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Weak partial metric spaces and some fixed point results

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Abstract

The concept of partial metric p on a nonempty set X was introduced by Matthews [8]. One of the most interesting properties of a partial metric is that $p(x, x)$ may not be zero for $x \in X$. Also, each partial metric p on a nonempty set X generates a T_0 topology on X. By omitting the small self-distance axiom of partial metric, Heckmann [7] defined the weak partial metric space. In the present paper, we give some fixed point results on weak partial metric spaces.

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1. Introduction

The notion of partial metric space was introduced by Matthews [8] as a part of the study of denotational semantics of data flow networks. It is widely recognized that partial metric spaces play an important role in constructing models in the theory of computation. In a partial metric spaces, the distance of a point in the self may not be zero. After the definition of partial metric space, Matthews proved a partial metric version of Banach's fixed point theorem. Then, Valero [11], Oltra and Valero [9] and Altun et al [1], [3] gave some generalizations of the result of Matthews. Recently, Romaguera [10] proved the Caristi type fixed point theorem on this space.

First, we recall some definitions of partial metric space and some properties of theirs. See [2, 7, 8, 9, 10, 11] for details.

A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ (nonnegative reals) such that for all $x, y, z \in X$:

(p$_1$) $x = y \Longleftrightarrow p(x,x) = p(x,y) = p(y,y)$ (T_0-separation axiom),
(p$_2$) $p(x,x) \leq p(x,y)$ (small self-distance axiom),
(p$_3$) $p(x,y) = p(y,x)$ (symmetry),
(p$_4$) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$ (modified triangular inequality).

A partial metric space (for short PMS) is a pair (X,p) such that X is a nonempty set and p is a partial metric on X. It is clear that, if $p(x,y) = 0$, then, from (p$_1$) and (p$_2$), $x = y$. But if $x = y$, $p(x,y)$ may not be 0. A basic example of a PMS is the pair $(\mathbb{R}^+, p)$, where $p(x,y) = \max\{x,y\}$ for all $x, y \in \mathbb{R}^+$. For another example, let I denote the set of all intervals $[a,b]$ for any real numbers $a \leq b$. Let $p : I \times I \to \mathbb{R}^+$ be the function such that $p([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}$. Then (I,p) is a PMS. Other examples of PMS which are interesting from a computational point of view may be found in [5], [8].

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family open p-balls

$$\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\},$$

where

$$B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$$

for all $x \in X$ and $\varepsilon > 0$.

It is easy to see that, a sequence $\{x_n\}$ in a PMS (X,p) converges with respect to τ_p to a point $x \in X$ if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$.

If p is a partial metric on X, then the functions $d_p, d_w : X \times X \to \mathbb{R}^+$ given by

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y) \tag{1.1}$$

and

$$\begin{aligned} d_w(x,y) &= \max\{p(x,y) - p(x,x), p(x,y) - p(y,y)\} \\ &= p(x,y) - \min\{p(x,x), p(y,y)\} \end{aligned} \tag{1.2}$$

are ordinary metrics on X.

Remark 1.1. Let $\{x_n\}$ be a sequence in a PMS (X,p) and $x \in X$, then

$$\lim_{n\to\infty} d_w(x_n, x) = 0$$

if and only if

$$p(x,x) = \lim_{n\to\infty} p(x_n,x) = \lim_{n,m\to\infty} p(x_n,x_m).$$

Proposition 1.2. *Let (X,p) be a PMS, then d_p and d_w are equivalent metrics on X.*

Proof. We obtain

$$\begin{aligned} d_p(x,y) &= 2p(x,y) - p(x,x) - p(y,y) \\ &= p(x,y) - p(x,x) + p(x,y) - p(y,y) \\ &\leq 2d_w(x,y). \end{aligned} \tag{1.3}$$

Again we obtain

$$\begin{aligned} d_w(x,y) &= p(x,y) - \min\{p(x,x), p(y,y)\} \\ &\leq p(x,y) - \min\{p(x,x), p(y,y)\} \\ &\quad + p(x,y) - \max\{p(x,x), p(y,y)\} \\ &= 2p(x,y) - p(x,x) - p(y,y) \\ &= d_p(x,y). \end{aligned} \tag{1.4}$$

From (1.3) and (1.4) we have

$$\frac{1}{2} d_p(x,y) \leq d_w(x,y) \leq d_p(x,y).$$

□

Definition 1.3. (i) A sequence $\{x_n\}$ in a PMS (X, p) is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m\to\infty} p(x_n, x_m)$.

