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The ε-approximated complete invariance property

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Dedicated to my teacher and friend Prof. Dr. Gaspar Mora

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Abstract

In the present paper we introduce a generalization of the complete invariance property (CIP) for metric spaces, which we will call the ε-approximated complete invariance property (ε-ACIP). For our goals, we will use the so called degree of nondensifiability (DND) which, roughly speaking, measures (in the specified sense) the distance from a bounded metric space to its class of Peano continua. Our main result relates the ε-ACIP with the DND and, in particular, proves that a densifiable metric space has the ε-ACIP for each $\varepsilon > 0$. Also, some essentials differences between the CIP and the ε-ACIP are shown.

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1. Introduction

In 1973 Ward [20] introduced the following concept:

Definition 1.1. A topological space X has the complete invariance property (CIP) if for every non-empty and closed $C \subset X$ there is a continuous mapping $f : X \longrightarrow X$ such that $\mathrm{Fix}(f) = C$, where $\mathrm{Fix}(f)$ stands for the set of fixed points of f.

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As is mentioned in [8], some spaces known to have the CIP include n-cells, dendrites, convex subsets of Banach spaces, compact manifolds without boundary, and all compact triangulable manifolds with or without boundary.

It is convenient to recall that a Peano continuum is a compact, connected and locally connected metric space (X, d), or equivalently, by the Hahn-Mazurkiewicz Theorem (see, for instance, [19, 21]), X is the continuous image of the unit interval $I = [0, 1]$.

In [20] was asked the following:

Has every Peano continuum the CIP?

The answer is negative: in [8, 9] are given some examples of n-dimensional Peano continua, with $n > 1$, that fail to have the CIP. However, for $n = 1$ the situation is very different:

Theorem 1.2 (Martin and Tymchatyn [10], 1980)**.** *Every 1-dimensional Peano continuum has the CIP.*

Since the publication of the Ward's paper, many others works have been devoted to the study and analysis of the CIP and other issues related with it, see [2, 5, 6, 7, 11, 12, 13, 22] and references therein. So, it seems that the study of the CIP problem, and its variants, is an interesting and actual topic.

On the other hand, the so called *degree of nondensifiability* (DND), explained in detail in Section 2, has been used to prove, under suitable conditions, the existence of fixed points of continuous self mappings defined into a non-empty, bounded, closed and convex subset of a Banach space (see [3] and references therein). In the present paper, for a given metric space (X, d), we introduce the concept of *ε-approximated complete invariance property* (ε-ACIP), which generalizes the CIP one and, by using the DND, we relate in our main result (see Theorem 3.2) this novel concept with the DND of a bounded metric space. In particular, our main result proves that densifiable metric spaces (and therefore every Peano continuum) have the ε-ACIP for each $\varepsilon > 0$.

Also, and as consequence of our main result, we derive some properties for the ε-ACIP which are not satisfied by the CIP, namely, that the ε-ACIP is preserved (in the specified sense) by the countable or finite products of bounded metric spaces or by the continuous image of a bounded metric space.

2. The degree of nondensifiability

In this section, and for a better comprehension of the manuscript, we recall the concepts of α-dense curves and densifiable sets and also that of degree of nondensifiability. As in Section 1, (X, d) will be a metric space and we denote by $\mathcal{B}(X)$ the class of non-empty and bounded subsets of X.

In 1997 Cherruault and Mora introduced in [15] the following concepts:

Definition 2.1. Let $\alpha \geq 0$ and $B \in \mathcal{B}(X)$. A continuous mapping $\gamma : I \longrightarrow (X, d)$ is said to be an α-dense curve in B if it satisfies:

(i) $\gamma(I) \subset B$.

(ii) For any $x \in B$ there is $y \in \gamma(I)$ such that $d(x, y) \leq \alpha$.

The class of α-dense curves in B is denoted by $\Gamma_{\alpha,B}$. The set B is said to be densifiable if $\Gamma_{\alpha,B} \neq \varnothing$ for each $\alpha > 0$.

