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Matkowski's type theorems for generalized contractions on (ordered) partial metric spaces

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Abstract

We obtain extensions of Matkowski's fixed point theorem for generalized contractions of Ćirić's type on 0-complete partial metric spaces and on ordered 0-complete partial metric spaces, respectively.

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1. Introduction and preliminaries

Partial metric spaces, and their equivalent weightable quasi-metric spaces, were introduced by Matthews [14] in the context of his studies on denotational semantics for dataflow networks. In fact, this class of spaces provides a suitable framework to construct computational models for metric spaces and related structures (see e.g. [10, 14, 21, 22, 24, 25, 26]). It also provides an appropriate setting to discuss, with the help of techniques of fixed points of denotational semantics, the complexity analysis of several algorithms which can be defined by recurrence equations (see e.g. [9, 20, 23]).

These facts explain, in part, the recent extensive research on fixed points for self maps in partial metric spaces ([1, 2, 3, 5, 8, 11, 13, 19], etc).

In this note we try to reach a new advance in this direction by obtaining extensions of Matkowski's fixed point theorem [15, Theorem 1.2] for generalized contractions on 0-complete partial metric space and on ordered 0-complete partial metric spaces, respectively. These results extend, generalize and unify some theorems from the current literature.

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Throughout this paper the letter ω will denote the set of all nonnegative integer numbers.

Let us recall [14] that a partial metric on a set X is a function $p : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$: (i) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$; (ii) $p(x, x) \leq p(x, y)$; (iii) $p(x, y) = p(y, x)$; (iv) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

A partial metric space is a pair (X, p) where p is a partial metric on X.

Each partial metric p on X induces a T_0 topology τ_p on X which has as a base the family of open balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Matthews observed in [14, p. 187] that a sequence $(x_n)_{n\in\omega}$ in a partial metric space (X, p) converges to some $x \in X$ with respect to τ_p if and only if $\lim_{n\to\infty} p(x, x_n) = p(x, x)$.

Next we recall some useful concepts and facts on completeness of partial metric spaces.

If p is a partial metric on X, then the function $p^s : X \times X \to [0, \infty)$ given by $p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$, is a metric on X.

Furthermore, a sequence $(x_n)_{n\in\omega}$ in X converges to some $x \in X$ with respect to τ_{p^s} if and only if $\lim_{n,m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(x, x_n) = p(x, x)$.

A sequence $(x_n)_{n\in\omega}$ in a partial metric space (X, p) is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m} p(x_n, x_m)$ [14, Definition 5.2].

A partial metric space (X, p) is said to be complete if every Cauchy sequence $(x_n)_{n\in\omega}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = \lim_{n,m} p(x_n, x_m)$ [14, Definition 5.3].

It is well known (see, for instance, [14, p. 194]) that a sequence in a partial metric space (X, p) is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, p^s), and that a partial metric space (X, p) is complete if and only the metric space (X, p^s) is complete.

Romaguera introduced in [18] the notions of a 0-Cauchy sequence in a partial metric space and of a 0-complete partial metric space.

A sequence $(x_n)_{n\in\omega}$ in a partial metric space (X, p) is called 0-Cauchy if $\lim_{n,m} p(x_n, x_m) = 0$.

We say that a partial metric space (X, p) is 0-complete if every 0-Cauchy sequence in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = 0$. In this case, p is said to be a 0-complete partial metric on X.

Note that every 0-Cauchy sequence in (X, p) is a Cauchy sequence in (X, p), and that every complete partial metric space is 0-complete.

On the other hand, the partial metric space $(\mathbb{Q} \cap [0, \infty), p)$, where $\mathbb{Q}$ denotes the set of rational numbers and the partial metric p is given by $p(x, y) = \max\{x, y\}$, provides an easy example of a 0-complete partial metric space which is not complete.

2. Fixed points for 0-complete partial metric spaces

Given a partial metric space (X, p) and $f : X \to X$ a map, we define

$$M(x, y) := \max\left\{p(x, y), p(x, fx), p(y, fy), \frac{1}{2}\left[p(x, fy) + p(y, fx)\right]\right\},$$

for all $x, y \in X$ (compare e.g. [5, 19]).

The proof of [19, Lemma 2] shows the following.

