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# Some fixed point theorems on the class of comparable partial metric spaces

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Abstract

*In this paper we present existence and uniqueness criteria of a fixed point for a self mapping on a non-empty set endowed with two comparable partial metrics.*

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## 1. Introduction and Preliminaries

In 1992, Matthews [10, 11] introduced the notion of a partial metric space which is a generalization of usual metric spaces in which $d(x,x)$ are no longer necessarily zero. After this remarkable contribution, many authors focused on partial metric spaces and its topological properties (see e.g. [15, 16, 2, 1, 3, 4, 5, 6]). Partial metric spaces have extensive potential applications in the research area of computer domains and semantics (see e.g. [7, 12, 8, 13, 14]). Consequently, the attention paid to such spaces rapidly increases.

A partial metric space (see e.g.[10, 11]) is a pair $(X,p)$ such that $X$ is non-empty set and $p : X \times X \to \mathbb{R}^+$ (where $\mathbb{R}^+$ denotes the set of all non negative real numbers) satisfies:

(PM1) $p(x,y) = p(y,x)$ (symmetry)
(PM2) If $p(x,x) = p(x,y) = p(y,y)$ then $x = y$ (equality)
(PM3) $p(x,x) \leq p(x,y)$ (small self-distances)
(PM4) $p(x,z) + p(y,y) \leq p(x,y) + p(y,z)$ (triangle inequality)

for all $x,y,z \in X$. We use the abbreviation PMS for the partial metric space $(X,p)$.

Notice that for a partial metric $p$ on $X$, the function $d_p : X \times X \to \mathbb{R}^+$ given by

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y) \tag{1.1}$$

is a (usual) metric on $X$. Observe that each partial metric $p$ on $X$ generates a $T_0$ topology $\tau_p$ on $X$ with a base the family open $p$-balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$. Similarly, closed $p$-ball is defined as $B_p[x,\varepsilon] = \{y \in X : p(x,y) \leq p(x,x) + \varepsilon\}$

**Definition 1.1** (see e.g.[10, 11, 1])**.**

- $(i)$ A sequence $\{x_n\}$ in a PMS $(X,p)$ converges to $x \in X$ if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$,
- $(ii)$ a sequence $\{x_n\}$ in a PMS $(X,p)$ is called a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists (and finite),
- $(iii)$ A PMS $(X,p)$ is said to be complete if every Cauchy sequence $\{x_n\}$ in $X$ converges, with respect to $\tau_p$, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.

**Lemma 1.2** (see e.g.[10, 11, 1] )**.**

- $(A)$ *A sequence $\{x_n\}$ in a PMS $(X,p)$ is Cauchy if and only if $\{x_n\}$ is Cauchy in a metric space $(X,d_p)$,*
- $(B)$ *A PMS $(X,p)$ is complete if and only if a metric space $(X,d_p)$ is complete. Moreover,*

$$\lim_{n\to\infty} d_p(x,x_n) = 0 \Leftrightarrow p(x,x) = \lim_{n\to\infty} p(x,x_n) = \lim_{n,m\to\infty} p(x_n,x_m) \tag{1.2}$$

In this manuscript, we present some new fixed point theorems on a non-empty set on which there exists two partial metrics with certain conditions.

## 2. Main Results

The following two lemmas will be used in the proof of the main theorem.

**Lemma 2.1** (see e.g. [3])**.** *Let $(X,p)$ be a complete PMS. Then*

- (A) *If $p(x,y) = 0$ then $x = y$,*
- (B) *If $x \neq y$, then $p(x,y) > 0$.*

**Lemma 2.2** (see e.g. [1, 3])**.** *Assume $x_n \to z$ as $n \to \infty$ in a PMS $(X,p)$ such that $p(z,z) = 0$. Then $\lim_{n\to\infty} p(x_n,y) = p(z,y)$ for every $y \in X$.*

The following theorem is an extension of the result of Maia [9].

**Theorem 2.3.** *Let $X$ be a non-empty set endowed with two partial metrics $p_1, p_2$, and let $T$ be a mapping of $X$ into itself. Suppose that*

- $(i)$ *$(X,p_1)$ is complete,*
- $(ii)$ *$p_1(x,y) \leq p_2(x,y)$ for all $x,y \in X$,*
- $(iii)$ *$T$ is continuous with respect to $\tau_{p_1}$,*
- $(iv)$ *$T$ is a contraction with respect to $p_2$, that is, $p_2(Tx,Ty) \leq kp_2(x,y)$ for all $x,y \in X$, where $0 \leq k < 1$.*

*Then $T$ has a unique fixed point in $X$.*

*Proof.* Fix $x \in X$. We construct a sequence $\{x_n\}$ in the following way:

(S1) $x_0 = x$,
(S2) $x_n = Tx_{n-1} = T^n x_0$ for each $n \in \mathbb{N}$.

