suppose $x_n \neq x_{n+1}$ for all $n \geq 0$. Then using (3.1) we have

$$\begin{aligned} F_{x_n,x_{n+1}}(t) &= F_{f_j x_{n-1}, f_j x_n}(t) > \min\{F_{x_{n-1},f_j x_{n-1}}(t), F_{x_n,f_j x_n}(t)\} \\ &= \min\{F_{x_{n-1},x_n}(t), F_{x_n,x_{n+1}}(t)\} \\ &= F_{x_{n-1},x_n}(t). \end{aligned}$$

Thus $\{F_{x_n,x_{n+1}}(t)\}$ is a strictly increasing sequence of positive real numbers in $[0, 1]$ and, hence, tends to a limit $r \leq 1$. Suppose $r < 1$. Then there exists a positive integer N with $n \geq N$ such that

$$r - \delta(r) < F_{x_n,x_{n+1}}(t) < r. \tag{3.2}$$

This further implies

$$r - \delta(r) < \min\left\{F_{x_n,x_{n+1}}(t), F_{x_{n+1},x_{n+2}}(t)\right\} < r,$$

that is,

$$r - \delta(r) < \min\left\{F_{x_n,f_jx_n}(t), F_{x_{n+1},f_jx_{n+1}}(t)\right\} < r.$$

By virtue of (ii), this yields $F_{f_jx_n,f_jx_{n+1}}(t) = F_{x_{n+1},x_{n+2}}(t) > r$. This contradicts (3.2). Hence $\liminf_{n\to+\infty} F_{x_n,x_{n+1}}(t) = 1$. Further, if q is any positive integer then for each $t > 0$, we have

$$\begin{aligned} F_{x_n,x_{n+q}}(t) &= F_{f_jx_{n-1},f_jx_{n+q-1}}(t) > \\ &> \min\left\{F_{x_{n-1},f_jx_{n-1}}(t), F_{x_{n+q-1},f_jx_{n+q-1}}(t)\right\} \\ &= \min\left\{F_{x_{n-1},x_n}(t), F_{x_{n+q-1},x_{n+q}}(t)\right\}. \end{aligned}$$

Since $\liminf_{n\to+\infty} F_{x_n,x_{n+1}}(t) = 1$, making limit as $n \to +\infty$, the above inequality yields

$$\liminf_{n\to+\infty} F_{x_n,x_{n+q}}(t) = 1.$$

Therefore, $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists a point z in X such that $\lim_{n\to+\infty} x_n = \lim_{n\to+\infty} f_j^n x_0 = z$. Moreover, if y_0 is any other point in X and $y_n = f_j y_{n-1} = f_j^n y_0$, then (3.1) yields

$$\begin{aligned} F_{x_n,y_n}(t) &= F_{f_jx_{n-1},f_jy_{n-1}}(t) > \min\left\{F_{x_{n-1},f_jx_{n-1}}(t), F_{y_{n-1},fy_{n-1}}(t)\right\} \\ &= \min\left\{F_{x_{n-1},x_n}(t), F_{y_{n-1},y_n}(t)\right\}. \end{aligned}$$

Letting $n \to +\infty$, we get $\liminf_{n\to+\infty} F_{z,y_n}(t) = 1$ for each $t > 0$. Therefore, $\lim_{n\to+\infty} y_n = \lim_{n\to+\infty} f_j^n y_0 = z$. Suppose that f_j is weakly orbitally continuous. Since $f_j^n x_0 \to z$ for each x_0, by virtue of weak orbital continuity of f_j we get, $f_j^n y_0 \to z$ and $f_j^{n+1} y_0 \to f_j z$ for some $y_0 \in X$. This implies that $z = f_j z$ since $f_j^{n+1} y_0 \to z$. Therefore z is a fixed point of f_j. Uniqueness of the fixed point follows from (i). Moreover, if v and w are the fixed points of f_j and f_s respectively, then by (iii) we have

$$F_{v,w}(t) = F_{f_jv,f_sw}(t) \geq \min\left\{F_{v,f_jv}(t), F_{w,f_sw}(t)\right\}.$$

