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Convergence semigroup actions: generalized quotients

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Abstract. Continuous actions of a convergence semigroup are investigated in the category of convergence spaces. Invariance properties of actions as well as properties of a generalized quotient space are presented

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1. Introduction

The notion of a topological group acting continuously on a topological space has been the subject of numerous research articles. Park [8, 9] and Rath [10] studied these concepts in the larger category of convergence spaces. This is a more natural category to work in since the homeomorphism group on a space can be equipped with a coarsest convergence structure making the group operations continuous. Moreover, unlike in the topological context, quotient maps are productive in the category of all convergence spaces with continuous maps as morphisms. This property plays a key role in the proof of several results contained herein; for example, Theorem 4.11.

Given a topological semigroup acting on a topological space, Burzyk et al. [1] introduced a "generalized quotient space." Elements of this space are equivalence classes determined by an abstraction of the method used to construct the rationals from the integers. General quotient spaces are used in the study of generalized functions [5, 6, 7].

Generalized quotients in the category of convergence spaces are studied in section 4. First, invariance properties of continuous actions of convergence semigroups on convergence spaces are investigated in section 3.

2. Preliminaries

Basic definitions and concepts needed in the area of convergence spaces are given in this section. Let X be a set, 2^X the power set of X, and let $\mathfrak{F}(X)$ denote the set of all filters on X. Recall that $\mathfrak{B} \subseteq 2^X$ is a base for a filter on X provided $\mathfrak{B} \neq \varnothing$, $\varnothing \notin \mathfrak{B}$, and $B_1, B_2 \in \mathfrak{B}$ implies that there exists $B_3 \in \mathfrak{B}$ such that $B_3 \subseteq B_1 \cap B_2$. Moreover, $[\mathfrak{B}]$ denotes the filter on X whose base is $\mathfrak{B}$; that is, $[\mathfrak{B}] = \{A \subseteq X : B \subseteq A \text{ for some } B \in \mathfrak{B}\}$. Fix $x \in X$, define $\dot{x}$ to be the filter whose base is $\mathfrak{B} = \{\{x\}\}$. If $f : X \to Y$ and $\mathcal{F} \in \mathfrak{F}(X)$, then $f^{\to}\mathcal{F}$ denotes the image filter on Y whose base is $\{f(F) : F \in \mathcal{F}\}$.

A **convergence structure** on X is a function $q : \mathfrak{F}(X) \to 2^X$ obeying :
(CS1) $x \in q(\dot{x})$ for each $x \in X$
(CS2) $x \in q(\mathcal{F})$ implies that $x \in q(\mathcal{G})$ whenever $\mathcal{F} \subseteq \mathcal{G}$.
The pair (X, q) is called a **convergence space**. The more intuitive notation $\mathcal{F} \xrightarrow{q} x$ is used for $x \in q(\mathcal{F})$. A map $f : (X, q) \to (Y, p)$ between two convergence spaces is called **continuous** whenever $\mathcal{F} \xrightarrow{q} x$ implies that $f^{\to}\mathcal{F} \xrightarrow{p} f(x)$. Let **CONV** denote the category whose objects consist of all the convergence spaces, and whose morphisms are all the continuous maps between objects. The collection of all objects in CONV is denoted by $|\text{CONV}|$. If p and q are two convergence structures on X, then $\boldsymbol{p \leq q}$ means that $\mathcal{F} \xrightarrow{p} x$ whenever $\mathcal{F} \xrightarrow{q} x$. In this case, $p(q)$ is said to be **coarser**(**finer**) than $q(p)$, respectively. Also, for $\mathcal{F}, \mathcal{G} \in \mathfrak{F}(X)$, $\boldsymbol{\mathcal{F} \leq \mathcal{G}}$ means that $\mathcal{F} \subseteq \mathcal{G}$, and $\mathcal{F}(\mathcal{G})$ is called **coarser**(**finer**) than $\mathcal{G}(\mathcal{F})$, respectively.

It is well-known that CONV possesses initial and final convergence structures. In particular, if $(X_j, q_j) \in |\text{CONV}|$ for each $j \in J$, then the **product convergence structure** r on $X = \underset{j \in J}{\times} X_j$ is given by $\mathcal{H} \xrightarrow{r} x = (x_j)$ iff $\pi_j^{\to}\mathcal{H} \xrightarrow{q_j} x_j$ for each $j \in J$, where π_j denotes the $\text{j}^{\underline{\text{th}}}$ projection map. Also, if $f : (X, q) \to Y$ is a surjection, then the **quotient convergence structure** σ on Y is given by $\mathcal{H} \xrightarrow{\sigma} y$ iff there exists $x \in f^{-1}(y)$ and $\mathcal{F} \xrightarrow{q} x$ such that $f^{\to}\mathcal{F} = \mathcal{H}$. In this case, σ is the finest convergence structure on Y making $f : (X, q) \to (Y, \sigma)$ continuous.

Unlike the category of all topological spaces, CONV is cartesian closed and thus has suitable function spaces. In particular, let (X, q), $(Y, p) \in |\text{CONV}|$ and let $C(X, Y)$ denote the set of all continuous functions from X to Y. Define $\omega : (X, q) \times C(X, Y) \to (Y, p)$ to be the evaluation map given by $\omega(x, f) = f(x)$. There exists a coarsest convergence structure **c** on $C(X, Y)$ such that w is jointly continuous. More precisely, c is defined by : $\Phi \xrightarrow{c} f$ iff $w^{\to}(\mathcal{F} \times \Phi) \xrightarrow{p} f(x)$ whenever $\mathcal{F} \xrightarrow{q} x$. This compatibility between (X, q) and $(C(X, Y), c)$ is an example of a continuous action in CONV discussed in section 3. Continuous actions which are invariant with respect to a convergence space property P are studied in section 3. Choices for P include : locally compact, locally bounded, regular, Choquet(pseudotopological), and first-countable.

An object $(X, q) \in |\text{CONV}|$ is said to be **locally compact** (**locally bounded**) if $\mathcal{F} \xrightarrow{q} x$ implies that $\mathcal{F}$ contains a compact (bounded) subset of X, respectively. A subset B of X is **bounded** provided that each ultrafilter containing B q-converges in X. Further, (X, q) is called **regular** (**Choquet**) provided $\text{cl}_q\mathcal{F} \xrightarrow{q} x$ ($\mathcal{F} \xrightarrow{q} x$) whenever $\mathcal{F} \xrightarrow{q} x$ (each ultrafilter containing $\mathcal{F}$ q-converges to x), respectively. Here $\text{cl}_q\mathcal{F}$ denotes the filter on X whose base is $\{\text{cl}_q F : F \in \mathcal{F}\}$. Some authors use the term "pseudotopological space" for a Choquet space. Finally, (X, q) is said to be **first-countable** whenever $\mathcal{F} \xrightarrow{q} x$ implies the existence of a coarser filter on X having a countable base and q-converging to x.

Let **SG** denote the category whose objects consist of all the semigroups (with an identity element), and whose morphisms are all the homomorphisms between objects. Further, $(S, ., p)$ is said to be a **convergence semigroup** provided : $(S, .) \in |\text{SG}|$, $(S, p) \in |\text{CONV}|$, and $\gamma : (S, p) \times (S, p) \to (S, p)$ is continuous, where $\gamma(x, y) = x.y$. Let **CSG** be the category whose objects consist of all the convergence semigroups, and whose morphisms are all the continuous homomorphisms between objects.

3. Continuous Actions

An action of a semigroups on a topological space is used to define "generalized quotients" in [1]. Below is Rath's [10] definition of an action in the convergence space context. Let $(X, q) \in |\text{CONV}|$, $(S, ., p) \in |\text{CSG}|$, $\lambda : X \times S \to X$, and consider the following conditions :

(a1) $\lambda(x, e) = x$ for each $x \in X$ (e is the identity element)
(a2) $\lambda(\lambda(x, g), h) = \lambda(x, g.h)$ for each $x \in X$, $g, h \in S$
(a3) $\lambda : (X, q) \times (S, ., p) \to (X, q)$ is continuous.

