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Alexandroff duplicate and $\beta\kappa$

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ABSTRACT

We discuss spaces and the Alexandroff duplicates of those spaces that admit a Č-S embedding into the Čech-Stone compactification of a discrete space.

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1. INTRODUCTION

The *Alexandroff duplicate* of a topological space X, denoted by $A(X)$, is the topological space defined as follows.

The underlying set consists of two disjoint copies of the set X, say $X \times \{0\}$ and $X \times \{1\}$; for the sake of some technical simplicity, elements and subsets of the first copy are going to be denoted the same as it was for the original set, whereas the elements and subsets of the second copy are going to be denoted by priming the corresponding symbols, i.e., x', Y', etc. If $Y \subseteq X$, $D(Y)$ stands for the *duplicate of* Y, i.e., for the set $D(Y) = Y \cup Y'$.

The topology of the space $A(X)$ is generated by the subsets of the form $D(U) - F' = U \cup (U - F)'$, where U is open in X and $F \subseteq X$ is finite, and by $\{x'\}$ where $x \in X$. Thus each point of X' is an isolated point of the space $A(X)$. The original space X is contained in $A(X)$ as its closed subset and also as its exactly two-to-one retract.

The concept itself originated with P.S. Alexandroff and P.S. Urysohn in 1929 [1] for $X =$ the unit circle; the generalization for arbitrary X, as defined above, is due to R. Engelking [12] - it has been extensively utilized and studied

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since (cf. [16], [7], [6], [14], [4], and [5]). Of the other possible generalizations, the most general generalization of Engelking's one was accomplished by R.P. Chandler et al. [8], but it didn't get as much traction as Engelking's.

If X is a *crowded* (i.e., without isolated points) space, then X' is a dense subset of isolated points and X is the remainder of $A(X)$. In this regard, the structure of $A(X)$ is akin to that of $\beta\kappa$, so the following question arises:

When is $A(X)$ embeddable into the Čech-Stone compactification of the discrete space X'?

Specifically, we say that $A(X)$ is *Č-S embeddable* if there exists an embedding $h : A(X) \to \beta(X')$ such that $h(x') = x'$ for each $x \in X$.

Suppose that $A(X)$ is Č-S embeddable. There are some obvious necessary conditions $A(X)$ has to satisfy. To list a few.

(0) X has to be a completely regular space;

(1) $A(X)$ has to be extremally disconnected (since $X' \subseteq A(X) \subseteq \beta(X')$;

(2) Any bounded real function on X' has to have a continuous extension to $A(X)$ (since it has a continuous extension to $\beta(X')$);

(3) Any two disjoint subsets of X cannot have a common accumulation point.

Finally, considering $A(X)$ as a subspace of $\beta(X')$, one can also observe the following phenomenon: Let $p \in X$ be a non-isolated point of X. Considering p as a free ultrafilter on X', if $A \subseteq X'$ is such that $A \in p$, then there is an open neighborhood U of p in X and a finite subset F of X such that $U - F \subseteq A$. It means that

(4) For each non-isolated point $p \in X$, the family

$$\{U - F : U \text{ is an open neighborhood of } p \text{ and } F \text{ is a finite subset of } X\}$$

is a base for the ultrafilter p.

The main goal of this note is to show that any of the properties (1) − (4) stated above constitutes also a sufficient (thus, equivalent) condition for any (completely) regular crowded space and its Alexandroff duplicate to be Č-S embedabble. Additional mutual relationships between those properties are discussed in Section 1 and Section 2.

Spaces satisfying condition (4), above, are defined and studied in Section 1 under the name *ultrafilter spaces*. They play a crucial role in establishing the aforementioned equivalences.

Spaces satisfying condition (3), above, are called *perfectly disconnected*. They were first defined and studied by E. van Douwen in [9].

The equivalence between perfectly disconnected space and extremal disconnectedness of Alexandroff duplicate, i.e., the equivalence of the conditions (1) and (3), above, was first established by P. Bhattacharjee, M. Knox, and W. McGovern, [3].

We refer to R. Engelking's book [13] for all undefined topological notions.

2. Topological Ultrafilter Spaces

All considered topological spaces are T_1 and let X be a topological space.