In view of $\liminf_{n\to+\infty} F_{v,w}(t) = 1$ for each $t > 0$, we get $v = w$ and each mapping $\{f_j\}$ has a unique fixed point which is also the unique common fixed point of the family of mappings. □

The following result is an easy consequence of Theorem 3.1:

Corollary 3.2. *Let $(X, \mathcal{F}, T)$ be a complete Menger PM-space, and let $\{f_j : 0 \leq j \leq 1\}$ be a family of self-mappings of X such that for any given f_j satisfying conditions (i)-(ii) of Theorem 3.1. If f_j is either $k-$continuous or f_j^k is continuous for some positive integer $k \geq 1$ or f_j is orbitally continuous, then f_j has a unique fixed point Moreover, if every pair of mappings (f_r, f_s)*

satisfies the condition (iii) of Theorem 3.1, then the mappings $\{f_j\}$ *have a unique common fixed point which is also the unique fixed point of each* f_r.

The triple $(X, \mathcal{F}, T_{min})$ is a complete Menger PM-space, for $X \subseteq \mathbb{R}$ (see Remark 2.3). The following example illustrates Theorem 3.1.

Example 3.3. Let $X = [0, 2]$ and d be the usual metric. For each $0 \leq j \leq 1\}$, we define $f_j : X \mapsto X$ by

$$f_j x = \begin{cases} 1, & \text{if } 0 \leq x \leq 1, \\ j(x-1), & \text{if } 1 < x \leq 2. \end{cases}$$

Then the mappings f_j satisfy all the conditions of Theorem 3.1 and have a unique common fixed point $x = 1$ which is also the unique fixed point of each mapping. The mapping f_j is discontinuous at the fixed point. The mapping f_j satisfies condition *(ii)* with $\delta(\epsilon) = 1 - \epsilon$, if $\epsilon < 1$, and $\delta(\epsilon) = \epsilon$, for $\epsilon \geq 1$. It is also easy to see that the mapping f_j is orbitally continuous and, hence, weak orbitally continuous [14].

Taking $f_j = g$ in Theorm 3.1, we get the following result as a corollary which is a probabilistic version of Theorem 2.1 of Pant et al. [12]:

Theorem 3.4. *Let* $(X, \mathcal{F}, T)$ *be a complete Menger PM-space, and let* g *be a self-mapping of* X *such that* $F_{gx,gy}(t) \geq \min\{F_{x,gx}(t), F_{y,gy}(t)\}$ *for all* $x, y \in X$*;*

(iv) given $\epsilon \in (0, 1)$ *there exists* $\delta \in (0, \epsilon]$ *such that* $x, y \in X$ *with*

$$\epsilon - \delta \leq \min\{F_{x,gx}(t), F_{y,gy}(t)\} < \epsilon \text{ implies } F_{gx,gy}(t) > \epsilon.$$

Then

(a) g possesses a unique fixed point if and only if g is weakly orbitally continuous.

(b) g possesses a unique fixed point provided g is either orbitally continuous or $k-$*continuous or* g^k *is continuous for some positive integer* $k \geq 1$.

(c) g possesses a unique fixed point provided g is either x_0*-orbitally continuous or almost orbitally continuous.*

In the next result, we show that Theorem 3.4 characterizes metric completeness of X. Various workers have proved fixed point theorems that characterize metric completeness [4, 17, 19, 25, 26]. In the next theorem, we show that completeness of the space is equivalent to fixed point property for a large class of mappings including both continuous and discontinuous mappings. In what follows we use the notation $a \gg b$ (or $a \ll b$) to show that the positive number a is much greater (smaller) than the positive number b.