For a detailed exposition of the α-dense curves and densifiable sets, see [1, 14, 17]. Some comments are necessary before to continue:

(I) Let us note that, given $B \in \mathcal{B}(X)$, $\Gamma_{\alpha,B} \neq \varnothing$ for each $\alpha \geq \mathrm{Diam}(B)$, the diameter of B. Indeed, fixed $x_0 \in B$, the mapping $\gamma(t) = x_0$ is an α-dense curve in B for each $\alpha \geq \mathrm{Diam}(B)$.

(II) If $B = I^n$ for some integer $n > 1$ then a 0-dense curve is, precisely, a *space-filling curve* (see [19]), i.e. a continuous mapping from I onto I^n. So, we can say that the α-dense curves are a generalization of the space-filling curves.

(III) By recalling that the Hausdorff distance between $B_1, B_2 \in \mathcal{B}(X)$ is given by

$$d_{\mathrm{H}}(B_1, B_2) = \max\big\{ \sup_{b_1 \in B_1} \inf_{b_2 \in B_2} d(b_1, b_2), \sup_{b_2 \in B_2} \inf_{b_1 \in B_1} d(b_1, b_2) \big\},$$

is clear that if γ is an α-dense curve in $B \in \mathcal{B}(X)$, then $d_{\mathrm{H}}(B, \gamma(I)) \leq \alpha$. We also recall that d_{H} is pseudometric, and is a metric if X is complete, and a metric in the class of non-empty, bounded and closed subsets of X.

Next, we show some examples.

Example 2.2 (A compact and connected but not densifiable set)**.** Let, in the Euclidean plane, the set

$$B = \big\{(x, \sin(1/x)) : x \in [-1, 0) \cup (0, 1]\big\} \bigcup \big\{(0, y) : y \in [-1, 1]\big\}.$$

Then, given any continuous $\gamma : I \longrightarrow \mathbb{R}^2$ with $\gamma(I) \subset B$, $\gamma(I)$ has to be contained is some of the three connected components of B. So, if $0 < \alpha < 1$, there is not an α-dense curve in B, and consequently B is not densifiable.

Example 2.3 (A densifiable set without the CIP)**.** Consider, in the Euclidean plane, the sets

$$B_1 = \big\{(x, \sin(1/x)) : x \in (0, 1]\big\}, \quad B_2 = \big\{(0, y) : y \in [-1, 1]\big\},$$

and let $B = B_1 \cup B_2$, often called *the topologist's sine*. Then, is easy to prove that B is densifiable. In the following lines we will show that B has not the CIP.

Define the set

$$C = B \cap (I \times [-1, 0]),$$

and assume that there is a continuous $f : B \longrightarrow B$ such that $f(C) = C$. As $C \cap B_1 = f(C \cap B_1) \subset f(B_1)$ and $f(B_1)$ is path-wise connected, $f(B_1) = B_1$. Hence, as B is compact, we have $B = \overline{B}_1 = \overline{f(B_1)} \subset \overline{f(B)} = f(B)$, where the bar stands for the closure. This means that f is surjective and therefore $f(B_2) = B_2$.

So, there is a continuous surjection $\varphi : [-1, 1] \longrightarrow [-1, 1]$ such that $f(0, y) = (0, \varphi(y))$ for all $y \in [-1, 1]$. Hence there exists $a \in [-1, 1]$ such that $\varphi(a) = 1$.

Set $b = \varphi(1)$. As $[-1,0]$ is the set of fixed points of φ, we conclude that $a \in (0,1)$ and $b \in [-1,1)$.

Next, define $\psi : [a,1] \longrightarrow [-1,1]^2$ as $\psi(x) = (x, \varphi(x))$ and denote by Δ the diagonal of $[-1,1]^2$. Let us note that $\psi([a,1]) \cap \Delta = \varnothing$ because φ does not have any fixed point in $[a,1]$. Hence, ψ is a *path* (see the below definition) in $[-1,1] \setminus \Delta$.