For $A \subseteq X$, $\partial(A)$ denotes the set of all *accumulation points* of the set A, i.e., $\partial(A) = \{x \in X : |U \cap A| \geq \omega$ for each open neighborhood U of $x\}$

An *ultrafilter* on a non-empty set X is a family ξ of non-empty subsets of X closed under finite intersections and maximal with respect to that property, i.e., if $X = A \cup B$, then $A \in \xi$ or $B \in \xi$.

We say that the space X is an *ultrafilter space at p* if the family

$\xi_p = \{A \subseteq X : U - F \subseteq A$, where $p \in U$ is open and $F \subseteq X$ is finite$\}$

is an ultrafilter on the set X.

Proposition 2.1. *A space X is an ultrafilter space at p if and only if the following condition holds true:*
(∂) *If Y and Z are disjoint subsets of X, then $p \notin \partial(Y) \cap \partial(Z)$.*

Proof. ($\Rightarrow$) Let X be an ultrafilter space at p and let Y and Z be disjoint subsets of X. We may assume that they cover X. There exists an open neighborhood U of p and a finite subset F of X such that $U - F \subseteq Y$ or $U - F \subseteq Z$; say the former holds true. Hence $U \cap Z \subseteq F$, which means that $p \notin \theta(Z)$.

($\Leftarrow$) Suppose to the contrary that X is not an ultrafilter space at p. Thus $X = A \cup B$, where $A \cap B = \varnothing$, and for each open neighborhood U of p and a finite subset of X, $U - F \subsetneq A$ and $U - F \subsetneq B$. Thus both sets $(U - F) \cap A$ and $(U - F) \cap B$ are infinite, i.e., $p \in \partial(A) \cap \partial(B)$; a contradiction. □

We say that the space X is an *ultrafilter space* if X is an ultrafilter space at each point $p \in X$. Thus.

Proposition 2.2. *A space X is an ultrafilter space if and only if the following condition holds true*
($\partial\partial$) *If Y and Z are disjoint subsets of X, then $\partial(Y) \cap \partial(Z) = \varnothing$.*

Condition ($\partial\partial$) implies:

Corollary 2.3. *Let X be an ultrafilter space. Then*
(a) *X is hereditarily extremally disconnected and nodec (= each nowhere dense subset of X is closed and, consequently, discrete).*
(b) *(see also van Douwen [9]) $\partial(E) = cl(intE)$ for arbitrary $E \subseteq X$.*

Proof. Part (a) follows immediately from ($\partial\partial$). Part (b) needs some additional argument.

Inclusion $\partial(E) \supseteq cl(intE)$ is obvious. To show the converse inclusion, let $x \notin cl(intE)$ and let U be an open neighborhood of x disjoint from $intE$. It means that $U \cap E$ is boundary which, in turn, implies that $U \cap E$ is nowhere dense. Hence, by (a), x is not an accumulation point of E. □

Topological spaces satisfying condition ($\partial\partial$) were introduced by E. van Douwen under the name *perfectly disconnected space*. He also gave the following characterization of such spaces. For the sake of completeness, we provide (a different) proof.

Theorem 2.4 (van Douwen [9])**.** *X is a perfectly disconnected space if and only if X is extremally disconnected and each dense subset of X is open, i.e., X is submaximal.*

Proof. If X is perfectly disconnected, then, X is (hereditarily) extremally disconnected. It is submaximal as well for if D is dense in X, then $X - D$ cannot have any accumulation point. Thus $X - D$ is closed and discrete and so D is open.

Let X be extremally disconnected and submaximal and assume, to the contrary, that X does not satisfy the condition $(\partial\partial)$. So let Y, Z be disjoint subsets of X such that there is a $p \in \partial(Y) \cap \partial(Z)$. Hence $p \in clY \cap clZ$. Set $U = int(cl(Y))$ and $V = int(cl(Z))$. It follows from submaximality that $U \cap V = \varnothing$ for no non-empty open subset of X can contain two disjoint dense subsets. The space X, being submaximal, is also nodec. Hence $Y - U$ and $Z - V$ are two disjoint closed discrete subsets of X. Hence