Then $(S, .)((S, ., p))$ is said to **act**(**act continuously**) on (X, q) whenever a1-a2 (a1-a3) are satisfied and, in this case, λ is called the **action** (**continuous action**), respectively. For sake of brevity, $(X, S) \in \mathbf{A}(\mathbf{AC})$ denotes the fact that $(S, ., p) \in |\text{CSG}|$) acts (acts continuously) on $(X, q) \in |\text{CONV}|$, respectively. Moreover, $\boldsymbol{(X, S, \lambda)} \in \text{A}$ indicates that the action is λ.

The notion of "generalized quotients" determined by commutative semigroup acting on a topological space is investigated in [1]. Elements of the semigroup in [1] are assumed to be injections on the given topological space.

Lemma 3.1 ([1]). *Suppose that $(S, X, \lambda) \in A$, $(S, .)$ is commutative and $\lambda(., g) : X \to X$ is an injection, for each $g \in S$. Define $(x, g) \sim (y, h)$ on $X \times S$ iff $\lambda(x, h) = \lambda(y, g)$. Then $\sim$ is an equivalence relation on $X \times S$.*

In the context of Lemma 3.1, let $\langle (x, g) \rangle$ be the equivalence class containing (x, g), $\boldsymbol{B(X, S)}$ denote the quotient set $(X \times S)/\sim$, and define $\varphi : (X \times S, r) \to B(X, S)$ to be the canonical map, where $r = q \times p$ is the product convergence structure. Equip $B(X, S)$ with the convergence quotient structure $\boldsymbol{\sigma}$. Then

$\mathcal{K} \xrightarrow{\sigma} \langle (y,h) \rangle$ iff there exist $(x,g) \sim (y,h)$ and $\mathcal{H} \xrightarrow{r} (x,g)$ such that $\varphi^{\rightarrow}\mathcal{H} = \mathcal{K}$. The space $(B(X,S),\sigma)$ is investigated in section 4.

Remark 3.2. Fix a set X. the set of all convergence structures on X with the ordering $p \leq q$ defined in section 2 is a complete lattice. Indeed, if $(X,q_j) \in |\mathrm{CONV}|$, $j \in J$, then $\sup\limits_{j\in J} q_j = q^1$ is given by $\mathcal{F} \xrightarrow{q^1} x$ iff $\mathcal{F} \xrightarrow{q_j} x$, for each $j \in J$. Dually, $\inf\limits_{j\in J} q_j = q^0$ is defined by $\mathcal{F} \xrightarrow{q^0} x$ iff $\mathcal{F} \xrightarrow{q_j} x$, for some $j \in J$. It is easily verified that if $((X,q_j),(S,.,p),\lambda) \in \mathrm{AC}$ for each $j \in J$, then both $((X,q^1),(S,.,p),\lambda)$ and $((X,q^0),(S,.,p),\lambda)$ belong to AC.

Theorem 3.3. *Assume that* $((X,q),(S,.,p),\lambda) \in AC$. *Then*

(a) *there exists a finest convergence structure* q^F *on* X *such that* $((X,q^F),(S,.,p),\lambda) \in AC$
(b) *there exists a coarsest convergence structure* p^c *on* S *for which* $((X,q),(S,.,p^c),\lambda) \in AC$
(c) $((B(X,S),\sigma),(S,.,p)) \in AC$ *provided* $(S,.)$ *is commutative and* $\lambda(.,g)$ *is an injection, for each* $g \in S$.

Proof. (a): Define q^F as follows: $\mathcal{F} \xrightarrow{q^F} x$ iff there exist $z \in X$, $\mathcal{G} \xrightarrow{p} g$ such that $x = \lambda(z,g)$ and $\mathcal{F} \geq \lambda^{\rightarrow}(\dot{z} \times \mathcal{G})$. Then $(X,q^F) \in |\mathrm{CONV}|$. Indeed, $\dot{x} \xrightarrow{q^F} x$ since $x = \lambda(x,e)$ and $\dot{x} = \lambda^{\rightarrow}(\dot{x} \times \dot{e})$. Hence (CS1) is satisfied. Clearly (CS2) is valid, and $(X,q^F) \in |\mathrm{CONV}|$.

It is shown that $\lambda : (X,q^F) \times (S,p) \to (X,q^F)$ is continuous. Suppose that $\mathcal{F} \xrightarrow{q^F} x$ and $\mathcal{H} \xrightarrow{p} h$; then there exist $z \in X$, $\mathcal{G} \xrightarrow{p} g$ such that $x = \lambda(z,g)$ and $\mathcal{F} \geq \lambda^{\rightarrow}(\dot{z} \times \mathcal{G})$. Hence, $\mathcal{F} \times \mathcal{H} \geq \lambda^{\rightarrow}(\dot{z} \times \mathcal{G}) \times \mathcal{H}$, and employing (a2), $\lambda^{\rightarrow}(\mathcal{F} \times \mathcal{H}) \geq \lambda^{\rightarrow}(\lambda^{\rightarrow}(\dot{z} \times \mathcal{G}) \times \mathcal{H}) = [\{\lambda(\{z\} \times G.H) : G \in \mathcal{G}, H \in \mathcal{H}\}] = \lambda^{\rightarrow}(\dot{z} \times \mathcal{G}.\mathcal{H})$. Since $\mathcal{G}.\mathcal{H} \xrightarrow{p} g.h$ and $\lambda(z,g.h) = \lambda(\lambda(z,g),h) = \lambda(x,h)$, it follows from the definition of q^F that $\lambda^{\rightarrow}(\mathcal{F} \times \mathcal{H}) \xrightarrow{q^F} \lambda(x,h)$. Hence $((X,q^F),(S,.,p),\lambda) \in \mathrm{AC}$.

Assume that $((X,s),(S,.,p),\lambda) \in \mathrm{AC}$. It is shown that $s \leq q^F$. Suppose that $\mathcal{F} \xrightarrow{q^F} x$; then there exist $z \in X$, $\mathcal{G} \xrightarrow{p} g$ such that $x = \lambda(z,g)$ and $\mathcal{F} \geq \lambda^{\rightarrow}(\dot{z}\times\mathcal{G})$. Since $\lambda^{\rightarrow}(\dot{z} \times \mathcal{G}) \xrightarrow{s} \lambda(z,g)$, it follows that $\mathcal{F} \xrightarrow{s} x$ and thus $s \leq q^F$. Hence q^F is the finest convergence structure on X such that $((X,q^F),(S,.,p),\lambda) \in \mathrm{AC}$.

(b): Define p^c as follows: $\mathcal{G} \xrightarrow{p^c} g$ iff for each $\mathcal{F} \xrightarrow{q} x$, $\lambda^{\rightarrow}(\mathcal{F} \times \mathcal{G}) \xrightarrow{q} \lambda(x,g)$. Then $(S,p^c) \in |\mathrm{CONV}|$. First, it is shown that $(S,.,p^c) \in |\mathrm{CSG}|$; that is, if $\mathcal{G} \xrightarrow{p^c} g$ and $\mathcal{H} \xrightarrow{p^c} h$, then $\mathcal{G}.\mathcal{H} \xrightarrow{p^c} g.h$. Assume that $\mathcal{F} \xrightarrow{q} x$; then using (a2), $\lambda^{\rightarrow}(\mathcal{F} \times \mathcal{G}.\mathcal{H}) = [\{\lambda(F \times G.H) : F \in \mathcal{F}, G \in \mathcal{G}, H \in \mathcal{H}\}] = [\{\lambda(\lambda(F \times G) \times H) : F \in \mathcal{F}, G \in \mathcal{G}, H \in \mathcal{H}\}] = \lambda^{\rightarrow}(\lambda^{\rightarrow}(\mathcal{F} \times \mathcal{G}) \times \mathcal{H})$. It follows from the definition of p^c that $\lambda^{\rightarrow}(\mathcal{F} \times \mathcal{G}) \xrightarrow{q} \lambda(x,g)$, and thus $\lambda^{\rightarrow}(\lambda^{\rightarrow}(\mathcal{F} \times \mathcal{G}) \times \mathcal{H}) \xrightarrow{q} \lambda(\lambda(x,g),h) = \lambda(x,g.h)$. Hence $\mathcal{G}.\mathcal{H} \xrightarrow{p^c} g.h$, and thus $(S, ., p^c) \in |\mathrm{CSG}|$. According to the construction, p^c is the coarsest convergence structure on S such that $\lambda : (X, q) \times (S, p^c) \to (X, q)$ is continuous.