But, the set $[-1,1] \setminus \Delta$ has two components Ω_1 and Ω_2 which are above and below of Δ, respectively. Then, $\psi(a) = (a,1) \in \Omega_1$ and $\psi(1) = (1,b) \in \Omega_2$, which is contradictory. So, B does not have the CIP as claimed.

Following Willard [21], we recall that a topological space Y is said to be *path-wise connected* (resp. *arc-wise connected*) if for any $x, y \in B$ there is a continuous (resp. a one-to-one continuous) $f : I \longrightarrow B$, often called *path* (resp. *arc*) such that $f(0) = x$ and $f(1) = y$. However, if Y is a Hausdorff space (and, in particular, a metric space), both concepts are equivalents (see [21, Corollary 31.6]. Here, as we work with metric spaces, for our goals is more convenient to use the term arc-wise connected.

At this point, we can state the following result (see [17]):

Proposition 2.4. *Let $B \in \mathcal{B}(X)$ be arc-wise connected. Then B is densifiable if, and only if, it is precompact.*

Although, by the Hahn-Mazurkiewicz Theorem, I^n is a Peano continuum and in particular densifiable, the above result also demonstrates us that I^n is densifiable. Moreover, we can give an explicit expression of an α-dense curve in I^n, γ, for an arbitrarily small $\alpha > 0$, such that $\gamma(I)$ is also a 1-dimensional Peano continuum:

Example 2.5 (1-dimensional Peano continua densifying I^n)**.** Fixed $n > 1$, for a given integer $k \geq 1$ define $\gamma_k : I \longrightarrow \mathbb{R}^n$ as

$$\gamma_k(t) = \big(t, \frac{1}{2}(1 - \cos(\pi m t)), \ldots, \frac{1}{2}(1 - \cos(\pi m^{k-1} t))\big),$$

for all $t \in I$. Then, γ_k is a $\frac{\sqrt{n-1}}{k}$-dense curve in I^n (see [1, Proposition 9.5.4])

Remark 2.6. Other examples of α-dense curves in more general subsets of $\mathbb{R}^n$ than I^n can be found in [18].

From the concepts of α-dense curves, we can define the so called *degree of nondensifiability*, which was introduced by Mora and Mira in [16] and analyzed in [4]:

Definition 2.7. Given $B \in \mathcal{B}(X)$, we define the degree of nondensifiability, DND, of B as

$$\phi_d(B) = \inf \big\{ \alpha \geq 0 : \Gamma_{\alpha,B} \neq \varnothing \big\}.$$

As we have pointed out above, $\Gamma_{\alpha,B} \neq \varnothing$ for each $\alpha \geq \mathrm{Diam}(B)$ and therefore the DND is well defined. Also, let us note that, for a given $B \in \mathcal{B}(X)$, $\phi_d(B)$

measures (in the specified sense) the distance from B to the class of its Peano continua.

Example 2.8 (see [16]). Let B be the closed unit ball of a Banach space V, and d the distance in V induced by its norm. Then,

$$\phi_d(B) = \begin{cases} 0, & \text{if } V \text{ is finite dimensional} \\ \\ 1, & \text{if } V \text{ is infinite dimensional} \end{cases} .$$

Some properties of the DND are given in the next result. (see [4, 16]).

Proposition 2.9. *The DND satisfies the following:*

(1) If $\phi_d(B) = 0$, then B is precompact. Moreover, if B is precompact and arc-wise connected then $\phi_d(B) = 0$.
(2) $\phi_d(B) = \phi_d(\overline{B})$, for each $B \in \mathcal{B}(X)$ where, as usual, the bar stands for the closure.