$cl(Y) \cap cl(Z) = [clU \cap (Z - V)] \cup [clV \cap cl(Y - U)]$. So if $p \in clY \cap clZ$, then $p \in clU \cap (Z - V)$ or $p \in clV \cap cl(Y - U)$; assume the former holds (similar argument applies when the latter holds true). There exists an open neighborhood W of p such that $W \cap (Z - V) = \{p\}$ and $W \subseteq clU$. Hence $W \cap clZ = \{p\}$, thus $p \notin \partial(Z)$; a contradiction. □

It is well known that the two aforementioned properties, i.e., that of submaximality and extremal disconnectedness, characterize maximal crowded topological spaces (cf. E. Hewitt [15] and M. Katětov [17]). By the Kuratowski-Zorn Lemma, maximal crowded topology majorizes any given T_1 or T_2 crowded topology. However examples of maximal crowded topologies which are T_3 are unknown with the exception of van Douwen's example from 1993 which happens to be countable (see [9]). Uncountable examples can be obtained by taking a disjoint union of the van Douwen's example.

The next characterization of ultrafilter spaces entails a new setting.

Lemma 2.5. *A set W is an open set in the space $A(X)$ if and only if $W = [D(U) - F'] \cup E'$, where $U = W \cap X$, $F \subseteq U$ is discrete in U, and E is a subset of $X - U$.*

Proof. Let $W \subseteq A(X)$ be open. By setting $U = X \cap W$ we have: $W = (W \cap D(U)) \cup [W - D(U)]$. Set $E = W - D(U)$. Since $W \cap D(U) = U \cup [W \cap U']$, we need to show that $U' - W = F'$ for some discrete in U subset F of U. Indeed, if $x \in U = W \cap X$, then there exists an open neighborhood V of x and a finite subset S of X such that $D(V) - S' \subseteq W$. Hence $V \cap F \subseteq S$, which shows that F is discrete in U.

The converse is obvious. □

Theorem 2.6. *A space X is an ultrafilter space viz. perfectly disconnected space if and only if $A(X)$ is extremally disconnected.*

Proof. Necessity is obvious. To prove sufficiency, pick any open subset W of $A(X)$ and let us show that ClW is open in $A(X)$. By Lemma 2.5, $W = [D(U) - F'] \cup E'$, where U is an open subset of X, $F \subseteq U$ is discrete in U, and

E is a subset of $X - U$. Thus $ClW = Cl\left[D\left(U\right) - F'\right] \cup ClE'$. One can easily verify that $Cl\left[D\left(U\right) - F'\right] = cl\left(U\right) \cup U' - F'$ and that $ClE' = E' \cup \partial\left(E\right)$. The perfect disconnectedness of X yields that $clE \cap clU = \varnothing$ and that clU is clopen. Subsequently, $Cl\left[D\left(U\right) - F'\right] = D\left(clU\right) - \left(clU - U\right)' - F'$ is an open subset of $A\left(X\right)$. Using part (b) of Corollary 2.3, we get: $ClE' = E' \cup cl\left(intE\right) = \left[D\left(cl\left(intE\right)\right) - \left(cl\left(intE\right) - intE\right)'\right] \cup \left(E - cl\left(intE\right)\right)'$ is an open subset of $A\left(X\right)$ too. □

The characterization of the extremal disconnectedness of the Alexandroff duplicate in terms of perfect disconnectedness (= Theorem 2.6) was first established by P. Bhattacharjee, M. L. Knox, and W. McGovern, [3], in 2020. The aforementioned van Douwen's example of a regular countable perfectly disconnected space provides an affirmative answer to a problem posed by K. Almontashery and L. Kalantan, [2]

The following corollary recaps the main results of this section,

Corollary 2.7. *For arbitrary T_1 crowded space X, the following conditions are pairwise equivalent.*
(j) X is an ultrafilter space;
(jj) X is a perfectly disconnected space;
(jjj) X is a maximal crowded space;
(Ij) $A\left(X\right)$ is an extremally disconnected space.