Theorem 3.5. *Let* $(X, \mathcal{F}, T)$ *be a Menger PM-space. If every* $k-$*continuous or almost orbitally continuous self-mapping of* X *satisfying the condition* (iv) *of Theorem 3.4 has a fixed point, then* X *is complete.*

Proof. Suppose that every $k-$continuous self-mapping of X satisfying condition (iv) of Theorem 3.4 possesses a fixed point. We will prove that X is complete. If possible, suppose X is not complete. Then there exists a Cauchy sequence in X, say $M = \{u_1, u_2, u_3, \ldots\}$, consisting of distinct points which does not converge. Let $x \in X$ be given. Then, since x is not a limit point of the Cauchy sequence M, there exists a least positive integer $N(x)$ such that $x \neq u_{N(x)}$ and for each $m \geq N(x)$ and $t > 0$ we have

$$1 - F_{x,u_{N(x)}}(t) \gg 1 - F_{u_{N(x)},u_m}(t). \tag{3.3}$$

Consider a mapping $g : X \mapsto X$ by $g(x) = u_{N(x)}$. Then, $g(x) \neq x$ for each x and, using (3.3), for any x, y in X and $t > 0$ we get

$$1 - F_{gx,gy}(t) = 1 - F_{u_{N(x)},u_{N(y)}}(t) \ll 1 - F_{x,u_{N(x)}}(t) = 1 - F_{x,gx}(t)$$

if $N(x) \leq N(y)$, or

$$1 - F_{gx,gy}(t) = 1 - F_{u_{N(x)},u_{N(y)}}(t) \ll 1 - F_{y,u_{N(y)}}(t) = 1 - F_{y,gy}(t)$$

if $N(x) > N(y)$.

This implies that

$$F_{gx,gy}(t) > \min\big\{F_{x,gx}(t), F_{y,gy}(t)\big\}. \tag{3.4}$$

In other words, given $\epsilon > 0$ we can select $\delta(\epsilon) = \epsilon$ such that

$$\epsilon - \delta \leq \min\big\{F_{x,gx}(t), F_{y,gy}(t)\big\} < \epsilon \text{ implies } F_{gx,gy}(t) > \epsilon. \tag{3.5}$$

It is clear from (3.4) and (3.5) that the mapping g satisfies condition (iv) of Theorem 3.4. Moreover, g is a fixed point free mapping whose range is contained in the non-convergent Cauchy sequence $M = \{u_n\}_{n\in\mathbb{N}}$. Hence, there exists no sequence $\{x_n\}_{n\in\mathbb{N}}$ in X for which $\{gx_n\}_{n\in\mathbb{N}}$ converges, that is, there exists no sequence $\{x_n\}_{n\in\mathbb{N}}$ in X for which the condition $gx_n \to t$ implies $g^2x_n \to gt$ is violated. Therefore, g is a 2-continuous mapping. In a similar manner it follows that g is almost orbitally continuous. Thus, we have a self-mapping g of X which satisfies condition (iv) of Theorems 3.4 but does not possess a fixed point. This contradicts the hypothesis of the theorem. Hence X is complete. □

We now give a weaker version of Theorem 3.4 which extends Theorem 3.2 of [17].

Theorem 3.6. *Let $(X, \mathcal{F}, T)$ be a complete Menger PM-space, and let g be a self-mapping of X such that*

(v) $F_{gx,gy}(t) > \min\big\{F_{x,gx}(t), F_{y,gy}(t)\big\}$ *for all $x, y \in X$;*
(vi) given $\epsilon \in (0, 1)$ there exists $\delta \in (0, \epsilon]$ such that $x, y \in X$ with

$$\epsilon - \delta < \min\big\{F_{x,gx}(t), F_{y,gy}(t)\big\} < \epsilon \textit{ implies } F_{gx,gy}(t) \geq \epsilon.$$

Then g possesses a unique fixed point if g is either weakly orbitally continuous or x_0-orbitally continuous or almost orbitally continuous.