(ii) A PMS (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.

The following lemma plays an important role to give fixed point results on a PMS.

Lemma 1.4 ([8], [9])**.** *Let (X, p) be a PMS.*

(a) *$\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, d_w).*

(b) *(X, p) is complete if and only if (X, d_w) is complete.*

Remark 1.5. Since d_p and d_w are equivalent, we can take d_p instead of d_w in Lemma 1.4.

2. Weak partial metric

Heckmann [7] introduced the concept of weak partial metric space (for short WPMS), which is a generalized version of Matthews' partial metric space by omitting the small self-distance axiom. That is, the function $p : X \times X \to \mathbb{R}^+$ is called weak partial metric on X if the conditions (p_1),(p_3) and (p_4) are satisfied. Also, Heckmann shows that, if p is a weak partial metric on X, then for all $x, y \in X$, we have the following weak small self-distance property

$$p(x,y) \geq \frac{p(x,x) + p(y,y)}{2}.$$

Weak small self-distance property shows that WPMS are not far from small-self distance axiom. It is clear that every PMS is a WPMS, but the converse may not be true. A basic example of a WPMS but not a PMS is the pair $(\mathbb{R}^+, p)$, where $p(x,y) = \frac{x+y}{2}$ for all $x, y \in \mathbb{R}^+$. For another example, let I denote the set of all intervals $[a,b]$ for any real numbers $a \leq b$. Let $p : I \times I \to \mathbb{R}^+$ be the function such that $p([a,b],[c,d]) = \frac{b+d-a-c}{2}$. Then (I,p) is a WPMS but not a PMS.

Remark 2.1. If (X,p) be a WPMS, but not a PMS, then the function d_p as in (1.1) may not be an ordinary metric on X. For example, let $X = \mathbb{R}^+$ and let $p : X \times X \to \mathbb{R}^+$ defined by $p(x,y) = \frac{x+y}{2}$. Then it is clear that $d_p(x,y) = 0$ for all $x, y \in X$, so d_p is not a metric on X. Note that, in this case $d_w(x,y) = \frac{1}{2}|x-y|$.

Proposition 2.2. *Let $a, b, c \in \mathbb{R}^+$, then we have*

$$\min\{a,c\} + \min\{b,c\} \leq \min\{a,b\} + c.$$

Proposition 2.3. *Let (X,p) be a WPMS, then $d_w : X \times X \to \mathbb{R}$ defined as in (1.2) is an ordinary metric on X.*

Proof. Since p is a weak partial metric, then we have

$$\begin{aligned} 2p(x,y) &\geq p(x,x) + p(y,y) \\ &\geq 2\min\{p(x,x), p(y,y)\}. \end{aligned}$$

Therefore $p(x,y) - \min\{p(x,x), p(y,y)\} \geq 0$. Again it is clear that, $d_w(x,y) = 0$ if and only if $x = y$ and $d_w(x,y) = d_w(y,x)$ for all $x, y \in X$. Now, let $x, y, z \in X$, then from Proposition 2.2, we have

$$\begin{aligned} d_w(x,z) &= p(x,z) - \min\{p(x,x), p(z,z)\} \\ &\leq p(x,y) + p(y,z) - p(y,y) - \min\{p(x,x), p(z,z)\} \\ &\leq p(x,y) - \min\{p(x,x), p(y,y)\} \\ &\quad + p(y,z) - \min\{p(y,y), p(z,z)\} \\ &= d_w(x,y) + d_w(y,z). \end{aligned}$$

□

In a WPMS, the convergence of a sequence, Cauchy sequence, completeness and continuity of a function are defined as PMS. To give some fixed point results on a WPMS, we need to prove Lemma 1.4 by omitting the small-self distance axiom.