Lemma 2.1. *Let (X, p) be a partial metric space, $f : X \to X$ a map and $x_0 \in X$ such that $f^n x_0 \neq f^{n+1} x_0$ and*

$$p(f^{n+1}x_0, f^{n+2}x_0) \leq \varphi(M(f^n x_0, f^{n+1}x_0)),$$

for all $n \in \omega$, where $\varphi : [0, \infty) \to [0, \infty)$ satisfies $\varphi(t) < t$ for all $t > 0$. Then the following hold:

(a) $M(f^n x_0, f^{n+1}x_0) = p(f^n x_0, f^{n+1}x_0\,)$ *for all* $n \in \omega$.

(b) $p(f^{n+1}x_0, f^{n+2}x_0) \leq \varphi(p(f^n x_0, f^{n+1}x_0)) < p(f^n x_0, f^{n+1}x_0)$ *for all* $n \in \omega$.

Remark 2.2. Recall [15, 16] that if $\varphi : [0, \infty) \to [0, \infty)$ is a nondecreasing function such that $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$, then $\varphi(t) < t$ for all $t > 0$ and thus $\varphi(0) = 0$.

Now we prove the main result of this section.

Theorem 2.3. *Let (X, p) be a 0-complete partial metric space and $f : X \to X$ a map such that*

$$p(fx, fy) \leq \varphi(M(x, y)),$$

for all $x, y \in X$, where $\varphi : [0, \infty) \to [0, \infty)$ is a nondecreasing function such that $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$. Then f has a unique fixed point $z \in X$. Moreover $p(z, z) = 0$.

Proof. Let $x_0 \in X$. For each $n \in \omega$ put $x_n = f^n x_0$. Thus $x_{n+1} = f x_n$ for all $n \in \omega$.

If there is $k \in \omega$ such that $x_k = x_{k+1}$, then x_k is a fixed point of f. Moreover if $fz = z$ for some $z \in X$, it follows that

$$p(z, x_k) = p(fz, fx_k) \leq \varphi(M(z, x_k)) = \varphi(p(z, x_k)),$$

so, $p(x_k, z) = 0$, i.e., $z = x_k$. So x_k is the unique fixed point of f, and, clearly, $p(x_k, x_k) = 0$ by the contraction condition.

Hence, we shall assume that $f^n x_0 \neq f^{n+1}x_0$ for all $n \in \omega$. Thus $p(x_n, x_{n+1}) > 0$ for all $n \in \omega$. By Lemma 2.1 (b), $p(x_n, x_{n+1}) \leq \varphi(p(x_{n-1}, x_n))$ for all $n \in \mathbb{N}$, and then

$$p(x_n, x_{n+1}) \leq \varphi^n(p(x_0, x_1)),$$

for all $n \in \omega$. So

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0.$$

Now choose an arbitrary $\varepsilon > 0$. Since, by Remark 2.2, $\varphi(\varepsilon) < \varepsilon$, then there is $n_\varepsilon \in \mathbb{N}$ such that

$$p(x_n, x_{n+1}) < \varepsilon - \varphi(\varepsilon),$$

for all $n \geq n_\varepsilon$. Therefore

$$\begin{aligned} p(x_n, x_{n+2}) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) \\ &< \varepsilon - \varphi(\varepsilon) + \varphi(p(x_n, x_{n+1})) \\ &\leq \varepsilon - \varphi(\varepsilon) + \varphi(\varepsilon) = \varepsilon, \end{aligned}$$

for all $n \geq n_\varepsilon$. So

$$\begin{aligned} p(x_n, x_{n+3}) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+3}) \\ &< \varepsilon - \varphi(\varepsilon) + \varphi(M(x_n, x_{n+2})), \end{aligned}$$

for all $n \geq n_\varepsilon$.

Now suppose that there is $n \geq n_\varepsilon$ such that $M(x_n, x_{n+2}) > \varepsilon$. Then, from

$$\begin{aligned} M(x_n, x_{n+2}) &= \max\{p(x_n, x_{n+2}), p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2}), \\ &\quad \frac{1}{2}[p(x_n, x_{n+3}) + p(x_{n+1}, x_{n+2})\} \\ &\leq \max\{\varepsilon, \frac{1}{2}[p(x_n, x_{n+3}) + \varphi(\varepsilon)]\}. \end{aligned}$$

it follows that

$$M(x_n, x_{n+2}) \leq \tfrac{1}{2}[p(x_n, x_{n+3}) + \varphi(\varepsilon)],$$

so

$$\begin{aligned} p(x_n, x_{n+3}) &< \varepsilon - \varphi(\varepsilon) + \varphi(M(x_n, x_{n+2})) \\ &< \varepsilon - \varphi(\varepsilon) + M(x_n, x_{n+2}) \\ &\leq \varepsilon - \varphi(\varepsilon) + \frac{1}{2}[p(x_n, x_{n+3}) + \varphi(\varepsilon)]. \end{aligned}$$

We deduce that $M(x_n, x_{n+2}) < \varepsilon$, a contradiction.

