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Hereditary separability in Hausdorff continua

D. Daniel and M. Tuncali*

Abstract

We consider those Hausdorff continua S such that each separable subspace of S is hereditarily separable. Due to results of Ostaszewski and Rudin, respectively, all monotonically normal spaces and therefore all continuous Hausdorff images of ordered compacta also have this property. Our study focuses on the structure of such spaces that also possess one of various rim properties, with emphasis given to rim-separability. In so doing we obtain analogues of results of M. Tuncali and I. Lončar, respectively.

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1. Introduction

It is straightforward that separability is inherited by open subspaces and therefore by regular closed sets [3], and is hereditary in second countable spaces. It has also been shown [14] that separability is inherited by closed sets in Luzin spaces. We consider those Hausdorff spaces S such that each separable subspace of S is hereditarily separable. Such a space will be said to be sub-hereditarily separable. A motivation for our interest in this property lies partly in its relationship to monotone normality and continuous Hausdorff images of ordered compacta. It has been shown that all monotonically normal spaces are sub-hereditarily separable [20]. The closed continuous image of a monotonically normal space is known [10] to be monotonically normal and is therefore sub-hereditarily separable. In particular, any continuous Hausdorff image of an

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ordered compactum [22] is sub-hereditarily separable. A first countable locally connected sub-hereditarily separable continuum need not be the continuous image of an arc as we demonstrate by example.

A central motivation for this study is a sequence of (still un-resolved) questions raised in [8]:

Question - Is each locally connected separable Suslinian continuum hereditarily separable?

Question - Is each (hereditarily) separable locally connected Suslinian continuum metrizable?

We first note that local connectivity is essential in each of these questions. There is constructed in [7] a compactification of the half-line $[0, \infty)$ whose remainder is a homeomorphic copy of a Souslin line S (a linearly ordered non-separable continuum having only countably many mutually disjoint subintervals). The resulting continuum Z of course fails to be locally connected at each point of S. We also note that in [2], Banakh, Fedorchuk, Nikiel and Tuncali proved that under Suslin Hypothesis, each Suslinian continuum is metrizable, and under the negation of Suslin Hypothesis, they constructed a hereditarily separable, non-metric continuum which is nowhere locally connected.

Arkhangelskii [1] and Shapirovskii [23] have independently shown that if each closed subspace of a separable compact space S is separable then S is hereditarily separable. In addition, it is shown in [8] that each closed zero-dimensional subset of a Suslinian continuum is metrizable. It therefore follows that it is sufficient in the initial question above to consider the separability of non-degenerate subcontinua.

In [26], it is shown that each separable and connected continuous image of a compact ordered space is metrizable. It is therefore sufficient in the second question above to consider whether the space is the continuous image of a compact ordered space. We also note that in [9] it is shown that if there is an example of a locally connected Suslinian continuum which is not the continuous image of an arc then there is such a continuum which is separable.

2. Preliminaries

All spaces are assumed to be Hausdorff. A *compactum* is a compact space. A *continuum* is a connected compactum. Recall that a continuum in *Suslinian* provided that it possesses only countably many pairwise disjoint non-degenerate subcontinua. An *arc* is a continuum X which admits a linear ordering such that the order topology coincides with the given topology. It is straightforward that each ordered compactum is contained in an arc. A space X is said to be an *IOK* if it is the continuous image of some compact ordered space K. If K is also connected, then X is said to be *IOC*.

A space S is *hereditarily separable* provided each subspace of S is separable. A space S is said to be *sub-hereditarily separable* if and only if each separable subspace of S is hereditarily separable. For brevity, we will also say that such a space is SHS.

A topological space X is said to be *monotonically normal* (see [10]) provided that there exists a function G which assigns, to each point $x \in X$ and each open set U of X containing x, an open set $G(x, U)$ such that

(1) $x \in G(x, U) \subseteq U$,

(2) if U' is open and $x \in U \subset U'$, then $G(x, U) \subseteq G(x, U')$,

(3) if x and y are distinct points of X, then $G(x, X - y) \cap G(y, X - x) = \varnothing$.