Lemma 2.4. *Let (X,p) be a WPMS.*

(a) *$\{x_n\}$ is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the metric space (X,d_w).*
(b) *(X,p) is complete if and only if (X,d_w) is complete.*

Proof. First we show that every Cauchy sequence in (X,p) is a Cauchy sequence in (X,d_w). Let $\{x_n\}$ be a Cauchy sequence in (X,p), then there exists $a \in \mathbb{R}$

such that, given $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ with $|p(x_n, x_m) - a| < \frac{\varepsilon}{2}$ for all $n, m \geq n_0$. Hence

$$\begin{aligned} d_w(x_n, x_m) &= p(x_n, x_m) - \min\{p(x_n, x_n), p(x_m, x_m)\} \\ &= p(x_n, x_m) - a + a - \min\{p(x_n, x_n), p(x_m, x_m)\} \\ &\leq |p(x_n, x_m) - a| + |a - \min\{p(x_n, x_n), p(x_m, x_m)\}| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

for all $n, m \geq n_0$. Therefore $\{x_n\}$ is a Cauchy sequence in (X, d_w).

Next we prove that completeness of (X, d_w) implies completeness of (X, p). Indeed, if $\{x_n\}$ is a Cauchy sequence in (X, p), then it is also a Cauchy sequence in (X, d_w). Since (X, d_w) is complete we deduce that there exists $x \in X$ such that $\lim_{n\to\infty} d_w(x_n, x) = 0$. Now we show that $\lim_{n,m\to\infty} p(x_n, x_m) = p(x, x)$. Since $\{x_n\}$ is a Cauchy sequence in (X, p) it is sufficient to show that $\lim_{n\to\infty} p(x_n, x_n) = p(x, x)$. Let $\varepsilon > 0$, then there exists $n_0 \in \mathbb{N}$ such that $d_w(x_n, x) < \frac{\varepsilon}{2}$ for all $n \geq n_0$. Thus

$$\begin{aligned} |p(x_n, x_n) - p(x, x)| &= \max\{p(x_n, x_n), p(x, x)\} - \min\{p(x_n, x_n), p(x, x)\} \\ &= 2\left\{\frac{\max\{p(x_n, x_n), p(x, x)\} + \min\{p(x_n, x_n), p(x, x)\}}{2}\right. \\ &\qquad \left. - \min\{p(x_n, x_n), p(x, x)\}\right\} \\ &= 2[\frac{p(x_n, x_n) + p(x, x)}{2} - \min\{p(x_n, x_n), p(x, x)\}] \\ &\leq 2[p(x_n, x) - \min\{p(x_n, x_n), p(x, x)\}] \\ &= 2d_w(x_n, x) < \varepsilon \end{aligned}$$

whenever $n \geq n_0$. This shows that (X, p) is complete.

Now we prove that every Cauchy sequence $\{x_n\}$ in (X, d_w) is a Cauchy sequence in (X, p). Let $\varepsilon = \frac{1}{2}$. Then there exists $n_0 \in \mathbb{N}$ such that $d_w(x_n, x_m) < \frac{1}{2}$ for all $m, n \geq n_0$. Therefore we have

$$\begin{aligned} p(x_n, x_n) &= p(x_n, x_n) - p(x_{n_0}, x_{n_0}) + p(x_{n_0}, x_{n_0}) \\ &\leq |p(x_n, x_n) - p(x_{n_0}, x_{n_0})| + p(x_{n_0}, x_{n_0}) \\ &\leq 2d_w(x_n, x_{n_0}) + p(x_{n_0}, x_{n_0}) \\ &< 1 + p(x_{n_0}, x_{n_0}). \end{aligned}$$