(c): Define $\lambda_B : (B(X, S), \sigma) \times (S, ., p) \to (B(X, S), \sigma)$ by $\lambda_B(\langle (x, g)\rangle, h) = \langle (x, g.h)\rangle$. It is shown that λ_B is a continuous action. Indeed, $\lambda_B(\langle (x, g)\rangle, e) = \langle (x, g)\rangle$, and $\lambda_B(\lambda_B(\langle (x, g)\rangle, h), k) = \lambda_B(\langle (x, g.h)\rangle, k) = \langle (x, g.h.k)\rangle = \lambda_B(\langle (x, g)\rangle, h.k)$. Hence λ_B is an action. It remains to show that λ_B is continuous. Suppose that $\mathcal{K} \xrightarrow{\sigma} \langle (x, g)\rangle$ and $\mathcal{L} \xrightarrow{p} l$. Since φ is a quotient map in CONV, there exists $\mathcal{H} \xrightarrow{r} (x_1, g_1) \sim (x, g)$ such that $\varphi^{\to}\mathcal{H} = \mathcal{K}$. Then $\lambda_B^{\to}(\mathcal{K} \times \mathcal{L}) = \lambda_B^{\to}(\varphi^{\to}\mathcal{H} \times \mathcal{L})$. Let $K \in \mathcal{K}$ and $L \in \mathcal{L}$, and note that $\lambda_B(\varphi(H) \times L) \subseteq \lambda_B(\varphi(\pi_1(H) \times \pi_2(H)) \times L) = \varphi(\pi_1(H) \times \pi_2(H).L)$. Hence $\lambda_B^{\to}(\varphi^{\to}\mathcal{H} \times \mathcal{L}) \geq \varphi^{\to}(\pi_1^{\to}\mathcal{H} \times \pi_2^{\to}\mathcal{H}.\mathcal{L}) \xrightarrow{\sigma} \varphi(x_1, g_1.l) = \langle (x_1, g_1.l)\rangle = \lambda_B(\langle (x_1, g_1)\rangle, l) = \lambda_B(\langle (x, g)\rangle, l)$. Therefore $(B(X, S), S, \lambda_B) \in \mathrm{AC}$. □

Remark 3.4. Let $(X, q) \in |\mathrm{CONV}|$ and let $(C(X, X), c)$ denote the space defined in section 2. Since c is the coarsest convergence structure for which the evaluation map $\omega : (X, q) \times (C(X, X), c) \to (X, q)$ is continuous, this is a particular case of Theorem 3.3(b), where $\lambda = \omega$, $(S, ., p^c) = (C(X, X), ., c)$, and the group operation is composition. Moreover, it is well-known that, in general, there fails to exist a coarsest topology on $C(X, X)$ for which $\omega : (X, q) \times C(X, X) \to (X, q)$ is jointly continuous (even when q is a topology).

Assume that $(X, S, \lambda) \in \mathrm{A}$; then λ is said to **distinguish elements in S** whenever $\lambda(x, g) = \lambda(x, h)$ for all $x \in X$ implies that $g = h$. In this case, define $\theta : S \to C(X, X)$ by $\theta(g)(x) = \lambda(x, g)$, for each $x \in X$. Note that θ is an injection iff λ separates elements in S. Moreover, θ is a homomorphism whenever the operation in $C(X, X)$ is $k.l = l \circ k$ is composition.

Theorem 3.5. *Suppose that $((X, q), (S, ., p), \lambda) \in AC$, and assume that λ distinguishes elements in S. Then the following are equivalent:*
(a) $\theta : (S, p) \to (C(X, X), c)$ *is an embedding*
(b) $p = p^c$
(c) *if* $\mathcal{G} \stackrel{p}{\not\to} g$, *then there exists* $\mathcal{F} \xrightarrow{q} x$ *such that* $\lambda^{\to}(\mathcal{F} \times \mathcal{G}) \stackrel{q}{\not\to} \lambda(x, g)$.

Proof. (a) ⇒ (b): Assume that $\theta : (S, p) \to (C(X, X), c)$ is an embedding. According to Theorem 3.3(b), $p^c \leq p$. Suppose that $\mathcal{G} \xrightarrow{p^c} g$; then if $\mathcal{F} \xrightarrow{q} x$, $\lambda^{\to}(\mathcal{F} \times \mathcal{G}) \xrightarrow{q} \lambda(x, g)$. It is shown that $\theta^{\to}\mathcal{G} \xrightarrow{c} \theta(g)$. Indeed, note that $\omega^{\to}(\mathcal{F} \times \theta^{\to}\mathcal{G}) = [\{\omega(F \times \theta(G)) : F \in \mathcal{F}, G \in \mathcal{G}\}] = [\{\lambda(F \times G) : F \in \mathcal{F}, G \in \mathcal{G}\}] = \lambda^{\to}(\mathcal{F} \times \mathcal{G}) \xrightarrow{q} \lambda(x, g) = \omega(x, \theta(g))$. Hence $\theta^{\to}\mathcal{G} \xrightarrow{c} \theta(g)$, and thus $\mathcal{G} \xrightarrow{p} g$. Therefore $p = p^c$.

(b) ⇒ (c): Verification follows directly from the definition of p^c.

(c) ⇒ (a): Suppose that $\mathcal{G} \xrightarrow{p} g$ and $\mathcal{F} \xrightarrow{q} x$. Since $\lambda : (X, q) \times (S, p) \to (X, q)$ is continuous, $\lambda^{\to}(\mathcal{F} \times \mathcal{G}) \xrightarrow{q} \lambda(x, g)$. Hence $\omega^{\to}(\mathcal{F} \times \theta^{\to}\mathcal{G}) = \lambda^{\to}(\mathcal{F} \times \mathcal{G}) \xrightarrow{q}$

$\lambda(x,g) = \omega(x,\theta(g))$, and thus $\theta^{\rightarrow}\mathcal{G} \xrightarrow{c} \theta(g)$. Conversely, if $\mathcal{G} \in \mathfrak{F}(S)$ such that $\theta^{\rightarrow}\mathcal{G} \xrightarrow{c} \theta(g)$, then the hypothesis implies that $\mathcal{G} \xrightarrow{p} g$. Hence $\theta : (S,p) \to (C(X,X),c)$ is an embedding. □

Remark 3.6. The map θ given in Theorem 3.5 is called a **continuous representation** of $(S,.,p)$ on (X,q). Rath [10] discusses this concept in the context of a group with $(C(X,X),.,c)$ replaced by $(H(X),.,\gamma)$, where $(H(X),.)$ is the group of all homeomorphisms on X with composition as the group operation, and γ is the coarsest convergence structure making the operations of composition and inversion continuous.

Quite often it is desirable to consider modifications of convergence structures. For example, given $(X,q) \in |\text{CONV}|$, there exists a finest regular convergence structure on X which is coarser than q [4]. The notation $\boldsymbol{Pq}$ denotes the P-modification of q. Generally, P represents a convergence space property; however, it is convenient to include the case whenever $Pq = q$. Let **PCONV** denote the full subcategory of CONV consisting of all the objects in CONV that satisfy condition P. Condition P is said to be **finitely productive**(**productive**) provided that for each collection $(X_j,q_j) \in |\text{CONV}|$, $j \in J$, $P(\underset{j\in J}{\times} q_j) = \underset{j\in J}{\times} Pq_j$ whenever J is a finite (arbitrary) set, respectively.