Such a function G is called a monotone normality operator on X. It follows from a result of Ostaszewski [20] that each monotonically normal compactum is SHS (see also [22]).

If $S \subseteq X$, $\mathrm{Int}_X(S)$ will denote the interior of S with respect to X or simply $\mathrm{Int}(S)$ if the superspace is clear. Similarly, $\mathrm{Bd}_X(S)$ (or simply $\mathrm{Bd}(S)$) and $\mathrm{Cl}_X(S)$ (or simply $\mathrm{Cl}(S)$), respectively, will denote the boundary of S and closure of S, respectively, with respect to X.

A space S is said to be *rim-separable* (*rim-metrizable*) at $s \in S$ if and only if S admits a basis of open sets at s with separable (metrizable) boundaries, and S is *rim-separable* (*rim-metrizable*) provided that it is rim-separable (rim-metrizable) at each point. It is clear that any rim-metrizable compact space is rim-separable. A relatively comprehensive survey of rim properties may be found in [5].

3. Examples and Fundamental Properties

Our interest in spaces that are sub-hereditarily separable was initially spurred by the aforementioned results of Ostaszewski [20] and Rudin [22], respectively. These results demonstrate that sub-hereditarily separability is a necessary condition in order for a compactum to be an IOK. We give a short direct proof of this below. The property is however not sufficient as the example below demonstrates.

Theorem 3.1. *If X is the continuous image of some compact ordered space then X is SHS.*

Proof. By the result of Arkhangelskii [1] and Shapirovskii [23] noted above, it is enough to consider $A \subseteq B \subseteq X$ with A closed in B and B separable and closed in X. From Nikiel, Purisch, and Treybig [18], B is the continuous image of the double arrow space ($D = ([0,1] \times \{0,1\}) - \{(0,0),(1,1)\}$ with the order topology given by lexicographic order). They have also noted that the double arrow space is hereditarily separable. Therefore, B is also hereditarily separable. □

Let $Y = [0,1]^2$ be equipped with the lexicographic order and let $X = Y \times [0,1]$. Then X is a first countable non-separable locally connected continuum that is both SHS and rim-separable. We also note that X is neither an IOK [25] nor rim-metrizable [28].

We also note that sub-hereditarily separability is not preserved under arbitrary continuous maps. The Sorgenfrey plane R_l^2 (where R_l is the real line with the topology generated by the basis $B = \{[a, b) : a, b \in R, a < b\}$) fails to be hereditarily separable although R_l is hereditarily separable. (Sub-hereditarily separability therefore fails to be preserved under products.) Let R_d^2 denote the plane with the discrete topology; R_d^2 is obviously SHS since each separable subset is countable. The identity map $id : R_d^2 \to R_l^2$ is clearly continuous.

Theorem 3.2. *A topological space X is SHS if and only if there exists an open continuous onto map $f : X \to Y$ such that Y is SHS and $f^{-1}(y)$ is separable for each $y \in Y$.*

Proof. Suppose $A \subseteq B \subseteq X$ and B is separable. Since $f(B)$ is separable and Y is SHS, there exists a countable dense subset $D_{f(A)}$ of $f(A)$. For each $y \in D_{f(A)}$, let D_y denote a countable dense subset of $f^{-1}(y)$. Consider $D = \cup\{D_y : y \in D_{f(A)}\}$. If D were not dense in A, there exists an A-open set U such that $U \cap D = \varnothing$. Assume that $U = A \cap U'$ with U' open in X. Then $f(U) = f(A) \cap f(U')$, and $f(U) \cap D_{f(A)} \neq \varnothing$, a contradiction. Therefore, $U \cap D \neq \varnothing$ and D is dense in A. □

Lemma 3.3. *Suppose $f : X \to Y$ is a closed continuous map and X is SHS. Then Y is SHS.*

Proof. Consider $A \subseteq B \subseteq Y$ with A closed in B and B separable and closed in Y. Let D be a countable dense subset in B. For each $d \in D$, select one point from $f^{-1}(d)$ and let D_X be the collection of points so selected. Then $\mathrm{Cl}(D_X)$ is a hereditarily separable subspace of X. Since $A \subseteq f(\mathrm{Cl}(D_X))$, it follows that A is separable. □

The following very simple theorem is an analogue to an important result of Treybig [25] - if the product $X \times Y$ of two infinite Hausdorff spaces X and Y is an IOK then each of X and Y is metrizable.