Consequently the sequence $\{p(x_n, x_n)\}$ is bounded in $\mathbb{R}$ and so there exists $a \in \mathbb{R}$ such that a subsequence $\{p(x_{n_k}, x_{n_k})\}$ is convergent to a. On the other hand, since $\{x_n\}$ is a Cauchy sequence in (X, d_w), given $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $d_w(x_n, x_m) < \frac{\varepsilon}{2}$ for all $m, n \geq n_\varepsilon$. Thus we have

$$|p(x_n, x_n) - p(x_m, x_m)| \leq 2d_w(x_n, x_m) < \varepsilon.$$

That is, the sequence $\{p(x_n, x_n)\}$ is Cauchy in $\mathbb{R}$. Therefore

$$\lim_{n\to\infty} p(x_n, x_n) = a.$$

On the other hand, since

$$\begin{aligned} |p(x_n,x_m)-a| &\le |p(x_n,x_m)-\min\{p(x_n,x_n),p(x_m,x_m)\}|+ \\ &\quad |\min\{p(x_n,x_n),p(x_m,x_m)\}-a| \\ &= d_w(x_n,x_m)+|\min\{p(x_n,x_n),p(x_m,x_m)\}-a|, \end{aligned}$$

we have $\lim_{n,m\to\infty} p(x_n,x_m)=a$ and so $\{x_n\}$ is a Cauchy sequence in (X,p).

Now we prove that completeness of (X,p) implies completeness of (X,d_w). Indeed, if $\{x_n\}$ is a Cauchy sequence in (X,d_w), then it is also a Cauchy sequence in (X,p). Since (X,p) is complete we deduce that there exists $x\in X$ such that $\lim_{n,m\to\infty} p(x_n,x_m)=\lim_{n\to\infty} p(x_n,x)=p(x,x)$. Then, given $\varepsilon>0$, there exists $n_\varepsilon\in\mathbb{N}$ such that

$$\max\{|p(x_n,x)-p(x_n,x_n)|,|p(x_n,x)-p(x,x)|\}<\varepsilon$$

whenever $n\ge n_\varepsilon$. As a consequence we have

$$\begin{aligned} d_w(x_n,x) &= p(x_n,x)-\min\{p(x_n,x_n),p(x,x)\} \\ &= |p(x_n,x)-\min\{p(x_n,x_n),p(x,x)\}| \\ &< \varepsilon \end{aligned}$$

whenever $n\ge n_\varepsilon$. Therefore (X,d_w) is complete. □

Remark 2.5. Remark 1.1 is still true for WPMS.

3. Fixed point results

In this section we give some fixed point results on weak partial metric spaces. We begin by giving Hardy and Rogers type [6] fixed point theorem.

Theorem 3.1. *Let (X,p) be a complete WPMS and let $F:X\to X$ be a map such that*

$$\begin{aligned} p(Fx,Fy) &\le ap(x,y)+bp(x,Fx)+cp(y,Fy)+ \\ &\quad dp(x,Fy)+ep(y,Fx) \end{aligned} \tag{3.1}$$

for all $x,y\in X$, where $a,b,c,d,e\ge 0$ and, if $d\ge e$, then $a+b+c+2d<1$, if $d<e$, then $a+b+c+2e<1$. Then F has a unique fixed point.

Proof. Let $x_0\in X$ be an arbitrary point. Define a sequence $\{x_n\}$ in X by $x_n=Fx_{n-1}$ for $n=1,2,\cdots$. Now if $x_{n_0}=x_{n_0+1}$ for some $n_0=0,1,2,\cdots$, then it is clear that x_{n_0} is a fixed point of F. Now assume $x_n\ne x_{n+1}$ for all n. Then we have from (3.1)

$$\begin{aligned} p(x_{n+1},x_n) &= p(Fx_n,Fx_{n-1}) \\ &\le ap(x_n,x_{n-1})+bp(x_n,Fx_n)+cp(x_{n-1},Fx_{n-1})+ \\ &\quad dp(x_n,Fx_{n-1})+ep(x_{n-1},Fx_n) \\ &= ap(x_n,x_{n-1})+bp(x_n,x_{n+1})+cp(x_{n-1},x_n)+ \\ &\quad dp(x_n,x_n)+ep(x_{n-1},x_{n+1}) \\ &\le (a+c+e)p(x_n,x_{n-1})+(b+e)p(x_n,x_{n+1})+ \\ &\quad (d-e)p(x_n,x_n). \end{aligned} \tag{3.2}$$