Theorem 3.7. *Assume that F_P : CONV $\to$ PCONV is a functor obeying $F_P(X,q) = (X,Pq)$, $F_P(f) = f$, and suppose that P is finitely productive. If $((X,q),(S,.,p),\lambda) \in AC$ and $h : (T,.,\xi) \to (S,.,p)$ is a continuous homomorphism in CSG, then $((X,Pq),(T,.,P\xi)) \in AC$; in particular, $((X,Pq),(S,.,Pp),\lambda) \in AC$.*

Proof. Given that $((X,q),(S,.,p),\lambda) \in$ AC, define $\Lambda : (X,q) \times (T,\xi) \to (X,q)$ by $\Lambda(x,t) = \lambda(x,h(t))$. Clearly Λ is an action; moreover, Λ is continuous. Indeed, suppose that $\mathcal{F} \xrightarrow{q} x$ and $\mathcal{G} \xrightarrow{\xi} t$; then $\Lambda^{\rightarrow}(\mathcal{F} \times \mathcal{G}) = [\{\Lambda(F \times G) : F \in \mathcal{F}, G \in \mathcal{G}\}] = [\{\lambda(F \times h(G)) : F \in \mathcal{F}, G \in \mathcal{G}\}] = \lambda^{\rightarrow}(\mathcal{F} \times h^{\rightarrow}\mathcal{G}) \xrightarrow{q} \lambda(x,h(t)) = \Lambda(x,t)$. Therefore Λ is continuous.

Since F_P is a functor and P is finitely productive, continuity of the operation $\gamma : (T,.,\xi) \times (T,.,\xi) \to (T,.,\xi)$, defined by $\gamma(t_1,t_2) = t_1.t_2$, implies continuity of $\gamma : (T,.,P\xi) \times (T,.,P\xi) \to (T,.,P\xi)$. Hence $(T,.,P\xi) \in |\text{CSG}|$. Likewise, $\Lambda : (X,Pq) \times (T,P\xi) \to (X,Pq)$ is continuous, and thus $((X,Pq),(T,.,P\xi),\Lambda) \in$ AC. □

Let $(S_j,.,p_j) \in |\text{CSG}|$, $j \in J$, and denote the product by $(S,.,p) = \underset{j\in J}{\times}(S_j,.,p_j)$. The **direct sum** of $(S_j,.)$, $j \in J$, is the subsemigroup of $(S,.)$ defined by $\oplus_{j\in J}S_j = \{(g_j) \in S : g_j = e_j \text{ for all but finitely many } j \in J\}$. Denote $\theta_j : S_j \to \oplus_{j\in J}S_j$ to be the map $\theta_j(g) = (g_k)$, where $g_j = g$ and $g_k = e_k$ whenever $k \neq j$, and let $\theta : \oplus_{j\in J}S_j \to \underset{j\in J}{\times} S_j$ be the inclusion map. Define $\mathcal{H} \xrightarrow{\eta} (g_j)$ in $\oplus_{j\in J}S_j$ iff $\mathcal{H} \geq \theta_{k_1}^{\rightarrow}\mathcal{G}_1.\theta_{k_2}^{\rightarrow}\mathcal{G}_2...\theta_{k_n}^{\rightarrow}\mathcal{G}_n$, where $\mathcal{G}_j \xrightarrow{p_{k_j}} g_{k_j}$ in $(S_{k_j},.,p_{k_j})$ and

$n \geq 1$. Then $(\oplus_{j\in J} S_j, ., \eta) \in |\mathrm{CSG}|$, and $\theta : (\oplus_{j\in J} S_j, ., \eta) \to (S, ., p)$ is a continuous homomorphism.

Theorem 3.8. *Suppose that $F_P : CONV \to PCONV$ is a functor satisfying $F_P(X, q) = (X, Pq)$, $F_P(f) = f$, and P is productive. Assume that $((X_j, q_j), (S_j, ., p_j), \lambda_j) \in AC$ for each $j \in J$. Then*
(a) $(\underset{j\in J}{\times}(X_j, Pq_j), \underset{j\in J}{\times}(S_j, ., Pp_j)) \in AC$
(b) $(\underset{j\in J}{\times}(X_j, Pq_j), (\oplus_{j\in J} S_j, ., P\eta)) \in AC$.

Proof. (a): Denote $(X, q) = \underset{j\in J}{\times}(X_j, q_j)$, $(S, ., p) = \underset{j\in J}{\times}(S_j, ., p_j)$, and define $\lambda : (X, q) \times (S, p) \to (X, q)$ by $\lambda((x_j), (g_j)) = (\lambda_j(x_j, g_j))$. Clearly λ is an action. Then, according to Theorem 3.7 and the assumption that P is productive, it suffices to show that $((X, q), (S, p), \lambda) \in \mathrm{AC}$. The latter follows from a routine argument, and thus $(\underset{j\in J}{\times}(X_j, Pq_j), \underset{j\in J}{\times}(S_j, ., Pp_j), \lambda) \in \mathrm{AC}$.

(b): Since $\theta : (\oplus S_j, ., \eta) \to (S, ., p)$ is a continuous homomorphism in CSG and P is productive, it follows from Theorem 3.7 that $(\underset{j\in J}{\times}(X_j, Pq_j), (\oplus S_j, ., P\eta)) \in$ AC. □

Corollary 3.9. *Assume that $F_P : CONV \to PCONV$ is a functor satisfying $F_P(X, q) = (X, Pq)$, $F_P(f) = f$, and P is finitely productive. Suppose that $((X_j, q_j), (S_j, ., p_j)) \in AC$ for each $j \in J$. Denote $(X, q) = \underset{j\in J}{\times}(X_j, q_j)$ and $(S, ., p) = \underset{j\in J}{\times}(S_j, ., p_j)$. Then*
(a) $((X, Pq), (S, ., Pp)) \in AC$
(b) $((X, Pq), (\oplus_{j\in J} S_j, ., P\eta)) \in AC$.

Verification of Corollary 3.9 follows the proof of Theorem 3.8 with the exception that since P is only finitely productive, (X, Pq) and $\underset{j\in J}{\times}(X_j, Pq_j)$, as well as $(S, ., Pp)$ and $\underset{j\in J}{\times}(S_j, ., Pp_j)$, may differ. Of course equality holds whenever the index set is finite. Choices of P that are finitely productive, and preserve continuity when taking P-modifications include: locally compact, locally bounded, regular, and first-countable. The property of being Choquet is productive, and continuity is preserved under taking Choquet modifications.

4. Generalized Quotients

Recall that if $((X, q), (S, ., p), \lambda) \in \mathrm{AC}$, $(S, .)$ is commutative, $\lambda(., g)$ is an injection, then by Lemma 3.1, $(x, g) \sim (y, h)$ iff $\lambda(x, h) = \lambda(y, g)$ is an equivalence relation. Denote $R = \{((x, g), (y, h)) : (x, g) \sim (y, h)\}$, $r = q \times p$, and $\varphi : (X \times S, r) \to ((X \times S)/\sim, \sigma)$ the convergence quotient map defined by $\varphi(x, g) = \langle(x, g)\rangle$. Then $\boldsymbol{(B(X,S),\sigma)}$:$= ((X \times S)/\sim, \sigma)$ is called the **generalized quotient space**. Convergence space properties of $(B(X, S), \sigma)$ are investigated in this section.

For ease of exposition, $((X,q),(S,.,p),\lambda) \in \mathbf{GQ}$ denotes that $((X,q),(S,.,p),\lambda) \in$ AC, $(S,.)$ is commutative, and $\lambda(.,g)$ is an injection, for each $g \in S$. The generalized quotient space $(B(X,S),\sigma)$ exists whenever $((X,q),(S,.,p),\lambda) \in$ GQ.