Proof. The proof follows on the similar lines as the proof of Theorem 3.5. □

3.2. **Fixed points of a family of $(\epsilon - \delta)$ non-expansive mappings in Menger PM-space.** We now prove a fixed point theorem for a family of $(\epsilon - \delta)$ non-expansive mappings in Menger PM-space.

Theorem 3.7. *Let $(X, \mathcal{F}, T)$ be a complete Menger PM-space, and let $\{f_j : 0 \le j \le 1\}$ be a family of self-mappings of X such that for any given f_j the following conditions are satisfied:*

(i') $F_{f_jx,f_jy}(t) \ge \min\{F_{x,f_jx}(t), F_{y,f_jy}(t)\}$ *for all* $x, y \in X$;

(ii') *given* $\epsilon \in (0,1)$ *there exists* $\delta \in (0,\epsilon]$ *such that* $x, y \in X$ *with*

$$\epsilon - \delta < \min\{F_{x,f_jx}(t), F_{y,f_jy}(t)\} < \epsilon \text{ implies } F_{f_jx,f_jy}(t) \le \epsilon;$$

If f_j is continuous, then f_j has a unique fixed point, say z, Moreover, if every pair of mappings (f_r, f_s) satisfies the condition

(iii') $F_{f_jx,f_sy}(t) \ge \min\{F_{x,f_jx}(t), F_{y,f_sy}(t)\}$;

then the mappings $\{f_j\}$ have a unique common fixed point which is also the unique fixed point of each f_r.

Proof. Let x_0 be any point in X. Define a sequence $\{x_n\}$ in X recursively by $x_n = f_jx_{n-1}, n = 1, 2, \ldots$. Then following the lines of Theorem 3.1, it can be shown that $\{x_n\}$ is a Cauchy sequence. Continuity of f_j now implies that $f_jz = z$ and z is a fixed point of f_j. Rest of the proof follows from Theorem 3.1. □

Taking $f_j = g$ in Theorm 3.7, we get the following result as a corollary:

Theorem 3.8. *Let $(X, \mathcal{F}, T)$ be a complete Menger PM-space, and let g be a continuous self-mapping of X such that*

(iv') $F_{gx,gy}(t) \ge \min\{F_{x,gx}(t), F_{y,gy}(t)\}$ *for all* $x, y \in X$;

(v') *given* $\epsilon \in (0,1)$ *there exists* $\delta \in (0,\epsilon]$ *such that* $x, y \in X$ *with*

$$\epsilon - \delta < \min\{F_{x,gx}(t), F_{y,gy}(t)\} < \epsilon \text{ implies } F_{gx,gy}(t) \le \epsilon.$$

Then g possesses a unique fixed point.

The triple $(X, \mathcal{F}, T_{min})$ is a complete Menger PM-space, for $X \subseteq \mathbb{R}$ (Remark 2.3). The following example [13] illustrates Theorem 3.8.

Example 3.9. Let $X = [-1, 1]$ and d be the usual metric. Define $g : X \mapsto X$ by

$$gx = -|x|x, \text{ for each } x \in X.$$

Then the mapping g satisfies all the conditions of Theorem 3.8 and has a unique fixed point $x = 0$. Also, g possesses two periodic points $x = 1$ and $x = -1$. The mapping g satisfies condition (v') with $\delta(\epsilon) = (\sqrt{(\epsilon/2)}) - (\epsilon/2)$, if $\epsilon < 2$, and $\delta(2) = 2$.

Remark 3.10. It is pertinent to mention here that uniqueness of the fixed point in Theorem 3.8 is because of the particular form (iv'). If we change (iv') by the following

$$F_{gx,gy}(t) \ge \min\{F_{x,y}(t), F_{x,gx}(t), F_{y,gy}(t)\} \text{ for all } x, y \in X,$$

then the fixed point need not be unique.

Remark 3.11. Theorem 3.1 provides a new answer to the once open question (see Rhoades [20], p. 242) on the existence of contractive mappings which admit discontinuity at the fixed point in the setting of Menger PM-space.

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