Theorem 3.4. *If X and Y are compacta and $X \times Y$ is SHS, then each of X and Y is SHS.*

Proof. If each of X and Y are countable, the result is obvious. So, without loss of generality, consider X to be uncountable. Suppose $A \subseteq B \subseteq X$ and B is separable. Then select a countable (or finite) closed subset Y' of Y. Then $B \times Y'$ is separable and thus $A \times Y'$ is separable since $X \times Y$ is SHS. Then the continuity and closedness of the projection π_X onto X yields that A is separable. □

A space Z is said to be *locally sub-hereditarily separable* (or locally SHS for brevity) at $z \in Z$ if and only if there is a SHS neighborhood of z in Z.

Theorem 3.5. *A compactum X is SHS if and only if it is locally SHS at each point.*

Proof. Consider $A \subseteq B \subseteq X$ with A closed in B and B separable and closed in X. For each point a of A, select a B-open separable neighborhood N_a of a. Since X is locally SHS, $A \cap N_a$ is separable. By compactness, A is separable. □

Theorem 3.6. *If a first countable compactum X is not SHS then the closed subspace $S = \{p \in X : X$ is not locally SHS at $p\}$ is not scattered and therefore not countable.*

Proof. Assume that S is scattered. S then has an isolated point q and therefore a local basis $\{U_n\}$ of open sets at q such that $\mathrm{Cl}(U_n) \cap S = \{q\}$ for each n and $U_1 \supset \mathrm{Cl}(U_2) \supset U_2 \supset \mathrm{Cl}(U_3) \supset \cdots$. Since X is not locally SHS at q, we may assume that there is a separable subset $B \subseteq U_1$ and non-separable subset A such that $A \subseteq B \subseteq U_1$. Select a countable dense subset $Q_k \subseteq A \cap (U_1 - \mathrm{Cl}(U_k))$ for each $k = 2, 3, \ldots$. Then $Q = \cup_{k=2}^{\infty} Q_k$ is countable dense in A. X is then locally SHS at q and therefore $q \notin S$, a contradiction. □

In [8], it is shown that each Suslinian continuum is first countable and that each closed zero-dimensional subset of a Suslinian continuum is metrizable. In view of the aforementioned questions (also in [8]), we then note that the following result follows immediately.

Corollary 3.7. *If X is a Suslinian continuum and X is not hereditarily separable then $S = \{p \in X : X$ is not locally hereditarily separable at $p\}$ contains a non-degenerate subcontinuum.*

4. The Class of Sub-hereditarily Separable Continua

Although several of the results hold in a more general setting (e.g. Lemma 4.3), we address in the main in this section spaces that are SHS continua. The initial result is a simple analogue to Theorem 3.4, p. 1052 of [4].

Theorem 4.1. *Suppose X is a locally connected rim-separable SHS continuum. Then X admits a basis $\mathcal{B}$ such that, for each $B \in \mathcal{B}$, B is F_σ in X, and $Bd(B)$ is separable.*

Proof. Let $\mathcal{B} = \{O : O$ is open and $O = \cup O_i$ with $O_1 \subseteq \mathrm{Cl}(O_1) \subseteq O_2 \subseteq \mathrm{Cl}(O_2) \subseteq \cdots$, each O_i is open in X, $\mathrm{Bd}(O_i)$ is separable for each i, $O \neq X\}$. Then $\mathrm{Bd}(O) \subseteq \mathrm{Cl}(\cup(\mathrm{Bd}(O_i))$ and is therefore separable by hypothesis. □

Lemma 4.2. *For a rim-separable locally connected continuum X and an F_σ-subset A of X, each compact subset K of $Cl(A) - A$ lies in some separable subspace of X. Such a K is therefore separable if X is SHS.*