On the other hand, for our main result we will use Theorem 1.2 and the DND. So, we will need that the α-dense curves used in the definition of the DND be 1-dimensional Peano continua. Note that, *a priori*, an α-dense curve is not necessarily a 1-dimensional Peano continua: for instance, a n-dimensional Peano continua or, in particular, the space-filling curves in I^n given in [19]. However, in the next result, we prove that the DND can be defined by means of α-curves such that the image of I under these curves be a 1-dimensional Peano continua.

Theorem 2.10. *Given $B \in \mathcal{B}(X)$ and $\alpha > 0$, let $\Gamma^{(1)}_{\alpha,B} \subset \Gamma_{\alpha,B}$ be the class of α-dense curves in B such that $\gamma^{(1)}(I)$ is a 1-dimensional Peano continuum for all $\gamma^{(1)} \in \Gamma^{(1)}_{\alpha,B}$. By putting*

$$\phi^{(1)}_d(B) = \inf \left\{ \alpha \geq 0 : \Gamma^{(1)}_{\alpha,B} \neq \varnothing \right\},$$

we have $\phi_d(B) = \phi^{(1)}_d(B)$.

Proof. Let α be such that $\alpha > \phi_d(B)$ and $\gamma : I \longrightarrow (X, d)$ an α-dense curve in B. So, by the compactness of $\gamma(I)$, given any $\varepsilon > 0$ there exists a finite set $\{y_1, \ldots, y_n\} \subset \gamma(I)$ (without loss of generality we assume $n > 1$) such that

$$B \subset \bigcup_{i=1}^{n} \bar{B}_d(y_i, \alpha + \varepsilon), \tag{2.1}$$

$\bar{B}_d(y_i, \alpha + \varepsilon)$ being the closed ball centered at y_i of radius $\alpha + \varepsilon$.

As $\gamma(I)$ is a Peano continuum it is arc-wise connected (see, for instance, [21, Theorem 31.2]), for each $i = 1, \ldots, n - 1$ there exists a one-to-one continuous $h_i : I \longrightarrow \gamma(I)$ with $h_i(0) = y_i$ and $h_i(1) = y_{i+1}$. In particular, each $h_i(I)$ is a 1-dimensional Peano continuum, for $i = 1, \ldots, n$. Define, for each $i = 1, \ldots, n-1$, the one-to-one continuous $\tau_i : I \longrightarrow [\frac{i-1}{n-1}, \frac{i}{n}]$ as $\tau_i(t) = \frac{i-1+t}{n-1}$ for all $t \in I$. Then, the mapping $\gamma^{(1)} : I \longrightarrow (X, d)$ given by

$$\gamma^{(1)}(t) = h_i\big(\tau_i(t)\big), \quad \text{for } t \in [\frac{i-1}{n-1}, \frac{i}{n}], \quad i = 1, \dots, n-1,$$

is continuous, $\gamma^{(1)}(I) \subset \gamma(I) \subset B$ and $\gamma^{(1)}(I)$ is a 1-dimensional Peano continuum because it is the finite union of 1-dimensional Peano continua. Also, from (2.1) we have $\gamma^{(1)} \in \Gamma^{(1)}_{\alpha+\varepsilon,B}$. By the arbitrariness of $\varepsilon > 0$, we conclude that $\phi^{(1)}_d(B) \leq \alpha$ and by the arbitrariness of $\alpha > \phi_d(B)$, the inequality $\phi^{(1)}_d(B) \leq \phi_d(B)$ holds.

On the other hand, if $\gamma \in \Gamma^{(1)}_{\alpha,B}$, from the inclusion $\Gamma^{(1)}_{\alpha,B} \subset \Gamma_{\alpha,B}$, we have $\gamma \in \Gamma_{\alpha,B}$. Thus, $\phi_d(B) \leq \phi^{(1)}_d(B)$ and the proof is now complete.

□

To conclude this section, we give a result for the DND of the product of bounded metric spaces.