Therefore $p(x_n, x_{n+3}) < \varepsilon$, and following this process,

$$p(x_n, x_{n+k}) < \varepsilon,$$

for all $n \geq n_\varepsilon$ and $k \in \mathbb{N}$. Consequently

$$\lim_{n,m\to\infty} p(x_n, x_m) = 0,$$

and thus $(x_n)_{n\in\omega}$ is a 0-Cauchy sequence in the 0-complete partial metric space (X, p). Hence there is $z \in X$ such that

$$\lim_{n,m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(z, x_n) = p(z, z) = 0.$$

Finally, the fact that z is the unique fixed point of f follows similarly to the last part of the proof of [19, Theorem 4]. $\square$

In a recent paper [12], Jachymski showed the equivalence between several generalized contractions on (ordered) metric spaces. Since the key of his study is Lemma 1 of the cited paper, then Jachymski's approach also holds in the partial metric framework. As an instance, we shall combine this lemma with Theorem 2.3 above to deduce the following (compare [1, Corollary 2.1]).

Corollary 2.4. *Let (X, p) be a 0-complete partial metric space and $f : X \to X$ a map such that*

$$\psi(p(fx, fy)) \leq \psi(M(x, y)) - \phi(M(x, y)),$$

for all $x, y \in X$, where $\psi, \phi : [0, \infty) \to [0, \infty)$ are nondecreasing functions such that ψ is continuous on $[0, \infty)$ and $\psi^{-1}(0) = \phi^{-1}(0) = \{0\}$. Then f has a unique fixed point $z \in X$. Moreover $p(z, z) = 0$.

Proof. By [12, Lemma 1 (ii)⇒(viii)], there exists a continuous and nondecreasing function $\varphi : [0, \infty) \to [0, \infty)$ such that $\varphi(t) < t$ for all $t > 0$, and $p(fx, fy) \leq \varphi(M(x, y))$ for all $x, y \in X$. From continuity of φ it follows that $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$. Theorem 2.3 concludes the proof. □

Remark 2.5. Theorem 2.3 and Corollary 2.4 extend several fixed point theorems for complete metric spaces due to Dutta and Choudhury, Khan, Swaleh and Sessa, and Rhoades, among others (see [12, Theorems 1 and 3, and the bibliography]). Theorem 2.3 also improves [11, Theorem 3.2], [5, Theorem 1] and [19, Theorem 4].

3. Fixed points for ordered 0-complete partial metric spaces

Our main purpose in this section is to prove an ordered counterpart of Theorem 1. In this way, we shall extend the main result of Agarwal, El-Gebeily and O'Regan in [6] (see also [17, Theorem 3.11]).

By an ordered (0-complete) partial metric space we mean a triple $(X, \preceq, p)$ such that $\preceq$ is a partial order on X and p is a (0-complete) partial metric on X.

An ordered partial metric space $(X, \preceq, p)$ is called regular if for any nondecreasing sequence $(x_n)_{n\in\omega}$ for $\preceq$, which converges to some $z \in X$ with respect to τ_p, it follows $x_n \preceq z$ for all $n \in \omega$.

We say that a self map f of a partial metric space (X, p) is continuous if it is continuous from (X, τ_p) into itself.

Theorem 3.1. *Let $(X, \preceq, p)$ be an ordered 0-complete partial metric space and $f : X \to X$ a nondecreasing map for $\preceq$, such that*

$$p(fx, fy) \leq \varphi(M(x, y)),$$

for all $x, y \in X$ with $x \preceq y$, where $\varphi : [0, \infty) \to [0, \infty)$ is a nondecreasing function such that $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$.

If there is $x_0 \in X$ such that $x_0 \preceq fx_0$, and f is continuous or $(X, \preceq, p)$ is regular, then f has a fixed point $z \in X$ such that $p(z, z) = 0$. Moreover, the set of fixed points of f is a singleton if and only it is well-ordered.