Theorem 4.1. *Assume that $((X,q),(S,.,p),\lambda) \in GQ$. Then the following are equivalent:*

(a) *(X,q) is Hausdorff*
(b) *R is closed in $((X \times S) \times (X \times S), r \times r)$*
(c) *$(B(X,S),\sigma)$ is Hausdorff.*

Proof. (a) $\Rightarrow$ (b): Let π_{ij} denote the projection map defined by : $\pi_{ij} : (X \times S) \times (X \times S) \to X \times S$ where $\pi_{ij}(((x,g),(y,h))) = (x,g)$ when $i,j = 1,2$ and $\pi_{ij}(((x,g),(y,h))) = (y,h)$ when $i,j = 3,4$. Suppose that $\mathcal{H} \overset{r\times r}{\to} ((x,g),(y,h))$ and $R \in \mathcal{H}$. Let $H \in \mathcal{H}$; then $H \cap R \neq \varnothing$, and thus there exists $((x_1,g_1),(y_1,h_1)) \in H \cap R$. Hence $\lambda(x_1,h_1) = \lambda(y_1,g_1)$, and consequently $\lambda((\pi_1 \circ \pi_{12})(H) \times (\pi_2 \circ \pi_{34})(H)) \cap \lambda((\pi_1 \circ \pi_{34})(H) \times (\pi_2 \circ \pi_{12})(H)) \neq \varnothing$, for each $H \in \mathcal{H}$. It follows that $\mathcal{K} := \lambda^{\to}((\pi_1 \circ \pi_{12})^{\to}\mathcal{H} \times (\pi_2 \circ \pi_{34})^{\to}\mathcal{H}) \vee \lambda^{\to}((\pi_1 \circ \pi_{34})^{\to}\mathcal{H} \times (\pi_2 \circ \pi_{12})^{\to}\mathcal{H})$ exists. However, $(\pi_1 \circ \pi_{12})^{\to}\mathcal{H} \overset{q}{\to} x$, $(\pi_2 \circ \pi_{34})^{\to}\mathcal{H} \overset{p}{\to} h$, $(\pi_1 \circ \pi_{34})^{\to}\mathcal{H} \overset{q}{\to} y$, $(\pi_2 \circ \pi_{12})^{\to}\mathcal{H} \overset{p}{\to} g$, and thus $\mathcal{K} \overset{q}{\to} \lambda(x,h), \lambda(y,g)$. Since (X,q) is Hausdorff, $\lambda(x,h) = \lambda(y,g)$ and thus $(x,g) \sim (y,h)$. Therefore, $((x,g),(y,h)) \in R$, and thus R is closed.

(b) $\Rightarrow$ (c): Assume that $\mathcal{K} \overset{\sigma}{\to} \langle (y_i,h_i) \rangle$, $i = 1,2$. Since $\varphi : (X \times S, r) \to (B(X,S),\sigma)$ is a quotient map in CONV, there exist $(x_i,g_i) \sim (y_i,h_i)$ and $\mathcal{H}_i \overset{r}{\to} (x_i,g_i)$ such that $\varphi^{\to}\mathcal{H}_i = \mathcal{K}$, $i = 1,2$. Then for each $H_i \in \mathcal{H}_i$, $\varphi(H_1) \cap \varphi(H_2) \neq \varnothing$ and thus there exists $(s_i,t_i) \in H_i$ such that $(s_1,t_1) \sim (s_2,t_2)$, $i = 1,2$. Hence the least upper bound filter $\mathcal{L} := (\mathcal{H}_1 \times \mathcal{H}_2) \vee \dot{R}$ exists, and $\mathcal{L} \overset{r\times r}{\to} ((x_1,g_1),(x_2,g_2))$. Since R is closed, $(x_1,g_1) \sim (x_2,g_2)$ and thus $\langle (y_1,h_1) \rangle = \langle (y_2,h_2) \rangle$. Therefore $(B(X,S),\sigma)$ is Hausdorff.

(c) $\Rightarrow$ (a): Suppose that $(B(X,S),\sigma)$ is Hausdorff and $\mathcal{F} \overset{q}{\to} x, y$. Then $\varphi^{\to}(\mathcal{F} \times \dot{e}) \overset{\sigma}{\to} \langle (x,e) \rangle, \langle (y,e) \rangle$, and thus $(x,e) \sim (y,e)$. Therefore, $x = \lambda(x,e) = \lambda(y,e) = y$, and thus (X,q) is Hausdorff. □

Conditions for which $(B(X,S),\sigma)$ is T_1 are given below. In the topological setting, sufficient conditions in order for the generalized quotient space to be T_2 are given in [1] whenever $(S,.)$ is equipped with the discrete topology.

Theorem 4.2. *Suppose that $((X,q),(S,.,p),\lambda) \in GQ$. Then $(B(X,S),\sigma)$ is T_1 iff $\varphi^{-1}(\langle (y,h) \rangle)$ is closed in $(X \times S, r)$, for each $(y,h) \in X \times S$.*

Proof. The "only if" is clear since $\{\langle (y,h) \rangle\}$ is closed and φ is continuous. Conversely, assume that $\varphi^{-1}(\langle (y,h) \rangle)$ is closed, for each $(y,h) \in X \times S$, and suppose that $\langle(\dot{x,g})\rangle \xrightarrow{\sigma} \langle(y,h)\rangle$. Since φ is a quotient map in CONV, there exist $(s,t) \sim (y,h)$ and $\mathcal{H} \xrightarrow{r} (s,t)$ such that $\varphi^{\rightarrow}\mathcal{H} = \langle(\dot{x,g})\rangle$. Then $\varphi^{-1}(\langle(x,g)\rangle) \in \mathcal{H}$, and thus $(s,t) \in \mathrm{cl}_r\varphi^{-1}(\langle(x,g)\rangle) = \varphi^{-1}(\langle(x,g)\rangle)$. Hence $(x,g) \sim (s,t) \sim (y,h)$, and thus $\langle(x,g)\rangle = \langle(y,h)\rangle$. Therefore $(B(X,S),\sigma)$ is T_1. □

Corollary 4.3. *Assume that $((X,q),(S,.,p),\lambda) \in GQ$, and let p denote the discrete topology. Then $(B(X,S),\sigma)$ is T_1 iff (X,q) is T_1.*

Proof. Suppose that $(B(X,S),\sigma)$ is T_1 and $\dot{x} \xrightarrow{q} y$. Then $(\dot{x,e}) \xrightarrow{r} (y,e)$, and thus $\langle(\dot{x,e})\rangle = \varphi^{\rightarrow}((\dot{x,e})) \xrightarrow{\sigma} \langle(y,e)\rangle$. It follows that $\langle(x,e)\rangle = \langle(y,e)\rangle$ and hence $x = y$. Therefore (X,q) is T_1.

Conversely, assume that (X,q) is T_1 and $(y,h) \in \mathrm{cl}_r\varphi^{-1}(\langle(x,g)\rangle)$. Then there exists $\mathcal{H} \xrightarrow{r} (y,h)$ such that $\varphi^{-1}(\langle(x,g)\rangle) \in \mathcal{H}$, $\pi_1^{\rightarrow}\mathcal{H} \xrightarrow{q} y$, $\pi_2^{\rightarrow}\mathcal{H} \xrightarrow{p} h$, and since p is the discrete topology, choose $H \in \mathcal{H}$ for which $\pi_2(H) = \{h\}$ and $\varphi(H) = \{\langle(x,g)\rangle\}$. If $(s,t) \in H$, then $(s,t) \sim (x,g)$, $t = h$, and thus $\lambda(s,g) = \lambda(x,h)$. Hence $\lambda(\pi_1(H) \times \{g\}) = \{\lambda(x,h)\}$, and thus $\lambda(\dot{x,h}) = \lambda^{\rightarrow}(\pi_1^{\rightarrow}\mathcal{H} \times \dot{g}) \xrightarrow{q} \lambda(y,g)$. Then $\lambda(x,h) = \lambda(y,g)$, $(x,g) \sim (y,h)$, and thus $\varphi^{-1}(\langle(x,g)\rangle)$ is r-closed. Hence it follows from Theorem 4.2 that $(B(X,S),\sigma)$ is T_1. □

Corollary 4.4 ([1])**.** *Suppose that the hypotheses of Corollary 4.3 are satisfied with the exception that (X,q) is a topological space and $B(X,S)$ is equipped with the quotient topology τ. Then $(B(X,S),\tau)$ is T_1 iff (X,q) is T_1.*

Proof. It follows from Theorem 2 [2] that since $\varphi : (X \times S, r) \to (B(X,S),\sigma)$ is a quotient map in CONV, $\varphi : (X \times S, r) \to (B(X,S), t\sigma)$ is a topological quotient map, where $t\sigma$ is the largest topology on $X \times S$ which is coarser than σ. Moreover, $\tau = t\sigma$, and $A \subseteq B(X,S)$ is σ-closed iff it is τ-closed. Hence the desired conclusion follows from Corollary 4.3. □

An illustration is given to show that the generalized quotient space may fail to be T_1 even though (X,q) is a T_1 topological space.