Proof. Let K be any compact subset of $\mathrm{Cl}(A) - A$. We express $A = \cup_{n=1}^{\infty} A_n$ where $\{A_n : n = 1, 2, \ldots\}$ is an increasing sequence of compact subsets of X. By the rim-separability of X, we may select for each $n = 1, 2, \ldots$ an open neighborhood U_n of A_n in X with separable boundary $\mathrm{Bd}(U_n)$ such that $\mathrm{Cl}(U_n) \cap K = \varnothing$. We now show that K lies in the closure of the separable subspace $S = \cup_{n=1}^{\infty} \mathrm{Bd}(U_n)$. Let $x \in K$ and V a connected open neighborhood of x in X. Since $x \in K \subseteq \mathrm{Cl}(A)$, there exists some $n = 1, 2, \ldots$ such that V

meets A_n in some point a. Since V is connected and $x \notin \text{Cl}(U_n)$, the set $V \cap U_n$ is not both closed and open in V. It then follows that $V \cap \text{Bd}(U_n) \subseteq V \cap S$ is non-empty and $x \in \text{Cl}(S)$. □

A proof of the following may be found in [6].

Lemma 4.3. *Let X denote a locally connected continuum and suppose that the set $T = \{p \in X : X$ is not locally an IOK at $p\}$ is totally disconnected. Then X is rim-metric (and therefore rim-separable).*

Corollary 4.4. *Let X denote a locally connected SHS continuum and suppose that the set $T = \{p \in X : X$ is not locally an IOK at $p\}$ is totally disconnected. Suppose that A is a compact subset of X such that there exists an F_σ-subset B of X with $A \subseteq Cl(B) - B$. Then A is separable.*

Theorem 4.5. *Suppose X is a rim-separable SHS continuum and X fails to be locally connected at $x \in X$. Then for each open set O containing x there exists a non-degenerate separable subcontinuum M of X such that $M \subseteq O$.*

Proof. Let $x \in O$ with O open in X. Since X fails to be locally connected at $x \in X$, X is not connected im kleinen at x. As such [27], there exist open sets U and U' and mutually disjoint continua $C'_1, C'_2, \dots$ so that $x \in U \subset Cl(U) \subset U' \subset Cl(U') \subset O$ and each $C'_i \subseteq O$ meets both U and $(X - U')$. For each i, select a component C_i of $C'_i \cap Cl(U')$ such that C_i meets both U and $(X - U')$. The limiting continuum of M of $\{C_i\}_{i=1}^{\infty}$ is separable by the reasoning of Lemma 4.2 and, by construction, is contained in O. □

Theorem 4.6. *Suppose X is a Suslinian SHS continuum and each pair of points x and y is contained in a separable subcontinuum $S(x, y)$ of X. Then X is (hereditarily) separable.*

Proof. Let $\mathcal{M}$ be a maximal family of non-degenerate pairwise disjoint separable subcontinua of X. Since X is Suslinian, $\mathcal{M} = \{M_i : i = 1, 2, 3, \dots\}$ is at most countable and therefore $\cup\mathcal{M}$ had countable dense subset D. D is then dense in X; else there exist open sets U and O such that $U \subset \text{Cl}(U) \subset O$ and $O \cap \text{Cl}(D) = O \cap (\cup\mathcal{M}) = \varnothing$. Then for $p \in U$ and $q \in (X - O)$, the closure of the component of $S(p, q) \cap U$ is a separable continuum, contradicting the maximality of $\mathcal{M}$. □

We note that the condition described in the hypothesis of the previous result - the existence of a separable continuum containing each pair of distinct points - may be generalized and that generalization has been shown to have wide applications in utility theory. In particular, space Z is said to be *separably connected* if for each pair of distinct points x and y in Z, there is a separable connected subspace $S \subset Z$ such that S contains both x and y. The reader is referred to the survey of Induráin [11] for results concerning such spaces and their applications.