Now if $d \geq e$, then adding the term $(d-e)p(x_{n+1}, x_{n+1})$ or $(d-e)p(x_{n-1}, x_{n-1})$ in the right side of (3.2) and using weak small self distance axiom, we have

$$p(x_{n+1}, x_n) \leq \max\{\frac{a+c+e}{1-b-2d+e}, \frac{a+c+2d-e}{1-b-e}\}p(x_n, x_{n-1}) \tag{3.3}$$

for all n. If $d < e$, then from (3.2) by omitting the term $(d-e)p(x_n, x_n)$, we have

$$p(x_{n+1}, x_n) \leq \frac{a+c+e}{1-b-e}p(x_n, x_{n-1}). \tag{3.4}$$

Hence from (3.3) and (3.4) we have for $n = 1, 2, \cdots$

$$p(x_{n+1}, x_n) \leq \lambda^n p(x_1, x_0),$$

where

$$\lambda = \begin{cases} \max\{\frac{a+c+e}{1-b-2d+e}, \frac{a+c+2d-e}{1-b-e}\} & , \quad d \geq e \\ \frac{a+c+e}{1-b-e} & , \quad d < e \end{cases}.$$

It is clear that $\lambda \in [0, 1)$, thus we have

$$\lim_{n\to\infty} p(x_{n+1}, x_n) = 0. \tag{3.5}$$

On the other hand, since

$$\begin{aligned} d_w(x_{n+1}, x_n) &= p(x_{n+1}, x_n) - \min\{p(x_n, x_n), p(x_{n+1}, x_{n+1})\} \\ &\leq p(x_{n+1}, x_n) \\ &\leq \lambda^n p(x_1, x_0) \end{aligned}$$

we have $\lim_{n\to\infty} d_w(x_n, x_{n+1}) = 0$. Therefore we have for $k = 1, 2, \cdots$

$$\begin{aligned} d_w(x_{n+k}, x_n) &\leq d_w(x_{n+k}, x_{n+k-1}) + \cdots + d_w(x_{n+1}, x_n) \\ &\leq \lambda^{n+k-1}p(x_1, x_0) + \cdots + \lambda^n p(x_1, x_0) \\ &= [\lambda^{n+k-1} + \cdots + \lambda^n]p(x_1, x_0) \\ &\leq \frac{\lambda^n}{1-\lambda}p(x_1, x_0). \end{aligned}$$

This shows that $\{x_n\}$ is a Cauchy sequence in the metric space (X, d_w). Since (X, p) is complete then from Lemma 2.4, the sequence $\{x_n\}$ converges in the metric space (X, d_w), say $\lim_{n\to\infty} d_w(x_n, x) = 0$. Again from Lemma 2.4, we have

$$p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_n, x_m). \tag{3.6}$$

Moreover since $\{x_n\}$ is a Cauchy sequence in the metric space (X, d_w), we have $\lim_{n,m\to\infty} d_w(x_n, x_m) = 0$. On the other hand since

$$p(x_n, x_n) + p(x_{n+1}, x_{n+1}) \leq 2p(x_n, x_{n+1})$$

we obtain by (3.5)

$$\lim_{n\to\infty} p(x_n, x_n) = 0.$$

Therefore from the definition d_w we have

$$p(x_n, x_m) = d_w(x_n, x_m) + \min\{p(x_n, x_n), p(x_m, x_m)\}$$

and so $\lim_{n,m\to\infty} p(x_n, x_m) = 0$. Thus from (3.6) we have

$$p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_n, x_m) = 0.$$

Now we show that $p(x, Fx) = 0$. Assume this is not true, then from (3.1) we obtain

