Applied General Topology
© Universidad Politécnica de Valencia
Volume 10, No. 1, 2009
pp. 1-12

Almost cl-supercontinuous functions

J. K. Kohli and D. Singh

Abstract. Reilly and Vamanamurthy introduced the class of 'clopen maps' (≡ 'cl-supercontinuous functions'). Subsequently generalizing clopen maps, Ekici defined and studied almost clopen maps (≡ almost cl-supercontinuous functions). Continuing in the spirit of Ekici, here basic properties of almost clopen maps are studied. Behavior of separation axioms under almost clopen maps is elaborated. The interrelations between direct and inverse transfer of topological properties under almost clopen maps are investigated. The results obtained in the process generalize, improve and strengthen several known results in literature including those of Ekici, Singh, and others.

2000 AMS Classification: Primary: 54C05, 54C10; Secondary: 54D10, 54D15, and 54D 20.

Keywords: almost clopen map, almost cl-supercontinuous function, (almost) z-supercontinuous function, clopen almost closed graphs, almost zero dimensional space, hyperconnected space.

1. Introduction

Variants of continuity occur in almost all branches of mathematics and applications of mathematics. The strong variants of continuity with which we shall be dealing in this paper include strongly continuous functions introduced by Levine [13], perfectly continuous functions considered by Noiri ([18], [19]), clopen maps (≡ cl-supercontinuous functions) defined by Reilly and Vamanamurthy [21], and studied by Singh [26], z-supercontinuous functions initiated by Kohli and Kumar [12], and supercontinuous functions introduced by Munshi and Bassan [16]. The variants of continuity which are independent of continuity and will be dealt with in this paper include regular set connected functions (≡ almost perfectly continuous functions) defined by Dontchev, Ganster and Reilly [3], almost clopen maps (≡ almost cl-supercontinuous functions) studied by Ekici [4], almost z-supercontinuous functions [11] and δ-continuous functions defined by Noiri [17]. Moreover, the weak forms of continuity which will crop up in our discussion include almost continuous functions due to Singal and Singal [24], θ-continuous functions [5], quasi θ-continuous functions [20], weakly continuous functions [14], faintly continuous functions [15], D_δ-continuous functions [9], z-continuous functions [23], and others.

The purpose of this paper is to study properties of almost cl-supercontinuous functions ($\equiv$ almost clopen maps). In the process we generalize, improve and refine several known results in the literature including those of Ekici [4], Singh [26], and others.

Section 2 is devoted to basic definitions, preliminaries and nomenclature. In Section 3 of this paper we study basic properties of almost cl-supercontinuous functions. It is shown that (i) almost cl-supercontinuity is preserved under the expansion of range as well as under the shrinking of range if $f(X)$ is δ-embedded in Y; (ii) A mapping into a product space is almost cl-supercontinuous if and only if its composition with each projection map onto the co-ordinate space is almost cl-supercontinuous; (iii) If X is almost zero-dimensional, then f is almost cl-supercontinuous if and only if the graph function is almost cl-supercontinuous.

Section 4 is devoted to the behavior of separation axioms under almost cl-supercontinuous functions wherein interrelations between direct and inverse transfer of separation properties are investigated. In the process we generalize and considerably improve upon certain results of Ekici [4], and Singh [26].

In Section 5, we interrelate (almost) cl-supercontinuity and connectedness. In the process we prove the existence and nonexistence of certain (almost) cl-supercontinuous functions. In Section 6, we consider clopen almost closed graphs and obtain refinements of certain results of Ekici [4].

2. Preliminaries and basic definitions

2.1. **Nomenclature.** Reilly and Vamanamurthy [21] call a function *clopen continuous* if for each open set V containing $f(x)$ there is a clopen (closed and open) set U containing x such that $f(U) \subset V$. Similarly, Ekici [4] calls a function *almost clopen* if for each $x \in X$ and each regular open set V containing $f(x)$ there is a clopen set U containing x such that $f(U) \subset V$. Moreover, Dontchev, Ganster and Reilly [3] call a function *regular set connected* if $f^{-1}(V)$ is clopen in X for every regular open set V in Y.

However, as was also pointed out in [26] that in the topological folklore the phrase "clopen map" is used for the functions which map clopen sets to open sets and hence therein the "clopen continuous maps" of Reilly and Vamanamurthy are renamed as "cl-supercontinuous functions", a better nomenclature since it represents a strong form of supercontinuity introduced by Munshi and Bassan [16]. In the same sprit in this paper we rename "almost clopen maps" studied by Ekici [4] as "almost cl-supercontinuous functions" and "regular set connected functions" defined by Dontchev, Ganster and Reilly [3] as "almost perfectly continuous functions", respectively.

For the convenience of the reader and for the clarity of presentation we give here the precise definitions of all these variants of continuity.