Separable continua may arise in locally connected Suslinian continua in a somewhat natural way by utilizing related equivalence relations. In particular, a locally connected SHS Suslinian continuum admits an upper semi-continuous decomposition such that the resulting decomposition space is the continuous image of an arc. The following is based on similar constructions employed in work of B. Pearson and J. Simone (e.g., [24]), respectively. Let X be a locally connected Suslinian continuum. For each $x \in X$, define $M_x = \{y \in X :$ there exists a separable subcontinuum of X containing x and $y\}$. Note that each M_x is separable by Theorem 4.6. Let $G = \{M_x : x \in X\}$. Then X/G is hereditarily locally connected and is thus the image of an arc [17], and each element $M_x \in G$ is a (separable) continuum.

Corollary 4.7. *Let X be a Suslinian SHS locally connected continuum. If X contains no non-degenerate separable subcontinuum then X is an IOC.*

Proof. X is first countable and rim-metrizable (and therefore rim-separable) by [7]. Let G denote the decomposition of X as indicated above. If X contains no non-degenerate separable subcontinuum then G is a decomposition of X into singletons. Therefore, X is an IOC. □

5. Inverse Limits of Separable Continua

In this section, we consider rim-separable continua as a sub-class of hereditarily separable continua. In particular, we demonstrate that each rim-separable continuum is the inverse limit of a σ-directed inverse system of separable continua with monotone surjective bonding mappings. The following four results are analogues of various results of M. Tuncali [28]. Where the proof is not provided, it is an obvious modification of that of Tuncali .

Lemma 5.1. *Let X be a continuum and let U be an open set in X with separable boundary. Let G denote the upper semi-continuous decomposition of Cl(U) into components. Then Cl(U)/G is separable.*

Theorem 5.2. *Let X be a rim-separable continuum and let Y be compact metric. Suppose $f : X \to Y$ is onto and light. Then X is separable.*

Proof. Let $f : X \to Y$ be a light mapping of a rim-separable continuum X onto a metrizable continuum Y. Let $\mathcal{B}$ be a countable basis for the topology on Y and let $\{(V_n, W_n) : n = 1, 2, 3, \dots\}$ be an enumeration of the set $\{(V, W) :$ each of V and W is in $\mathcal{B}$ and $\mathrm{Cl}(V) \subseteq W\}$. For each $n = 1, 2, 3, \dots$ there is by the rim-separability of X an open neighborhood $U_n \subseteq X$ of the compact subset $f^{-1}(\mathrm{Cl}(V_n))$ such that $\mathrm{Cl}(U_n) \subseteq f^{-1}(\mathrm{Cl}(W_n))$ and $\mathrm{Bd}(U_n)$ is separable.

We now show that the separable set $S = \cup_{n=1}^{\infty} \mathrm{Bd}(U_n)$ is dense in X so that X is separable. Let $x \in X$ and let U be an open neighborhood of x in X. As the mapping f is light, the pre-image $f^{-1}(y)$ of the point $y = f(x)$ is zero-dimensional. We may assume, by replacing U by a suitable smaller neighborhood if necessary, that $U \cap f^{-1}(y)$ is both closed and open in $f^{-1}(y)$ and that $X - U$ is non-empty. $X - \mathrm{Bd}(U)$ is then a neighborhood of $f^{-1}(y)$ and we can find a basic neighborhood $W \in \mathcal{B}$ of y such that $f^{-1}(\mathrm{Cl}(W)) \subseteq X - \mathrm{Bd}(U)$. Select any basic neighborhood $V \in \mathcal{B}$ of y such that $\mathrm{Cl}(V) \subseteq W$.

Then $(V,W) = (V_n, W_n)$ for some $n = 1,2,3,\ldots$ By the choice of the open neighborhood U_n, we have $x \in f^{-1}(\mathrm{Cl}(V)) \subseteq U_n \subseteq f^{-1}(W) \subseteq f^{-1}(\mathrm{Cl}(W)) \subseteq X - \mathrm{Bd}(U)$. Since X is a continuum, the open neighborhood $U \cap U_n$ of x in X has non-empty boundary in X. This boundary lies in U so that $U \cap S \neq \varnothing$. □

Lemma 5.3. *A compactum X is rim-separable if and only if, for each pair p,q of distinct elements of X, there exists a closed separable subset S of X such that S separates X between p and q.*

Theorem 5.4. *Let X be a rim-separable continuum and let $f : X \to Y$ be onto and monotone. Then Y is rim-separable.*

Recall that an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is σ-directed provided that for each sequence $\{a_i : i = 1,2,\ldots\}$ there is an $a \in A$ such that $a \geq a_i$ for each $i = 1,2,\ldots$. In Theorem 18 of [13] (see also [12]), I. Lončar demonstrates that each rim-metrizable continuum X admits a σ-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that each X_a is a metrizable continuum, each p_{ab} is monotone and onto, and $X = \varprojlim \mathbf{X}$.