Example 4.5. Denote $X = (0,1)$, q the cofinite topology on X, and define $f : X \to X$ by $f(x) = ax$, where $0 < a < 1$ is fixed. Let $S = \{f^n : n \geq 0\}$, where $f^0 = \mathrm{id}_X$ and f^n denotes the n-fold composition of f with itself. Then $(S,.) \in |\mathrm{SG}|$ is commutative with composition as the operation. Also equip $(S,.)$ with the cofinite topology p. It is shown that the operation $\gamma : (S,p) \times (S,p) \to (S,p)$ defined by $\gamma(g,h) = g.h := h \circ g$ is continuous at (f^m, f^n). Define $C = \{f^k : k \geq k_0\}$; then $\{f^{m+n}\} \cup C$ is a basic p-neighborhood of f^{m+n}, where $k_0 \geq 0$. Observe that if $A = \{f^m\} \cup C$ and $B = \{f^n\} \cup C$, then $\gamma(A \times B) \subseteq C \cup \{f^{m+n}\}$. Therefore γ is continuous, and $(S,.,p) \in |\mathrm{CSG}|$.

Define $\lambda : X \times S \to X$ by $\lambda(x,g) = g(x)$, for each $x \in X$, $g \in S$, and note that λ is an action. It is shown that $\lambda : (X,q) \times (S,p) \to (X,q)$ is continuous at (x_0, f^n) in $X \times S$. A basic q-neighborhood of $\lambda(x_0, f^n) = f^n(x_0)$ is of the form $W = X - F$, where $f^n(x_0) \notin F$ and F is a finite subset of X. Let y_0 be the smallest member of F, and choose k_0 to be a natural number such that $a^{k_0} < y_0$. Then for each $k \geq k_0$, $f^k(x) = a^k x < y_0$ for each $x \in X$. Since f^n is injective, $F_0 = (f^n)^{-1}(F)$ is a finite subset of X. Then $U = X - F_0$ is a q-neighborhood of x_0, $V = \{f^n\} \cup \{f^k : k \geq k_0\}$ is a p-neighborhood of f^n, and $\lambda(U \times V) \subseteq W$. Indeed, if $x \in U$ and $k \geq k_0$, then $\lambda(x, f^k) = f^k(x) < y_0$, and thus $f^k(x) \in W$. Further, if $x \in U$, then $f^n(x) \notin F$, and hence $f^n(x) \in W$. It follows that $\lambda(U \times V) \subseteq W$, and thus λ is a continuous action.

It is shown that $\varphi^{-1}(\langle(x_0, \mathrm{id}_X)\rangle)$ is not closed in $(X \times S, r)$. Note that $(x, f^n) \in \varphi^{-1}(\langle(x_0, \mathrm{id}_X)\rangle)$ iff $\mathrm{id}_X(x) = f^n(x_0)$. Hence $\varphi^{-1}(\langle(x_0, \mathrm{id}_X)\rangle) = \{(f^n(x_0), f^n) : n \geq 0\}$. Since $\mathrm{id}_X = f^0 > f^1 > f^2 > ...$, it easily follows that $\mathrm{cl}_r \varphi^{-1}(\langle(x_0, \mathrm{id}_X)\rangle) = X \times S$, and thus $\varphi^{-1}(\langle(x_0, \mathrm{id}_X)\rangle)$ is not r-closed. It follows from Theorem 4.2 that $(B(X,S), \sigma)$ is not T_1 even though both (X, q) and (S, p) are T_1 topological spaces.

A continuous surjection $f : (X, q) \to (Y, p)$ in CONV is said to be **proper map** provided that for each ultrafilter $\mathcal{F}$ on X, $f^{\to}\mathcal{F} \xrightarrow{p} y$ implies that $\mathcal{F} \xrightarrow{q} x$, for some $x \in f^{-1}(y)$. Proper maps in CONV are discussed in [3]; in particular, proper maps preserve closures. A proper convergence quotient map is called a **perfect map** [4].

Remark 4.6. Assume that $((X, q), (S, ., p), \lambda) \in$ GQ, (X, q) and (S, p) are regular, and $\varphi : (X \times S, r) \to ((B(X,S), \sigma)$ is a perfect map. Then $(B(X,S), \sigma)$ is also regular. Indeed, suppose that $\mathcal{H} \in \mathfrak{F}(B(X,S))$ such that $\mathcal{H} \xrightarrow{\sigma} \langle(y,h)\rangle$. Since φ is a quotient map in CONV, there exists $(x, g) \sim (y, h)$ and $\mathcal{K} \xrightarrow{r} (x, g)$ such that $\varphi^{\to}\mathcal{K} = \mathcal{H}$. Moreover, the regularity of $(X \times S, r)$ implies that $\mathrm{cl}_r\mathcal{K} \xrightarrow{r} (x, g)$. Since φ is a proper map and thus preserves closures, $\varphi^{\to}(\mathrm{cl}_r\mathcal{K}) = \mathrm{cl}_\sigma \varphi^{\to}\mathcal{K} = \mathrm{cl}_\sigma\mathcal{H} \xrightarrow{\sigma} \langle(y,h)\rangle$. Hence $(B(X,S), \sigma)$ is regular.

The proof of the following result is straightforward to verify.

Lemma 4.7. *Suppose that* $(S, ., p) \in |CSG|$ *and* $(T, .) \in |SG|$. *Assume that* $f : (S, ., p) \to (T, ., \sigma)$ *is both a homomorphism and a quotient map in CONV. Then* $(T, ., \sigma) \in |CSG|$.

Assume that $((X, q), (S, ., p), \lambda) \in$ AC. Recall that λ distinguishes elements in S whenever $\lambda(x, g) = \lambda(x, h)$ for each $x \in X$ implies $g = h$. This property was needed in the verification of Theorem 3.5. In the event that λ fails to distinguish elements in S, define $g \sim h$ iff $\lambda(x, g) = \lambda(x, h)$ for each $x \in X$. Then $\sim$ is an equivalence relation on S; denote $\boldsymbol{S_1} = S/\sim = \{[g] : g \in S\}$, and define the operation $[g].[h] = [g.h]$, for each $g, h \in S$. The operation is well defined and $(S_1, .) \in |\mathrm{SG}|$. Let $\boldsymbol{p_1}$ denote the quotient convergence structure on S_1 determined by $\rho : (S, p) \to S_1$, where $\rho(g) = [g]$. Then $\rho : (S, .) \to (S_1, .)$ is a homomorphism, and it follows from Lemma 4.7 that $(S_1, ., p_1) \in |\mathrm{CSG}|$. Define $\lambda_1 : X \times S_1 \to X$ by $\lambda_1(x, [g]) = \lambda(x, g)$.

Theorem 4.8. *Assume* $((X,q),(S,.,p),\lambda\in)$ *GQ,* λ *fails to distinguish elements in* S*, and let* $(B(X\times S),\sigma),(B(X\times S_1),\sigma_1)$ *denote the generalized quotient spaces corresponding to* $(X\times S,r)$ *and* $(X\times S_1,r_1)$*, where* $r=q\times p$ *and* $r_1=q\times p_1$*. Then*

(a) $\lambda_1:(X\times S_1,r_1)\to(X,q)$ *is a continuous action*
(b) λ_1 *separates elements in* S_1
(c) $(B(X,S),\sigma)$ *and* $(B(X,S_1),\sigma_1)$ *are homeomorphic.*

Proof. (a): It is routine to verify that λ_1 is an action. Let us show that λ_1 is continuous. Suppose that $\mathcal{F}\xrightarrow{q}x$ and $\mathcal{G}\xrightarrow{p_1}[g]$; then since p_1 is a quotient structure in CONV, there exists $\mathcal{G}_1\xrightarrow{p}g_1\sim g$ such that $\rho^{\to}\mathcal{G}_1=\mathcal{G}$. Hence $\lambda_1^{\to}(\mathcal{F}\times\mathcal{G})=\lambda_1^{\to}(\mathcal{F}\times\rho^{\to}\mathcal{G}_1)=[\{\lambda_1(F\times\rho(G_1)):F\in\mathcal{F},G_1\in\mathcal{G}_1\}]=[\{\lambda(F\times G_1):F\in\mathcal{F},G_1\in\mathcal{G}_1\}]=\lambda^{\to}(\mathcal{F}\times\mathcal{G}_1)\xrightarrow{q}\lambda(x,g_1)=\lambda_1(x,[g])$, and thus λ_1 is continuous.