Remark 2.8. The ultrafilters ξ_p induced by the topology on an ultrafilters space X may not be uniform, i.e., that any member of ξ_p has to be of cardinality $|X|$. However there exists an open subset G of X such that if G is considered as an ultrafilter space, then the ultrafilters ξ_p are going to be uniform. To see this, let's recall some (known) definitions.
If Z is an arbitrary topological space, $Y \subseteq Z$, and $p \in Z$ is its non-isolated point, then:
$\Delta\left(p, Z\right) = \min\left\{|U| : U \text{ is an open neighborhood of } p\right\} -$ the *dispersion character of Z at p*; and $\Delta\left(Z\right) = \min\left\{\Delta\left(p, Z\right) :\ p \in Z \text{ and } p \text{ is non-isolated}\right\} -$ the *dispersion character* of Z.
Thus any open subset G of X such that $|G| = \Delta\left(X\right)$ will yield an ultrafilter space with all the ultrafilters ξ_p to be uniform.

There may be a variety of maximal filters of open sets, The question arises whether all types of maximal filters of open sets on an ultrafilter space X are ultrafilters on the set X. It turns out, it depends on the separation axioms of X.

A space X is said to be a *strong ultrafilter space* if any maximal filter of open sets on the space X generates an ultrafilter on the set X.

Proposition 2.9. *If X is normal ultrafilter space, then X is a strong ultrafilter space.*

Proof. Let X be an ultrafilter space with an underlying set being a cardinal number κ. Let ξ be a maximal filter of open subsets of X, and let us show that ξ is an ultrafilter on κ.

Suppose to the contrary that $\kappa = A \cup B$, where A, B are disjoint and $U \cap A \neq \varnothing \neq U \cap B$ for each $U \in \xi$. By Proposition 2.2, $int\,(clA) \cap int\,(clB) = \varnothing$. Let us assume $int\,(clA) \in \xi$. Since $B \cap int\,(clA)$ is a nowhere dense subset of X, it is closed and nowhere dense by Corollary 2.3. Hence $int\,(clA) - B \in \xi$ and $(int\,(clA) - B) \cap B = \varnothing$, which contradicts the initial assumption. By the similar argument, $int\,(clB) \notin \xi$. By maximality of ξ, there exists $V \in \xi$ such that $V \cap int\,(clA) = \varnothing = V \cap int\,(clB)$. Hence $U \cap (A - int\,(clA)) \neq \varnothing \neq U \cap (B - int\,(clB))$ for each $U \in \xi$. The sets $E = A - int\,(clA)$ and $F = B - int\,(clB)$ are nowhere dense, thus closed, and disjoint. By normality of X, there are disjoint open subsets W_1, W_2 of X such that $E \subseteq W_1$ and $F \subseteq W_2$. But then W_1, $W_2 \in \xi$; a contradiction. □

3. Ultrafilter Spaces vs. Spaces of Ultrafilters

Let κ be an infinite cardinal number. $\beta\kappa$ stands, as usual, for the set of all ultrafilters (free or principal) on the set κ endowed with the topology generated by the sets $\widehat{A} = \{\xi \in \beta\kappa : A \in \xi\}$, where $\varnothing \neq A \subseteq \kappa$.

In what follows, all considered ultrafilters spaces are assumed to be crowded.

Let X be an ultrafilter space with the underlying set κ, where κ is a cardinal number. For each $\alpha \in \kappa$, let

$\xi_\alpha = \{U - F : U$ is an open neighborhood of α and F is a finite subset of $\kappa\}$.

Thus $\xi_\alpha \in \beta\kappa - \kappa$ for each $\alpha \in \kappa$. We can define a function $\varphi : X \to \beta\kappa - \kappa$ by setting: $\varphi\,(\alpha) = \xi_\alpha$ for each $\alpha \in \kappa = X$. Thus $\varphi\,(X)$ can be thought off as a pointless copy of X. Let $E\,(X) = \kappa \cup \varphi\,(\kappa) \subseteq \beta\kappa$. In what follows, both $\varphi\,(X)$ and $E\,(X)$ are considered to be subspaces of the space $\beta\kappa$.

Proposition 3.1.
(a) *The function* $\varphi : X \to \beta\kappa - \kappa$ *is continuous;*
(b) φ *is one-to-one if and only if* X *is* T_2*;*
(c) φ *is an embedding if and only if* X *is* T_3.