$$\begin{aligned} p(x, Fx) &\leq p(x, Fx_n) + p(Fx_n, Fx) - p(Fx_n, Fx_n) \\ &\leq p(x, x_{n+1}) + p(Fx_n, Fx) \\ &\leq p(x, x_{n+1}) + ap(x, x_n) + bp(x, Fx) + cp(x_n, x_{n+1}) + \\ &\quad dp(x, x_{n+1}) + ep(x_n, Fx) \\ &\leq p(x, x_{n+1}) + ap(x, x_n) + bp(x, Fx) + cp(x_n, x_{n+1}) + \\ &\quad dp(x, x_{n+1}) + ep(x_n, x) + ep(x, Fx) \end{aligned}$$

letting $n \to \infty$, we have

$$p(x, Fx) \leq (b + e)p(x, Fx),$$

which is a contradiction. Thus $p(x, Fx) = 0$ and so $x = Fx$. Moreover $p(x, x) = 0$.

For the uniqueness, suppose y is another fixed point of F. Then we have

$$\begin{aligned} p(y, y) &= p(Fy, Fy) \\ &\leq (a + b + c + d + e)p(y, y). \end{aligned}$$

This shows that $p(y, y) = 0$. Now, if $p(x, y) > 0$, then we have

$$p(x, y) = p(Fx, Fy) \leq (a + d + e)p(x, y),$$

which is a contradiction. Therefore, the fixed point is unique. □

We can have the following corollaries from Theorem 3.1.

Corollary 3.2 (Banach type)**.** *Let (X, p) be a complete WPMS and let $F : X \to X$ be a map such that*

$$p(Fx, Fy) \leq \alpha p(x, y)$$

for all $x, y \in X$, where $0 \leq \alpha < 1$. Then F has a unique fixed point.

Corollary 3.3 (Kannan type)**.** *Let (X, p) be a complete WPMS and let $F : X \to X$ be a map such that*

$$p(Fx, Fy) \leq \beta p(x, Fx) + \gamma p(y, Fy)$$

for all $x, y \in X$, where $\beta, \gamma \geq 0$ and $\beta + \gamma < 1$. Then F has a unique fixed point.

Corollary 3.4 (Reich type)**.** *Let (X,p) be a complete WPMS and let $F : X \to X$ be a map such that*

$$p(Fx,Fy) \leq \alpha p(x,y) + \beta p(x,Fx) + \gamma p(y,Fy)$$

for all $x,y \in X$, where $\alpha,\beta,\gamma \geq 0$ and $\alpha+\beta+\gamma < 1$. Then F has a unique fixed point.

Next we state a nonlinear contractive type fixed point theorem.

Let $\phi : [0,\infty) \to [0,\infty)$ be a function. In the connection with the function ϕ we consider the following properties:

(i) ϕ is nondecreasing,

(ii) $\phi(t) < t$ for all $t > 0$,

(iii) $\phi(0) = 0$,

(iv) ϕ is continuous,

(v) $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t \geq 0$,

(vi)$\sum_{n=0}^{\infty} \phi^n(t)$ convergent for all $t > 0$.

It is easy to see that, (i) and (ii) imply (iii), (ii) and (iv) imply (iii), (i) and (v) imply (ii).

Definition 3.5 ([4])**.** A function ϕ satisfying (i) and (v) is said to be a comparison function and a function ϕ satisfying (i) and (vi) is said to be (c)-comparison function.

It is clear that, any (c)-comparison function is a comparison function and any comparison function satisfies (iii).

Theorem 3.6. *Let (X,p) be a complete WPMS and let $F : X \to X$ be a map such that*

$$\begin{aligned} p(Fx,Fy) \quad &\leq \quad \phi(\max\{p(x,y),p(x,Fx),p(y,Fy), \\ &\qquad \frac{1}{2}[p(x,Fy)+p(y,Fx)]\}) \end{aligned} \tag{3.7}$$

for all $x,y \in X$, where $\phi : [0,\infty) \to [0,\infty)$ is a (c)-comparison function. Then F has a unique fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point. Define a sequence $\{x_n\}$ in X by $x_n = Fx_{n-1}$ for $n = 1,2,\cdots$. Now if $x_{n_0} = x_{n_0+1}$ for some $n_0 = 0,1,2,\cdots$, then it is clear that x_{n_0} is a fixed point of F. Now assume $x_n \neq x_{n+1}$ for all