Proposition 2.11. *Let Λ be a finite set or $\Lambda = \mathbb{N}$, and $(X_\lambda, d_\lambda)_{\lambda\in\Lambda}$ a family of metric spaces such that* $\mathrm{Diam}(X_\lambda) \leq M$ *for certain $M > 0$ and all $\lambda \in \Lambda$. Put $\phi^* = \sup\{\phi_{d_\lambda}(X_\lambda) : \lambda \in \Lambda\}$, $X^* = \prod_{\lambda\in\Lambda} X_\lambda$ and $d^*(x, y) = \max\{d_\lambda(x, y) : \lambda \in \Lambda\}$ if Λ is finite or $d^*(x, y) = \sum_{k\geq1} 2^{-k} d_k(x, y)$ if $\Lambda = \mathbb{N}$, for all $x, y \in X^*$. Then,*

$$\phi_{d^*}(X^*) \leq \phi^*.$$

Moreover if Λ is finite, then the equality holds.

Proof. Firstly, note that (X^*, d^*) is, effectively, a bounded metric space and therefore $\phi_{d^*}(X^*)$ is well defined (in fact, $\phi_{d^*}(X^*) \leq M$).

Assume, $\Lambda = \mathbb{N}$ and let $\alpha > \phi^*$. Let, for each $k \geq 1$, $\gamma_k : I \longrightarrow X_k$ an α-dense curve in X_k. So, for each $k \geq 1$, given $x_k \in X_k$ there is $t_k \in I$ such that

$$d_k(x_k, \gamma_k(t_k)) \leq \alpha. \tag{2.2}$$

Let $\omega = (\omega_k)_{k\geq1} : I \longrightarrow I^{\mathbb{N}}$ be a space-filling curve (see [19, Section 7.5]). That is, ω (and hence each *coordinate function* ω_k) is continuous and $\omega(I) = I^{\mathbb{N}}$. Define $\gamma : I \longrightarrow X^*$ as

$$\gamma(t) = \big(\gamma_k(\omega_k(t))\big)_{k\geq1}, \quad \text{for all } t \in I.$$

It is clear that γ is continuous and $\gamma(I) \subset X^*$. Also, given $(x_k)_{k\geq1} \in X^*$ take $(t_k)_{k\geq1} \subset I$ satisfying (2.2) and $t \in I$ such that $\omega(t) = (t_k)_{k\geq1}$. So, we have

$$d^*\big((x_k)_{k\geq1}, \gamma(t)\big) = \sum_{k\geq1} \frac{d_k\big(x_k, \gamma_k(\omega_k(t))\big)}{2^k} = \sum_{k\geq1} \frac{d_k\big(x_k, \gamma_k(t_k)\big)}{2^k} \leq \alpha.$$

and consequently γ is an α-dense curve in X^*. Then, $\phi_{d^*}(X^*) \leq \alpha$ and letting $\alpha \to \phi^*$, we conclude $\phi_{d^*}(X^*) \leq \phi^*$.

If Λ is finite, without loss of generality we assume $\Lambda = \{1, \dots, n\}$ for some $n > 1$, we take $\omega = (\omega_1, \dots, \omega_n) : I \longrightarrow I^n$ a space-filling curve (again, [19]) and the proof follows in a totally analogous way that above.

Assume $\phi_{d^*}(X^*) < \phi^*$ and take $\phi_{d^*}(X^*) < \alpha < \phi^*$ and an α-dense curve in X^*, put $\gamma = (\gamma_1, \dots, \gamma_n) : I \longrightarrow (X^*, d^*)$. Then, fixed $1 \leq k \leq n$, the mapping $\gamma_k : I \longrightarrow (X_k, d_k)$ is continuous and one can check straightforwardly that it is an α-dense curve in X_k. But, this is not possible as $\alpha < \phi^* \leq \phi_{d_k}(X_k)$.

□

3. The main result

We start this section with the following definition:

Definition 3.1. Given $\varepsilon \geq 0$, we will say that a metric space (X, d) has the ε-approximated complete invariance property (ε-ACIP) if for each non-empty and closed $C \subset X$ there is a continuous $f_\varepsilon : X \longrightarrow X$ such that $d_{\mathrm{H}}(C, \mathrm{Fix}(f_\varepsilon)) \leq \varepsilon$.