Then, by assumption $(iv)$ we have

$$p_2(x_{n+1}, x_n) = p_2(Tx_n, Tx_{n-1}) \leq kp_2(x_n, x_{n-1}) \leq \cdots \leq k^n p_2(Tx_0, x_0).$$

Hence, by standard calculations, we get that $\lim_{n,m\to\infty} p_2(x_n, x_m) = 0$, and by assumption $(ii)$, $\lim_{n,m\to\infty} p_1(x_n, x_m) = 0$, i.e., $\{x_n\}$ is a Cauchy sequence in $(X, p_1)$. So, by assumption $(i)$ and Lemma 1.2, it converges in $(X, d_{p_1})$ to a point $z \in X$. Again by Lemma 1.2,

$$p_1(z, z) = \lim_{n\to\infty} p_1(x_n, z) = \lim_{n,m\to\infty} p_1(x_n, x_m) \tag{2.1}$$

Since $\lim_{n,m\to\infty} p_1(x_n, x_m) = 0$, then by (2.1) we have $p_1(z, z) = 0$.

By the continuity of $T$ and also Lemma 2.2, one can get

$$\begin{aligned} p_1(z, z) &= \lim_{n\to\infty} p_1(z, x_{n+1}) = \lim_{n\to\infty} p(z, T^{n+1}x_0) \\ &= p_1(z, T(\lim_{n\to\infty} T^n x_0)) = p_1(z, T(\lim_{n\to\infty} x_n)) = p_1(z, Tz). \end{aligned}$$

Hence $P(Tz, z) = p(z, z) = 0$. Due to Lemma 2.1 the point $z$ is a unique fixed point of $T$. Suppose not, that is, there exist $z, y \in X$ such that $Tz = z$ and $Ty = y$. Then, $p_2(z, y) = p_2(Tz, Ty) \leq kp_2(z, y)$. Thus, $p_2(z, y) = 0$.Regarding Lemma 2.1, $z = y$. □

**Theorem 2.4.** *Let $(X, p_1)$ be a PMS and $T : X \to X$ a mapping. Consider the series:*

$$\sum_{n=0}^{\infty} t^n p_1(T^n x, T^n y) \tag{2.2}$$

*Suppose that for some $t > 1$, the series (2.2) converges for every $x, y \in X$. Then, for such a point $t$, the function $p_2 : X \times X \to \mathbb{R}^+$ defined by*

$$p_2(x, y) = \sum_{n=0}^{\infty} t^n p_1(T^n x, T^n y)$$

*is a partial metric on $X$, moreover,*

(i) *$p_2$ is an upper bound partial metric for $p_1$,*
(ii) *$T$ is a contraction with respect to $p_2$.*

*Proof.* Since $t > 1$ and $p_1(T^n x, T^n y) \geq 0$ for all $x, y \in X$ and $n \in \mathbb{N}$, then $p_2(x, y) \geq 0$. It is clear that $p_2$ satisfies (PM1). For the proof of (PM2), assume $p_2(x, x) = p_2(x, y) = p_2(x, y)$ which is equivalent to

$$\sum_{n=0}^{\infty} t^n p_1(T^n x, T^n x) = \sum_{n=0}^{\infty} t^n p_1(T^n x, T^n y) = \sum_{n=0}^{\infty} t^n p_1(T^n y, T^n y)$$

Hence

$$\sum_{n=0}^{\infty} t^n \left(p_1(T^n y, T^n x) - p_1(T^n x, T^n x)\right) = \sum_{n=0}^{\infty} t^n \left(p_1(T^n y, T^n x) - p_1(T^n y, T^n y)\right) = 0,$$

so $p_1(T^n y, T^n x) = p_1(T^n y, T^n y) = p_1(T^n x, T^n x)$ for all $n \in \mathbb{N} \cup \{0\}$. In particular, $p_1(x,y) = p_1(x,x) = p_1(y,y)$, and hence, $x = y$. Moreover, (PM3) and (PM4) are obtained by definition.