Theorem 5.5. *Let X be a rim-separable continuum. There exists a σ-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that each X_a is a separable continuum, each p_{ab} is monotone and onto, and $X = \varprojlim \mathbf{X}$.*

Proof. By Mardešič [15] and by Nikiel, Tuncali, and Tymchatyn (Theorem 9.4 of [19]), there is a σ-directed inverse system $\mathbf{Y} = \{Y_a, q_{ab}, A\}$ such that each Y_a is compact metric, each q_{ab} is onto, and $X = \varprojlim \mathbf{Y}$. For each $a \in A$, let $q_a : X \to Y_a$ denote the natural projection. For each $a \in A$, apply the monotone-light factorization of q_a to obtain a space X_a, a monotone mapping $q'_a : X \to X_a$, and a light mapping $q''_a : X_a \to Y_a$ such that $q_a = q''_a \circ q'_a$.

By Mardešič [15], for each pair a,b of elements of A such that $a \leq b$ in A, there is a monotone mapping $p_{ab} : X_b \to X_a$. We therefore obtain an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that $X = \varprojlim \mathbf{X}$. Applying Theorem 5.4 to $q'_a : X \to X_a$, each X_a is rim-separable. Applying Theorem 5.2 to $q''_a : X_a \to Y_a$, each X_a is separable. □

From this result, a number of analogues of results of Lončar follow. Furthermore, the proofs are essentially identical to their proofs. In particular, the two following results (and their proofs) are analogues of Theorem 19 and Theorem 20, respectively, of [13]. Recall that a metric continuum is said to have the *property of Kelley* if and only if, given $\epsilon > 0$, there is a $\delta > 0$ such that if a and b are in X, $d(a,b) < \epsilon$ and $a \in A \in C(X)$ then there exists $B \in C(X)$ such that $b \in B$ and $H(A,B) < \epsilon$. It is known that each locally connected metric continuum has the property of Kelley (see e.g. [16]). The following topological generalization of this property is due to W. J. Charatonik. A continuum X is defined to have the *property of Kelley* if, for each $a \in X$, each A in $C(X)$ containing a, and each open set $V \in C(X)$ containing A, there exists an open set W containing a and, if $b \in W$, then there exists $B \in C(X)$ such that $b \in B$ and $B \in V$. Lončar (Theorem 9 of [12]) has shown that each locally connected continuum has the property of Kelley. A continuum X is *smooth at the point* $p \in X$ if for each convergent net $\{x_n\}$ of points of X and for each subcontinuum K of X such that both p and $x = \lim\{x_n\}$ are in K, there exists a net $\{K_i\}$ of subcontinua of X such that each K_i contains p and some x_n and $K = \lim\{K_i\}$. Rakowski [21] has shown that a continuum X is smooth at p if and only if for each subcontinuum N of X containing p and for each open set V such that $N \subseteq V$ there exists an open connected set K such that $N \subseteq K \subseteq V$. Recall also that a *dendroid* is an arcwise connected and hereditarily unicoherent continuum.

Corollary 5.6. *Every rim-separable dendroid with the property of Kelley is smooth.*

Corollary 5.7. *Every rim-separable dendroid is the inverse limit of an inverse system of separable dendroids.*

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D. Daniel (*dale.daniel@lamar.edu*)
Lamar University, Department of Mathematics, Beaumont, Texas 77710, USA

M. Tuncali (*muratt@nipissingu.ca*)
Nipissing University, Faculty of Arts and Sciences, North Bay, Ontario P1B 8L7, Canada