(b): Suppose that $\lambda_1(x,[g])=\lambda_1(x,[h])$ for each $x\in X$. Then $\lambda(x,g)=\lambda(x,h)$ for each $x\in X$, and thus $[g]=[h]$. Hence λ_1 distinguishes elements in S_1.

(c): It easily follows that the diagram below is commutative:

$$\begin{array}{ccc} X\times S & \xrightarrow{\varphi_1} & B(X,S) \\ \Big\downarrow{\scriptstyle\psi_1} & & \Big\downarrow{\scriptstyle\psi_2} \\ X\times S_1 & \xrightarrow{\varphi_2} & B(X,S_1) \end{array}$$

where φ_1, φ_2 are quotient maps, $\psi_1(x,g)=(x,[g])$, and $\psi_2(\langle x,g\rangle)=\langle(x,[g])\rangle$. Moreover, ψ_2 is an injection. Indeed, assume that $\langle(x,[g])\rangle=\psi_2(\langle(x,g)\rangle)=\psi_2(\langle(y,h)\rangle)=\langle(y,[h])\rangle$; then $\lambda_1(x,[h])=\lambda_1(y,[g])$ and thus $\lambda(x,h)=\lambda(y,g)$. Hence $\langle(x,g)\rangle=\langle(y,h)\rangle$ and ψ_2 is an injection. Clearly ψ_2 is a surjection.

It is shown that ψ_2 is continuous. Indeed, suppose that $\mathcal{H}\xrightarrow{\sigma}\langle(y,h)\rangle$; then there exist $(x,g)\sim(y,h)$ and $\mathcal{K}\xrightarrow{r}(x,g)$ such that $\varphi_1^{\to}\mathcal{K}=\mathcal{H}$. Since the diagram above commutes with ψ_1 and φ_2 continuous, it follows that $\psi_2^{\to}\mathcal{H}=(\psi_2\circ\varphi_1)^{\to}\mathcal{K}=(\varphi_2\circ\psi_1)^{\to}\mathcal{K}\xrightarrow{\sigma_1}(\varphi_2\circ\psi_1)(x,g)=(\psi_2\circ\varphi_1)(x,g)=\psi_2(\langle(x,g)\rangle)=\psi_2(\langle(y,h)\rangle)$. Hence ψ_2 is continuous.

Finally, let us show that ψ_2^{-1} is continuous. Assume that $\mathcal{H}\xrightarrow{\sigma_1}\langle(y,[h])\rangle$. Since φ_2 is a quotient map, there exist $(x,[g])\sim(y,[h])$ and $\mathcal{K}\xrightarrow{r_1}(x,[g])$ such that $\varphi_2^{\to}\mathcal{K}=\mathcal{H}$. In particular, $\mathcal{F}=\pi_1^{\to}\mathcal{K}\xrightarrow{q}x$ and $\mathcal{G}=\pi_2^{\to}\mathcal{K}\xrightarrow{p_1}[g]$. Since $\rho:(S,p)\to(S_1,p_1)$ is a quotient map, there exist $g_1\sim g$ and $\mathcal{G}_1\xrightarrow{p}g_1$ such that $\rho^{\to}\mathcal{G}_1=\mathcal{G}$. Then $\mathcal{F}\times\mathcal{G}_1\xrightarrow{r}(x,g_1)$, and thus $\psi_1^{\to}(\mathcal{F}\times\mathcal{G}_1)=\mathcal{F}\times\rho^{\to}\mathcal{G}_1=\mathcal{F}\times\mathcal{G}\leq\mathcal{K}$. Hence $(\varphi_2\circ\psi_1)^{\to}(\mathcal{F}\times\mathcal{G}_1)\leq\varphi_2^{\to}\mathcal{K}=\mathcal{H}$, and since the diagram commutes, $\psi_2^{\leftarrow}\mathcal{H}\geq(\psi_2^{-1}\circ\varphi_2\circ\psi_1)^{\to}(\mathcal{F}\times\mathcal{G}_1)=\varphi_1^{\to}(\mathcal{F}\times\mathcal{G}_1)\xrightarrow{\sigma}\langle(x,g)\rangle=\psi_2^{-1}(\langle(y,[h])\rangle)$. Therefore ψ_2 is a homeomorphism. □

Sufficient conditions in order for (X,q) to be embedded in $(B(X,S),\sigma)$ are presented below.

Theorem 4.9. *Suppose that* $((X,q),(S,.,p),\lambda) \in GQ$. *Define* $\beta : (X,q) \to (B(X,S),\sigma)$ *by* $\beta(x) = \langle (x,e) \rangle$, *for each* $x \in X$. *Then*

- (a) β *is a continuous injection*
- (b) β *is an embedding provided that* (X,q) *is a Choquet space,* p *is discrete, and* λ *is a proper map.*

Proof. (a): Clearly β is an injection. Next, assume that $\mathcal{F} \xrightarrow{q} x$; then $\beta^{\to}\mathcal{F} = [\{\beta(F) : F \in \mathcal{F}\}] = [\{\varphi(F \times \{e\}) : F \in \mathcal{F}\}] = \varphi^{\to}(\mathcal{F} \times \dot{e}) \xrightarrow{\sigma} \varphi(x,e) = \beta(x)$. Therefore β is continuous.

(b): First, suppose that $\mathcal{F}$ is an ultrafilter on X such that $\beta^{\to}\mathcal{F} \xrightarrow{\sigma} \beta(x) = \langle (x,e) \rangle$. Since $\varphi : (X \times S, r) \to (B(X,S),\sigma)$ is a quotient map in CONV, there exist $(y,g) \sim (x,e)$ and $\mathcal{K} \xrightarrow{r} (y,g)$ such that $\varphi^{\to}\mathcal{K} = \beta^{\to}\mathcal{F}$. Denote $\mathcal{F}_1 = \pi_1^{\to}\mathcal{K} \xrightarrow{q} y$ and $\mathcal{G}_1 = \pi_2^{\to}\mathcal{K} \xrightarrow{p} g$. Since p is the discrete topology, $\mathcal{G}_1 = \dot{g}$, and thus $\mathcal{K} \geq \pi_1^{\to}\mathcal{K} \times \pi_2^{\to}\mathcal{K} = \mathcal{F}_1 \times \dot{g}$. Let $F_1 \in \mathcal{F}_1$; then $\varphi^{\to}(\mathcal{F}_1 \times \dot{g}) \leq \varphi^{\to}\mathcal{K} = \beta^{\to}\mathcal{F}$ implies that there exists $F \in \mathcal{F}$ such that $\beta(F) \subseteq \varphi(F_1 \times \{g\})$. If $z \in F$, then $\beta(z) = \langle (z,e) \rangle = \langle (z_1,g) \rangle$, for some $z_1 \in F_1$, and thus $\lambda(z,g) = \lambda(z_1,e) = z_1 \in F_1$. It follows that $\lambda(F \times \{g\}) \subseteq F_1$, and thus $\lambda^{\to}(\mathcal{F} \times \dot{g}) \geq \mathcal{F}_1 \xrightarrow{q} y$. Since $\mathcal{F} \times \dot{g}$ is an ultrafilter on $X \times S$ and λ is a proper map, $\mathcal{F} \times \dot{g} \xrightarrow{r} (s,t)$, for some $(s,t) \in \lambda^{-1}(y)$. Then $\mathcal{F} \xrightarrow{q} s$ and $g = t$ since p is discrete. It follows that $\lambda(y,e) = y = \lambda(s,t) = \lambda(s,g)$, and thus $(s,e) \sim (y,g)$. As shown above, $(y,g) \sim (x,e)$, and thus $(x,e) \sim (s,e)$. Therefore $x = s$, and $\mathcal{F} \xrightarrow{q} x$.