n. In this case $p(x_n, x_{n+1}) > 0$ for all n. Then we have from (3.7)

$$\begin{aligned} p(x_{n+1}, x_n) &= p(Fx_n, Fx_{n-1}) \\ &\leq \phi(\max\{p(x_n, x_{n-1}), p(x_n, Fx_n), p(x_{n-1}, Fx_{n-1}), \\ &\quad \frac{1}{2}[p(x_n, Fx_{n-1}) + p(x_{n-1}, Fx_n)]\}) \\ &\leq \phi(\max\{p(x_n, x_{n-1}), p(x_n, x_{n+1}), \\ &\quad \frac{1}{2}[p(x_{n-1}, x_n) + p(x_n, x_{n+1})]\}) \\ &= \phi(\max\{p(x_n, x_{n-1}), p(x_n, x_{n+1})\}), \end{aligned} \tag{3.8}$$

since

$$p(x_n, x_n) + p(x_{n-1}, x_{n+1}) \leq p(x_{n-1}, x_n) + p(x_n, x_{n+1})$$

and ϕ is nondecreasing. Now if

$$\max\{p(x_n, x_{n-1}), p(x_n, x_{n+1})\} = p(x_n, x_{n+1})$$

for some n, then from (3.8) we have

$$p(x_{n+1}, x_n) \leq \phi(p(x_n, x_{n+1})) < p(x_{n+1}, x_n)$$

which is a contradiction since $p(x_n, x_{n+1}) > 0$. Thus

$$\max\{p(x_n, x_{n-1}), p(x_n, x_{n+1})\} = p(x_n, x_{n-1})$$

for all n. Then from (3.8) we have

$$p(x_{n+1}, x_n) \leq \phi(p(x_n, x_{n-1}))$$

and hence

$$p(x_{n+1}, x_n) \leq \phi^n(p(x_1, x_0)). \tag{3.9}$$

This shows that

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0. \tag{3.10}$$

On the other hand, since

$$\begin{aligned} d_w(x_{n+1}, x_n) &= p(x_{n+1}, x_n) - \min\{p(x_n, x_n), p(x_{n+1}, x_{n+1})\} \\ &\leq p(x_{n+1}, x_n) \\ &\leq \phi^n(p(x_1, x_0)) \end{aligned}$$

we have $\lim_{n\to\infty} d_w(x_n, x_{n+1}) = 0$. Therefore we have for $m > n$

$$\begin{aligned} d_w(x_m, x_n) &\leq d_w(x_m, x_{m-1}) + \cdots + d_w(x_{n+1}, x_n) \\ &\leq \phi^{m-1}(p(x_1, x_0)) + \cdots + \phi^n(p(x_1, x_0)) \\ &\leq \sum_{k=n}^{\infty} \phi^k(p(x_1, x_0)). \end{aligned}$$

Since ϕ is (c)-comparison function, then $\{x_n\}$ is a Cauchy sequence in the metric space (X, d_w). Since (X, p) is complete then from Lemma 2.4, the sequence

$\{x_n\}$ converges in the metric space (X, d_w), say $\lim_{n\to\infty} d_w(x_n, x) = 0$. Again from Lemma 2.4, we have

$$p(x,x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_n, x_m). \tag{3.11}$$

Moreover since $\{x_n\}$ is a Cauchy sequence in the metric space (X, d_w), we have $\lim_{n,m\to\infty} d_w(x_n, x_m) = 0$. On the other hand since

$$p(x_n, x_n) + p(x_{n+1}, x_{n+1}) \leq 2p(x_n, x_{n+1})$$

we obtain by (3.10)

$$\lim_{n\to\infty} p(x_n, x_n) = 0.$$

Therefore from the definition d_w we have

$$p(x_n, x_m) = d_w(x_n, x_m) + \min\{p(x_n, x_n), p(x_m, x_m)\}$$

and so $\lim_{n,m\to\infty} p(x_n, x_m) = 0$. Thus from (3.11) we have

$$p(x,x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_n, x_m) = 0. \tag{3.12}$$