Proof. (a) Let V be an open set in $\beta\kappa - \kappa$ and let $\alpha \in \kappa = X$ be such that $\xi_\alpha \in V$. There exists $A \subseteq \kappa$ such that $\xi_\alpha \in \{\xi \in \beta\kappa - \kappa : A \in \xi\} \subseteq V$. Since $A \in \xi_\alpha$, there exists an open set U of X such that $\alpha \in U \subseteq A$. Thus $\alpha \in U \subseteq \varphi^{-1}\,(V)$.

(b) Assume that φ is one-to-one and let $\alpha \neq \beta \in \kappa = X$. Since $\xi_\alpha \neq \xi_\beta$, there exist disjoint sets A and B such that $\xi_\alpha \in \{\xi \in \beta\kappa - \kappa : A \in \xi\}$ and $\xi_\beta \in \{\xi \in \beta\kappa - \kappa : B \in \xi\}$. Hence there exist open sets U, V in X such that $\alpha \in U \subseteq A$ and $\beta \in V \subseteq B$. Thus $\alpha \in U$ and $\beta \in V$ and $U \cap V = \varnothing$. The converse implication is obvious.

(c) Assume that φ is an embedding and let $\alpha \in U \subseteq X$, where U is open. Thus $\xi_\alpha \in \varphi\,(U)$ and since $\varphi\,(U)$ is open, there exists $A \subseteq \kappa$ such that $\xi_\alpha \in \{\xi \in \beta\kappa - \kappa : A \in \xi\} \subseteq \varphi\,(U)$. Since the set $\{\xi \in \beta\kappa - \kappa : A \in \xi\}$ is clopen and since $A \in \xi_\alpha$, there exists an open set V of X such that $\alpha \in V \subseteq A$. Thus $\alpha \in V \subseteq clV \subseteq \varphi^{-1}\,(U) = U$. Conversely, let U be open in X and let $\xi_\alpha \in \varphi\,(U)$. There exists an open set V in X such that $\alpha \in V \subseteq clV \subseteq U$.

Hence clV is clopen and such that $\xi_\alpha \in \{\xi \in \beta\kappa - \kappa : clV \in \xi\} \subseteq \varphi(U)$. Thus $\xi_\alpha \in int\varphi(U)$. □

Theorem 3.2. *Let X be an ultrafilter space with an underlying set being a cardinal number κ. If X is regular, then $E(X)$ is homeomorphic to $A(X)$.*

Proof. Let $h : A(X) \to E(X)$ be defined as follows: $h(\alpha') = \alpha$ for each $\alpha' \in X'$ and $h(\alpha) = \xi_\alpha$ for each $\alpha \in X$. Let's show that h is a homeomorphism.

Clearly, h is a bijection. To see that h is continuous, let $A \subseteq \kappa$ and $\alpha \in X$ satisfy $\xi_\alpha \in \widehat{A}$ (i.e., $h(\alpha) = \xi_\alpha \in \widehat{A}$). There exists an open neighborhood U of α and a finite set $F \subseteq X$ such that $U - F \subseteq A$. It is obvious that $h(D(U) - F') \subseteq \widehat{A} \cap E(X)$.

To see that h^{-1} is continuous, let $W \subseteq A(X)$ be open and let $\xi_\alpha \in h(W)$. Since h and φ coincide on X, $h|X$ is a homeomorphism (cf. Proposition 3.1 (c)). There exists $A \subseteq \kappa$ such that $\xi_\alpha \in \widehat{A} \cap h(X) \subseteq h(W)$. By Lemma 2.5, $W = [D(U) - H'] \cup G'$, where H is a discrete and closed (in U) subset of U and $G \subseteq X - U$ is arbitrary. There exists an open set $V \subseteq X$ and a finite set $F \subseteq X$ such that $\alpha \in V$, $V \cap H = \varnothing$, and $U - F \subseteq A$. Hence $\xi_\alpha \in \widehat{U} \subseteq h(W)$. □

Corollary 3.3. *X is a regular ultrafilter space if and only if $A(X)$ is Č-S embeddable.*

Let us consider a function $\psi : E(X) \to \varphi(\kappa) \subseteq \beta\kappa - \kappa$, given by: $\psi(\alpha) = \varphi(\alpha) = \xi_\alpha$ for each $\alpha \in \kappa$, and $\psi(\xi_\alpha) = \xi_\alpha$ for each $\alpha \in \kappa$. Clearly, ψ is continuous (see also Proposition 3.1 (a)): $\psi^{-1}(U^*) = \widehat{U} \cap E(X)$ for each open subset U of X. Thus ψ is an exactly two-to-one retraction onto $\varphi(\kappa)$.