The following facts are clear from the definitions:

(I) If (X, d) is bounded, then (X, d) has ε-ACIP for every $\varepsilon \geq \mathrm{Diam}(X)$.

(II) The 0-ACIP is, precisely, the CIP. Also, the CIP implies the ε-ACIP for each $\varepsilon > 0$, but as we will see below, the inverse implication does not hold in general. That is to say, there are metric spaces with the ε-ACIP for all $\varepsilon > 0$, but such metric spaces do not have the CIP.

Now, we are ready to state and prove our main result:

Theorem 3.2. *Let (X, d) a bounded metric space. Then, (X, d) has the ε-ACIP for each $\varepsilon > \phi_d(X)$. In particular, if X is densifiable then it has the ε-ACIP for each $\varepsilon > 0$.*

Proof. Let ε be such that $\varepsilon > \phi(X)$. Let any $C \subset X$ non-empty and closed, and $\gamma_\varepsilon : I \longrightarrow (X, d)$ and ε-dense curve such that $\gamma_\varepsilon(I)$ is a 1-dimensional Peano continuum. Such ε-dense curve exists by virtue of Theorem 2.10.

Define the set

$$G_C = \overline{\left\{x \in \gamma_\varepsilon(I) : d(x, c) \leq \varepsilon, \text{ for some } c \in C\right\}} \subset X.$$

It is clear that the set G_C is non-empty and closed. Thus, by Theorem 1.2, there is $f_\varepsilon : X \longrightarrow X$ with $\mathrm{Fix}(f_\varepsilon) = G_C$.

Now, let $c \in C$. As γ_ε is an ε-dense curve in X, there is $x \in \gamma_\varepsilon(I)$ with $d(x, c) \leq \varepsilon$. Then, $x \in G_C$ and therefore $x = f_\varepsilon(x)$. So, we have $\inf_{x \in \mathrm{Fix}(f_\varepsilon)} d(c, x) \leq \varepsilon$ and from the arbitrariness of $c \in C$, we infer

$$\sup_{c \in C} \inf_{x \in \mathrm{Fix}(f_\varepsilon)} d(c, x) \leq \varepsilon. \tag{3.1}$$

Likewise for a given $x \in \mathrm{Fix}(f_\varepsilon)$, as $x \in G_C$, $d(c, x) \leq \varepsilon$ for some $c \in C$. Consequently, $\inf_{c \in C} d(c, x) \leq \varepsilon$ and noticing the arbitrariness of $x \in \mathrm{Fix}(f_\varepsilon)$

$$\sup_{x \in \mathrm{Fix}(f_\varepsilon)} \inf_{c \in C} d(c, x) \leq \varepsilon. \tag{3.2}$$

So, from (3.1) and (3.2), we have $d_{\mathrm{H}}(C, \mathrm{Fix}(f_\varepsilon)) \leq \varepsilon$.

If X is densifiable then, by the definition of the DND, $\phi_d(X) = 0$ and therefore has the ε-ACIP for each $\varepsilon > 0$.

□

An immediate consequence of the above result is the following:

Corollary 3.3. *Every Peano continuum has the ε-ACIP for each $\varepsilon > 0$.*

As we have said above, in general, the ε-ACIP for each $\varepsilon > 0$ does not imply the CIP. We illustrate this fact in the following examples.

Example 3.4. Let X be the topologist's sine of Example 2.3. Then, X is densifiable but does not have the CIP. However, by Theorem 3.2 X has the ε-ACIP for each $\varepsilon > 0$.

Example 3.5. Let X be a n-dimensional Peano continuum without the CIP (see [8, 9]). Then, by Corollary 3.3, X has the ε-ACIP for each $\varepsilon > 0$.