Now we show that $p(x, Fx) = 0$. Suppose that $p(x, Fx) > 0$, as $\lim_{n\to\infty} p(x_{n+1}, x_n) = 0$ and $\lim_{n\to\infty} p(x_n, x) = 0$, there exists $n_0 \in \mathbb{N}$ such that for $n > n_0$,

$$p(x_{n+1}, x_n) < \frac{1}{3}p(x, Fx) \tag{3.13}$$

and there exist $n_1 \in \mathbb{N}$ such that for $n > n_1$,

$$p(x_n, x) < \frac{1}{3}p(x, Fx). \tag{3.14}$$

If we take $n > \max\{n_0, n_1\}$ then, by (3.13), (3.14) and triangular inequality, we have

$$\begin{aligned} \frac{1}{2}[p(x_n, Fx) + p(x, Fx_n)] &\leq \frac{1}{2}[p(x_n, x) + p(x, Fx) - p(x,x) + p(x, Fx_n)] \\ &\leq \frac{1}{2}[\frac{1}{3}p(x, Fx) + p(x, Fx) + \frac{1}{3}p(x, Fx)] \\ &= \frac{5}{6}p(x, Fx). \end{aligned} \tag{3.15}$$

Now for $n > \max\{n_0, n_1\}$, then, by (3.13), (3.14) and (3.15), we have

$$\begin{aligned} p(x_{n+1}, Fx) &= p(Fx_n, Fx) \\ &\leq \phi(\max\{p(x_n, x), p(x_n, Fx_n), p(x, Fx), \\ &\quad \frac{1}{2}[p(x_n, Fx) + p(x, Fx_n)]\}) \\ &\leq \phi(p(x, Fx)). \end{aligned}$$

Letting $n \to \infty$ in the last inequality, we have $p(x, Fx) \leq \phi(p(x, Fx))$, which is a contradiction. Thus $p(x, Fx) = 0$ and so x is a fixed point of F. Moreover by (3.12) $p(x,x) = 0$. The uniqueness follows easily from (3.7). □

Example 3.7. Let $X = \{0, 1, \cdots, 10\}$ and $p(x, y) = \frac{x+y}{2}$, then $d_w(x, y) = \frac{1}{2}|x - y|$. Therefore, since (X, d_w) is complete, then by Lemma 2.4 (X, p) is complete WPMS. Let $F : X \to X$,

$$Fx = \begin{cases} x - 1 & , \quad x \neq 0 \\ \\ 0 & \quad x = 0 \end{cases}.$$

We claim that the condition (3.7) of Theorem 3.6 is satisfied with $\phi(t) = \frac{9}{10}t$. For this, we consider the following cases.

Case 1. If $x = y = 0$, then

$$p(Fx, Fy) = 0 \leq \frac{9}{10}p(x, y).$$

Case 2. If $x = y > 0$, then

$$\begin{aligned} p(Fx, Fy) &= p(x-1, x-1) = x - 1 \\ &\leq \frac{9}{10}x = \frac{9}{10}p(x, y) \end{aligned}$$

Case 3. If $x > y = 0$, then

$$\begin{aligned} p(Fx, Fy) &= p(x-1, 0) = \frac{x-1}{2} \\ &\leq \frac{9}{10}\frac{x}{2} = \frac{9}{10}p(x, y). \end{aligned}$$

Case 4. If $x > y > 0$, then

$$\begin{aligned} p(Fx, Fy) &= p(x-1, y-1) = \frac{x+y-2}{2} \\ &\leq \frac{9}{10}\frac{x+y}{2} = \frac{9}{10}p(x, y). \end{aligned}$$

This shows that all conditions of Theorem 3.6 are satisfied and so F has a unique fixed point in X. Note that, if we use the usual metric on X, then the contractive condition is not satisfied.

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