There exists a continuous extension $\widehat{\psi}$ of ψ to $\beta\kappa$ into $cl\varphi(\kappa) \subseteq \beta\kappa - \kappa$. Hence $\widehat{\psi}$ is a retraction from the space $\beta\kappa$ onto its subspace $cl\varphi(\kappa)$. Let us note the following.

Lemma 3.4. *Let X be a regular ultrafilter space and $A \subseteq \kappa$, $\widehat{A} \cap cl\varphi(\kappa) = \varnothing$ if and only if A is a discrete and closed (in the topology on X). Consequently, if $\xi \in A^*$, then $\widehat{\psi}(\xi) \in cl\varphi(A) - \varphi(\kappa)$.*

Proof. Since $\varphi(\kappa)$ consists only of those ultrafilters that contain an open subset of X, $\widehat{A} \cap cl\varphi(\kappa) = \varnothing$ means that A (considered as a subspace of X) is boundary and so it is an infinite closed and nowhere dense subset of X.
Since $\xi \in \widehat{A} = clA$ (when considered in $\beta\kappa$), $\widehat{\psi}(\xi) \in cl\widehat{\psi}(A) = cl\varphi(A)$. Since A is closed in X, $\varphi(A)$ is closed in $\varphi(\kappa)$ (by Proposition 3.1). Hence $\widehat{\psi}(\xi) \in cl\varphi(A) - \varphi(\kappa)$. □

Theorem 3.5. *Let X be a regular ultrafilter space., The retraction $\widehat{\psi}$ is one-to-one on the subspace $\beta\kappa - cl\varphi(\kappa)$ if and only if X is a normal space.*

Proof. Assume that X is normal and let $\xi \neq \zeta \in \beta\kappa - cl\varphi(\kappa)$. The three cases when at least one of the two points ξ, ζ belongs to κ are obvious. So assume that $\xi, \zeta \in \beta\kappa - \kappa$. There exist disjoint sets $A, B \subseteq \kappa$ such that $A \in \xi$, $B \in \zeta$, and $\widehat{A} \cap cl\varphi(\kappa) = \varnothing = \widehat{B} \cap cl\varphi(\kappa)$. By Lemma 3.4, $\widehat{\psi}(\xi) \in cl\varphi(A) - \varphi(\kappa)$

and $\widehat{\psi}(\varsigma) \in cl\varphi(B) - \varphi(\kappa)$. Since X is normal, $cl\varphi(A) \cap cl\varphi(B) = \varnothing$. Hence $\widehat{\psi}(\xi) \neq \widehat{\psi}(\zeta)$.

Assume that the retraction ψ is a bijection outside of $\varphi(\kappa)$. To show that then X is a normal space (given that X is perfectly disconnected space), it suffices to show that any two disjoint discrete subsets of X can be separated.

For suppose that $A, B \subseteq X$ are disjoint and discrete but they cannot be separated. It follows that $cl\varphi(A) \cap cl\varphi(B) \neq \varnothing$; let $p \in cl\varphi(A) \cap cl\varphi(B)$. By Lemma 3.4, ψ restricted to $\widehat{A}$ is a homeomorphism between $\widehat{A}$ and $cl\varphi(A)$. Similarly, ψ restricted to $\widehat{B}$ is a homeomorphism between $\widehat{B}$ and $cl\varphi(B)$. There exist $\xi \in \widehat{A}$ and $\zeta \in \widehat{B}$ such that $\widehat{\psi}(\xi) = p = \widehat{\psi}(\zeta)$. Since $\widehat{A} \cap \widehat{B} = \varnothing$ and $\xi, \zeta \notin cl\varphi(\kappa)$, the retraction ψ is not a bijection outside of $cl\varphi(\kappa)$; a contradiction. □

Remark 3.6. The retraction $\widehat{\psi}$ in Theorem 3.5 is an instance of a $\leq$ two-to-one continuous maps on $\beta\kappa$ onto a compact space of density κ. An extensive and deep study of such maps was done by E. van Douwen in [9]. A. Dow, among others, has published several papers on related subjects (cf. [10]).
The retraction $\widehat{\psi}$ is also an instance of a one-to-one retraction. A. Dow did a thorough and profound study of one-to-one retractions on $\beta\omega$ in [11].