Finally, let $\mathcal{F}$ be any filter on X such that $\beta^{\to}\mathcal{F} \xrightarrow{\sigma} \beta(x)$. If $\mathcal{H}$ is any ultrafilter on X containing $\mathcal{F}$, then $\beta^{\to}\mathcal{H} \xrightarrow{\sigma} \beta(x)$, and from the previous case, $\mathcal{H} \xrightarrow{q} x$. Since (X,q) is a Choquet space, $\mathcal{F} \xrightarrow{q} x$ and hence β is an embedding. □

Assume that $((X,q),(S,.,p),\lambda) \in$ GQ, $(X,\bar{q})$ is the finest Choquet space such that $\bar{q} \leq q$, $\bar{r} = \bar{q} \times p$, and let $\bar{\sigma}$ denote the quotient convergence structure on $B(X,S)$ determined by $\varphi : (X \times S, \bar{r}) \to B(X,S)$.

Corollary 4.10. *Assume* $((X,q),(S,.,p),\lambda) \in GQ$, p *is discrete, and* λ *is a proper map. Then, using the above notations,* $\beta : (X,\bar{q}) \to (B(X,S),\bar{\sigma})$ *is an embedding.*

Proof. It follows from Theorem 3.7 that $((X,\bar{q}),(S,.,p),\lambda) \in$ AC. Since q and $\bar{q}$ agree on ultrafilter convergence, $\lambda : (X,\bar{q}) \times (S,p) \to (X,\bar{q})$ is also a proper map, and $(X,\bar{q})$ is a Choquet space. Then according to Theorem 4.9, $\beta : (X,\bar{q}) \to (B(X \times S),\bar{\sigma})$ is an embedding. □

Let us conclude by showing that the generalized quotient of a product is homeomorphic to the product of the generalized quotients. Assume that $((X_j, q_j), (S_j, ., p_j), \lambda_j) \in$ GQ, for each $j \in J$. Let $(X, q) = \underset{j \in J}{\times} (X_j, q_j)$ and $(S, ., p) = \underset{j \in J}{\times} (S_j, ., p_j)$ denote the product spaces, and define $\lambda : X \times S \to X$ by $\lambda((x_j), (g_j)) = (\lambda_j(x_j, g_j))$. According to Corollary 3.9, $((X, q), (S, ., p), \lambda) \in$ AC. Moreover, since each $(S_j, ., p_j)$ is commutative and $\lambda_j(., g)$ is an injection for each $j \in J$, $(S, ., p)$ is commutative and $\lambda(., g)$ is an injection. Hence $((X, q), (S, ., p), \lambda) \in$ GQ. Let $\varphi_j : (X_j, q_j) \times (S_j, ., p_j) \to (B(X_j, S_j), \sigma_j)$ denote the convergence quotient map, $r_j = q_j \times p_j$, $\varphi = \underset{j \in J}{\times} \varphi_j$, for each $j \in J$. Since the product of quotient maps in CONV is again a quotient map, $\varphi : \underset{j \in J}{\times} (X_i \times S_j, r_j) \to \underset{j \in J}{\times} (B(X_j, S_j), \sigma_j)$ is also a quotient map. Denote $\sigma = \underset{j \in J}{\times} \sigma_j$.

Define $((x_j), (g_j)) \sim ((y_j), (h_j))$ in $X \times S$ iff $\lambda((x_j), (h_j)) = \lambda((y_j), (g_j))$. This is an equivalence relation on $X \times S$, and it follows from the definition of λ that $((x_j), (g_j)) \sim ((y_j), (h_j))$ iff $(x_j, g_j) \sim (y_j, h_j)$, for each $j \in J$. Let $(B(X, S), \Sigma)$ denote the corresponding generalized quotient space, where $\Phi : (X \times S, r) \to (B(X, S), \Sigma)$ is the quotient map and $r = \underset{j \in J}{\times} r_j$.

Theorem 4.11. *Suppose that $((X_j, q_j), (S_j, ., p_j), \lambda_j) \in GQ$, for each $j \in J$. Then, employing the notations defined above, $\underset{j \in J}{\times} (B(X_j, S_j), \sigma_j)$ and $(B(X, S), \Sigma)$ are homeomorphic.*

Proof. Consider the following diagram:

$$\begin{array}{ccc} \underset{j \in J}{\times} (X_j \times S_j, r_j) & \xrightarrow{\ \delta\ } & (X \times S, r) \\ \Big\downarrow \varphi & & \Big\downarrow \Phi \\ \underset{j \in J}{\times} (B(X_j, S_j), \sigma_j) & \xrightarrow{\Delta} & (B(X, S), \Sigma), \end{array}$$

where $\delta(((x_j, g_j)_j)) = ((x_j), (g_j))$ and $\Delta((\langle (x_j, g_j) \rangle_j)) = \langle ((x_j), (g_j)) \rangle$. Then δ is a homeomorphism, and the diagram commutes. Note that Δ is a bijection. Indeed, if $\Delta((\langle (x_j, g_j) \rangle_j)) = \Delta((\langle (y_j, h_j) \rangle_j))$, then $((x_j), (g_j)) \sim ((y_j), (h_j))$ and thus $(x_j, g_j) \sim (y_j, h_j)$, for each $j \in J$. Hence $\langle (x_j, g_j) \rangle_j = \langle y_j, g_j \rangle_j$ for each $j \in J$, and thus Δ is an injection. Clearly Δ is a surjection.

It is shown that Δ is continuous. Assume that $\mathcal{H} \xrightarrow{\sigma} (\langle (y_j, h_j) \rangle_j)$; then since φ is a quotient map, there exist $((x_j), (g_j)) \sim ((y_j), (h_j))$ and $\mathcal{K} \xrightarrow{r} ((x_j, g_j)_j)$ such that $\varphi^{\to}\mathcal{K} = \mathcal{H}$. However, the diagram commutes, and thus $\Delta^{\to}\mathcal{H} = (\Delta \circ \varphi)^{\to}\mathcal{K} = (\Phi \circ \delta)^{\to}\mathcal{K} \xrightarrow{\Sigma} \Phi((x_j), (g_j)) = \Phi((y_j), (h_j)) = \langle ((y_j), (h_j)) \rangle$. Hence Δ is continuous.

Conversely, suppose that $\mathcal{H} \xrightarrow{\Sigma} \langle ((y_j), (h_j)) \rangle$; then since Φ is a quotient map, there exist $((x_j),(g_j)) \sim ((y_j),(h_j))$ and $\mathcal{K} \xrightarrow{r} ((x_j),(g_j))$ such that $\Phi^{\rightarrow}\mathcal{K} = \mathcal{H}$. Using the fact that δ is a homeomorphism and that the diagram commutes, $\Delta^{\leftarrow}\mathcal{H} = (\varphi \circ \delta^{-1})^{\rightarrow}\mathcal{K} \xrightarrow{\sigma} \varphi((x_j,g_j)_j) = \varphi((y_j,h_j)_j) = (\langle(y_j,h_j)\rangle_j)$, and thus Δ^{-1} is continuous. Therefore Δ is a homeomorphism. □

Remark 4.12. In general, quotient maps are not productive in the category of all topological spaces with the continuous maps as morphisms. Whether or not Theorem 4.11 is valid in the topological context is unknown to the authors.

References

[1] J. Burzyk, C. Ferens and P. Mikusiński, *On the topology of generalized quotients*, Applied Gen. Top. **9** (2008), 205–212.

[2] D. Kent, *Convergence quotient maps*, Fund. Math. **65** (1969), 197–205.

[3] D. Kent and G. Richardson, *Open and proper maps between convergence spaces*, Czech. Math. J. **23**(1973), 15–23.

[4] D. Kent and G. Richardson, *The regularity series of a convergence space*, Bull. Austral. Math. Soc. **13** (1975), 21–44.

[5] M. Khosravi, Pseudoquotients: Construction, applications, and their Fourier transform, Ph.D. dissertation, Univ. of Central Florida, Orlando, FL, 2008.

[6] P. Mikusiński, *Boehmians and generalized functions*, Acta Math. Hung. **51** (1988), 271–281.

[7] P. Mikusiński, *Generalized quotients with applications in analysis*, Methods and Applications of Anal. **10** (2003), 377–386.

[8] W. Park, *Convergence structures on homeomorphism groups*, Math. Ann. **199** (1972), 45–54.

[9] W. Park, *A note on the homeomorphism group of the rational numbers*, Proc. Amer. Math. Soc. **42** (1974), 625–626.

[10] N. Rath, *Action of convergence groups*, Topology Proceedings **27** (2003), 601–612.

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