Proof. For each $n \in \omega$ put $x_n = f^n x_0$. Thus $x_{n+1} = f x_n$ for all $n \in \omega$. Since $x_0 \preceq f x_0$ and f is nondecreasing for $\preceq$, it follows that $x_n \preceq x_{n+1}$ for all $n \in \omega$, so $(x_n)_{n\in\omega}$ is a nondecreasing sequence in $(X, \preceq)$.

If there is $k \in \omega$ such that $x_k = x_{k+1}$, then x_k is a fixed point of f.

Hence, we shall assume that $f^n x_0 \neq f^{n+1} x_0$ for all $n \in \omega$. Thus $p(x_n, x_{n+1}) > 0$ for all $n \in \omega$. Then, the proof of Theorem 2.3 shows (note, in particular, that $x_n \preceq x_{n+k}$ for all $n, k \in \omega$) that $(x_n)_{n\in\omega}$ is a 0-Cauchy sequence in (X, p). Hence there is $z \in X$ such that

$$\lim_{n,m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(z, x_n) = p(z, z) = 0.$$

We show that z is a fixed point of f.

Indeed, if f is continuous, we deduce that $\lim_{n\to\infty} p(fz, x_n) = p(fz, fz)$. Since

$$p(z, fz) \leq p(z, x_n) + p(x_n, fz),$$

for all $n \in \omega$, it follows, taking limits as $n \to \infty$, that $p(z, fz) \leq p(fz, fz)$, so $p(z, fz) = p(fz, fz)$. Hence, since $z \preceq z$, we have

$$p(z, fz) \leq \varphi(M(z, z)) = \varphi(0) = 0,$$

and thus $z = fz$, and $p(z, z) = 0$.

If f is not continuous, it follows from regularity of $(X, \preceq, p)$ that $x_n \preceq z$ for all $n \in \omega$. Assume $p(z, fz) > 0$. Then, there is $n_0 \in \mathbb{N}$ such that $M(z, x_{n-1}) = p(z, fz)$ for all $n \geq n_0$. Thus

$$\begin{aligned} p(z, fz) &\leq p(z, x_n) + p(x_n, fz) \leq p(z, x_n) + \varphi(M(z, x_{n-1})) \\ &= p(z, x_n) + \varphi(p(z, x_{n-1})) \leq p(z, x_n) + p(z, x_{n-1}), \end{aligned}$$

for all $n \geq n_0$. Taking limits as $n \to \infty$, we deduce that $p(z, fz) = 0$, a contradiction. We conclude that $z = fz$.

Finally, if the set of fixed point is well-ordered and $u \in X$ is a fixed point of f, we deduce, assuming $u \preceq z$, that

$$p(u, z) = p(fu, fz) \leq \varphi(M(u, z)) = \varphi(p(u, z)),$$

so $p(u, z) = 0$, i.e., $u = z$. This concludes the proof. □

From [12, Lemma 1] and Theorem 3.1 we deduce the following ordered counterpart of Corollary 2.4.

Corollary 3.2. *Let $(X, \preceq, p)$ be an ordered 0-complete partial metric space and $f : X \to X$ a nondecreasing map for $\preceq$, map such that*

$$\psi(p(fx, fy)) \leq \psi(M(x, y)) - \phi(M(x, y)),$$

for all $x, y \in X$ with $x \preceq y$, where $\psi, \phi : [0, \infty) \to [0, \infty)$ are nondecreasing functions such that ψ is continuous on $[0, \infty)$ and $\psi^{-1}(0) = \phi^{-1}(0) = \{0\}$.

If there is $x_0 \in X$ such that $x_0 \preceq fx_0$, and f is continuous or $(X, \preceq, p)$ is regular, then f has a fixed point $z \in X$ such that $p(z, z) = 0$. Moreover, the set of fixed points of f is a singleton if and only it is well-ordered.

Remark 3.3. Theorem 3.1 improves [4, Theorems 2.1 and 2.2]. Note also that regularity of $(X, \preceq, p)$ can be replaced, in Theorem 3.1, by the more general condition that for any nondecreasing sequence $(x_n)_{n\in\omega}$ for $\preceq$, which converges to some $z \in X$ with respect to τ_{p^s}, it follows $x_n \preceq z$ for all $n \in \omega$. Moreover, continuity of f from (X, τ_p) into itself can be replaced by continuity from (X, τ_{p^s}) into itself.

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