Let $f : X \to Y$ be a function. The *duplicate* of f, $D(f)$, is a function $D(f) : A(X) \to Y$ defined as follows: $D(f)(x) = f(x)$ and $D(f)(x') = f(x)$. It is easy to see that the duplicate of a continuous function on X yields a continuous function on $A(X)$, regardless of the separation axioms or other properties of its domain. In particular, X is C/C^*- embedded into $A(X)$. Can X' be C^*- embedded into $A(X)$? In this regard, we have the following.

Corollary 3.7. *If X' is C^*- embedded into $A(X)$, then X is an ultrafilter space. Conversely, if X is an ultrafilter space, then X' is C^*- embedded into $A(X)$ provided that X is a regular space.*

Proof. Let A and B disjoint subsets of X. Without loss of generality, we may also assume that $A \cup B = X$. Define $f : X' \to \mathbf{R}$ setting $f(x') = 0$ if $x \in A$ and $f(x') = 1$ if $x \in B$. If $\widetilde{f}$ is a continuous extension of f to $A(X)$, then $\widetilde{f}(x) = 0$ if $x \in \partial(A)$ and $\widetilde{f}(x) = 1$ if $x \in \partial(B)$. Hence $\partial(A) \cap \partial(B) = \varnothing$, which means, X is perfectly disconnected, viz. X is an ultrafilter space.

The converse part follows immediately from Theorem 3.2. □

Proposition 3.8. *Let X be a regular ultrafilter space with the underlying set κ. Then βX is homeomorphic to $cl(\varphi(\kappa))$. In particular, $cl(\varphi(\kappa))$ is extremally disconnected.*

Proof. Let $f : \varphi(\kappa) \to \mathbf{R}$ be a continuous bounded function. Take the duplicate $D(f)$ of f, i.e., $D(f)(\varphi(\alpha)) = f(\varphi(\alpha))$ and $D(f)(\alpha) = f(\varphi(\alpha))$ for each $\alpha \in \kappa$. Since $D(f)$ is continuos on $E(X)$, it has a continuous extension onto $\beta\kappa$ since $E(X) = \kappa \cup \varphi(\kappa) \subseteq \beta\kappa$. Since $D(f)|\varphi(\kappa) = f$, f has a continuous extension to $cl(\varphi(x))$. Thus $cl(\varphi(\kappa)) = \beta(\varphi(\kappa)) = \beta X$ since $\varphi(\kappa)$ and X are homeomorphic (see Proposition 3.1(c)). Since the Čech-Stone compactification of an extremally disconnected space is extremally disconnected, the proof is finished. □

Corollary 3.9. *Let X be a regular ultrafilter space.*
(a) *If $H \subseteq \beta X - X$ is a discrete set of remote points such that $X \cap clH = \varnothing$, then $Y = X \cup H$ is an ultrafilter space.*
(b) *If X is normal and $H \subseteq \beta X - X$ is a countable discrete set of remote points such that $X \cap clH = \varnothing$, then $Y = X \cup H$ is a strong ultrafilter space.*

Proof. (a) Y is a perfectly disconnected space. Indeed, let A and B be disjoint subsets of Y, and suppose that $p \in \partial(A) \cap \partial(B)$. If $p \in H$, then $p \in \partial(A \cap X) \cap \partial(B \cap X)$. Since p is remote, $p \notin cl\,(A \cap X - int\,(A \cap X))$ and $p \notin cl\,(B \cap X - int\,(X \cap X))$. Hence $p \in cl\,(int\,(A \cap X)) \cap cl\,(int\,(B \cap X))$, which is impossible since Y is extremally disconnected.

Suppose that $p \in X$. Then $p \notin clH$ and therefore $p \in \partial(A \cap X) \cap \partial(B \cap X)$. This is impossible since X is perfectly normal.

(b) It follows momentarily from part (a) and Proposition 2.9 since H is normal. □

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