So, in general, we cannot replace the condition $\varepsilon > \phi_d(X)$ by $\varepsilon \geq \phi_d(X)$ in Theorem 3.2. This fact is explained by the following ones:

(I) There is not necessarily a $\phi_d(X)$-dense curve in X. Indeed, for instance, the topologist's sine X of Example 2.3 satisfies $\phi_d(X) = 0$ but there is not a 0-dense curve in X: otherwise, X would be a Peano continuum, which is not possible because it is not locally connected.

(II) Even if $X = \gamma(I)$, for certain continuous $\gamma : I \longrightarrow X$, if $\gamma(I)$ is not a 1-dimensional Peano continuum, we cannot apply Theorem 1.2 in the proof of Theorem 3.2 to derive that X has the CIP (see also Example 3.5).

We have remarked above that if (X, d) is bounded, then (X, d) has ε-ACIP for every $\varepsilon \geq \mathrm{Diam}(X)$. This bound can be improved by Theorem 3.2:

Example 3.6. Let X be the set given in Example 2.2. Then, $\mathrm{Diam}(X) = 2$ and $\phi_d(X) = 1$. So, by Theorem 3.2, X has ε-ACIP for every $\varepsilon > 1$.

As was proved in [6], the CIP need not be preserved by self-products. However, bearing in mind Proposition 2.11 and Theorem 3.2, we have the following result for the product of bounded metric spaces:

Corollary 3.7. *With the notation of Proposition 2.11, (X^*, d^*) has the ε-ACIP for each $\varepsilon > \phi^*$. In particular, the finite or countable product of Peano continua has the ε-ACIP for each $\varepsilon > 0$.*

Example 3.8. Let (X, d) be the 1-dimensional Peano continuum given in [6, Theorem 2.2]. Then, $X \times X$ does not have the CIP. However, by Corollary 3.7, $X \times X$ has the ε-ACIP for each $\varepsilon > 0$.

Also, the CIP need not to be preserved by continuous mappings. Indeed, take any metric space (X, d) that does not have the CIP and τ the discrete topology on X. Then, (X, τ) has the CIP and the identity mapping $g : (X, \tau) \longrightarrow (X, d)$ is continuous. However, for the ε-ACIP we have the following:

Corollary 3.9. *Let (X,d) and (Y,d') be bounded metric spaces and $g:(X,d)\longrightarrow (Y,d')$ continuous. Then (Y,d') has the ε-ACIP for each $\varepsilon>\omega_{\phi_d(X)}(g)$, where*

$$\omega_r(g)=\sup\left\{d'(f(x),f(y)):x,y\in X, d(x,y)\le r\right\},$$

is the modulus of continuity of g of order r, for $r\ge 0$.

Proof. It is immediate to check that if $\gamma: I\longrightarrow (X,d)$ is an α-dense curve in (X,d), then $g\circ\gamma: I\longrightarrow (Y,d')$ is a $\omega_\alpha(g)$-dense curve in (Y,d'). Therefore, we infer that $\phi_{d'}(Y)\le\omega_{\phi_d(X)}(g)$ and by Theorem 3.2, (Y,d') has the ε-ACIP for each $\varepsilon>\omega_{\phi_d(X)}(g)$.

□

On the other hand, it is important to stress that the reciprocal of Theorem 3.2 is not true in general: there are metric spaces with the ε-ACIP for all $\varepsilon>0$ (in fact, with the CIP) that are not densifiable:

Example 3.10. Let X be the closed unit ball of an infinite dimensional Banach space. From the comments of Section 1, X has the CIP and therefore the ε-ACIP for all $\varepsilon>0$. However, from Example 2.8, $\phi_d(X)=1$ and noticing Proposition 2.9 X is not densifiable.

So, we conclude our exposition with the following question:

If (X,d) is a bounded metric space having the ε-ACIP, for some $\varepsilon>0$, under what conditions can we relate, in some way, ε and $\phi_d(X)$?

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