Let us prove $(i)$ and $(ii)$.

$$\begin{aligned} p_2(x,y) &= \textstyle\sum_{n=0}^{\infty} t^n p_1(T^n x, T^n y) = p_1(x,y) + \sum_{n=1}^{\infty} t^n p_1(T^n x, T^n y) \\ &= p_1(x,y) + t\left(\textstyle\sum_{n=0}^{\infty} t^n p_1(T^{n+1}x, T^{n+1}y)\right) = p_1(x,y) + tp_2(Tx,Ty) \end{aligned}$$

Thus,

$$p_2(Tx,Ty) = \frac{1}{t}(p_2(x,y) - p_1(x,y)) \leq \frac{1}{t} p_2(x,y).$$

□

**Theorem 2.5.** *Suppose $(X,p_1)$ is a PMS and $T: X \to X$ is a mapping such that $p_1(T^m x, T^m y) \leq kp_1(x,y)$ for some $m \in \mathbb{N}$, where $0 \leq k < 1$. Then the series $p_2(x,y) = \sum_{n=0}^{\infty} t^n p_1(T^n x, T^n y)$ converges for $t > 1$, whatever the points $x, y \in X$.*

*Proof.* By assumption,

$$p_1(T^m x, T^m y) \leq kp_1(x,y)$$

for some $m \in \mathbb{N}$, and $0 \leq k < 1$. It yields that

$$p_1(T^{mn}x, T^{mn}y) \leq k^n p_1(x,y)$$

for every $n$ integer. Then,

$$\begin{aligned} &p_2(x,y) = \textstyle\sum_{n=0}^{\infty} t^n p_1(T^n x, T^n y) \\ &= \textstyle\sum_{n=0}^{\infty} t^{mn} p_1(T^n x, T^n y) + \sum_{n=0}^{\infty} t^{mn+1} p_1(T^{mn+1}x, T^{mn+1}y) \\ &\quad + \cdots + \textstyle\sum_{n=0}^{\infty} t^{mn+n-1} p_1(T^{mn+n-1}x, T^{mn+n-1}y) \\ &\leq \textstyle\sum_{n=0}^{\infty} t^{mn} k^n p_1(x,y) + t\sum_{n=0}^{\infty} t^{mn} k^n p_1(Tx,Ty) \\ &\quad + \cdots + t^{n-1} \textstyle\sum_{n=0}^{\infty} t^{mn} k^n p_1(T^{n-1}x, T^{n-1}y) \end{aligned}$$

Just then take $t$ such that: $1 < t^n < \frac{1}{k}$, because the series converges regardless of the points $x, y \in X$. □

**Theorem 2.6.** *Let $X$ be a non-empty set endowed with two partial metrics $p_1, p_2$, and let $T$ be a mapping of $X$ into itself. Suppose that*

$(i)$ *There exists a point $x_0 \in X$ such that the sequence of iterates $\{T^n(x_0)\}$ has a subsequence $\{T^{n_i}(x_0)\}$ converging to a point $z \in X$ for $\tau_{p_1}$,*

$(ii)$ *$p_1(x,y) \leq p_2(x,y)$ for all $x, y \in X$,*

$(iii)$ *$T$ is continuous at $z$ with respect to $p_1$,*

$(iv)$ *$T$ is contraction with respect to $p_2$, that is, $p_2(Tx,Ty) \leq kp_2(x,y)$ for all $x, y \in X$, where $0 \leq k < 1$.*

*Then $T$ has a unique fixed point in $X$.*

*Proof.* Fix $x_0 \in X$ and define $x_{n+1} = Tx_n$ for $n \in \mathbb{N} \cup \{0\}$. As it shown in the proof of Theorem 2.3, this sequence $\{x_n\}$ is Cauchy with respect to $p_2$. By $(ii)$, the sequence $\{x_n\}$ is also Cauchy with respect to $p_1$. By $(i)$, Cauchy sequence $\{x_n\}$ has a subsequence $\{x_{n_i}\}$ which converges $z \in X$ for $\tau_{p_1}$. Thus, $\{x_n\}$ converges to $z$ for $\tau_{p_1}$. By the continuity of $T$ and also Lemma 2.2 one can get

$$\begin{aligned} p_1(z,z) &= \lim_{n\to\infty} p_1(z, x_{n+1}) = \lim_{n\to\infty} p(z, T^{n+1}x_0) \\ &= p_1(z, T(\lim_{n\to\infty} T^n x_0)) = p_1(z, T(\lim_{n\to\infty} x_n)) = p_1(z, Tz). \end{aligned}$$

Hence $P(Tz, z) = p(z, z) = 0$. Due to Lemma 2.1 the point $z$ is a unique fixed point of $T$. To show uniqueness, assume the contrary. Let $z$ and $w$ be two different fixed points. Then, by $(iv)$,

$$p_2(z, w) = p_2(Tz, Tw) \leq kp_2(z, w)$$

Since $0 \leq k < 1$, one can get a contradiction. Thus, $T$ has a unique fixed point. $\square$

**Remark 2.7.** Consider the following condition:

$(iv)^*$ There is a point $x_0 \in X$ such that the iterated sequence $\{T^n(x_0)\}$ is a Cauchy sequence with respect to $p_2$.

If the condition $(iv)$ is replaced by $(iv)^*$ in Theorem 2.6, the theorem will still guarantee the existence of fixed point.

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