Definition 2.2. *A function $f : X \to Y$ from a topological space X into a topological space Y is said to be*

(i) strongly continuous [13] *if $f(\bar{A}) \subset f(A)$ for each subset A of X.*
(ii) perfectly continuous [18] *if $f^{-1}(V)$ is clopen in X for every open set $V \subset Y$.*
(iii) almost perfectly continuous *(*$\equiv$ regular set connected [3]*) if $f^{-1}(V)$ is clopen for every regular open set V in Y.*
(iv) cl-supercontinuous [26] *(*$\equiv$ clopen map [21]*) if for each open set V containing $f(x)$ there is a clopen set U containing x such that $f(U) \subset V$.*
(v) almost cl-supercontinuous *(*$\equiv$ almost clopen map [4]*) if for each $x \in X$ and each regular open set V containing $f(x)$ there is a clopen set U containing x such that $f(U) \subset V$.*
(vi) ***z***-supercontinuous [12] *if for each $x \in X$ and each open set V containing $f(x)$ there is a cozero set U containing x such that $f(U) \subset V$.*
(vii) *almost z-supercontinuous* [11] *if for each $x \in X$ and each regular open set V containing $f(x)$ there is a cozero set U containing x such that $f(U) \subset V$.*
(viii) supercontinuous [16] *if for each $x \in X$ and each open set V containing $f(x)$ there is a regular open set U containing x such that $f(U) \subset V$.*
(ix) δ-continuous [17] *if for each $x \in X$ and each regular open set V containing $f(x)$ there is a regular open set U containing x such that $f(U) \subset V$.*
(x) almost continuous [24] *if for each $x \in X$ and each regular open set V containing $f(x)$ there is an open set U containing x such that $f(U) \subset V$.*

Remark 2.3. *The original definitions of the concepts* (v), (vii), (viii), (ix) *and* (x) *in Definitions* 2.2 *are slightly different from the ones which first appeared in the literature but are equivalent to ones given here, and are the simplest and most convenient to work with.*

The following implications are immediate from the definitions and well known (or easily verified).

$$
\begin{array}{ccc}
\text{strongly continuous} & & \\
\Downarrow & & \\
\text{perfectly continuous} & \Rightarrow & \text{almost perfectly continuous} \\
\Downarrow & & \Downarrow \\
\text{cl-supercontinuous} & \Rightarrow & \text{almost cl-supercontinuous} \\
\Downarrow & & \Downarrow \\
z\text{-supercontinuous} & \Rightarrow & \text{almost } z\text{-supercontinuous} \\
\Downarrow & & \Downarrow \\
\text{supercontinuous} & \Rightarrow & \delta\text{-continuous} \\
\Downarrow & & \Downarrow \\
\text{continuous} & \Rightarrow & \text{almost continuous}
\end{array}
$$

However, it is well known that none of the above implications is reversible.

3. Basic properties of almost cl-supercontinuous functions

Definition 3.1. *A set G in a topological space X is said to be* cl-open [26] *(*δ-open *[29]) if for each $x \in G$, there exist a clopen (regular open) set H such that $x \in H \subseteq G$, equivalently G is the union of clopen (regular open) sets. The complement of a cl-open (δ-open) set is referred to as* cl-closed *(*δ-closed*) set.*

Theorem 3.2. *Let $f : X \to Y$ and $g : Y \to Z$ be functions. Then the following statements are true.*

(a) *If f is cl-supercontinuous and g is continuous, then $g \circ f$ is cl-supercontinuous.*
(b) *If f is cl-supercontinuous and g is almost continuous, then $g \circ f$ is almost cl-supercontinuous.*
(c) *If f is almost cl-supercontinuous and g is δ-continuous, then $g \circ f$ is almost cl-supercontinuous.*
(d) *If f is almost cl-supercontinuous and g is supercontinuous, then $g \circ f$ is cl-supercontinuous.*

Proof. The assertion (a) is due to Singh (see [26, Theorem 2.10]) and (b) is due to Ekici [4, Theorem 13(2)].

To prove (c); let $W \subset Z$ be a regular open set. Since g is δ-continuous, $g^{-1}(W)$ is a δ-open set in Y, i.e. $g^{-1}(W) = \bigcup\limits_{\alpha} V_\alpha$, where each V_α is a regular open set in Y (see [17]). Since f is almost cl-supercontinuous, each $f^{-1}(V_\alpha)$ is cl-open in X. Thus $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W)) = f^{-1}(\bigcup\limits_{\alpha} V_\alpha) = \bigcup\limits_{\alpha} f^{-1}(V_\alpha)$ being the union of cl-open sets is cl-open in X and so $g \circ f$ is almost cl-supercontinuous.

To prove (d); let W be an open set in Z. Since g is supercontinuous, $g^{-1}(W)$ is δ-open set in Y, i.e. $g^{-1}(W) = \bigcup\limits_{\alpha} V_\alpha$, where each V_α is a regular open set in Y(see[16]). Since f is almost cl-supercontinuous, $f^{-1}(V_\alpha)$ is a cl-open set in X for each α. Thus $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W)) = f^{-1}(\bigcup\limits_{\alpha} V_\alpha) = \bigcup\limits_{\alpha} f^{-1}(V_\alpha)$ being the union of cl-open sets is cl-open in X. Hence $g \circ f$ is cl-supercontinuous. □

Remark 3.3. *The assertion* (c) *of Theorem* 3.2 *represents a simultaneous generalization of parts* (1), (4), (5) *and* (6) *of Theorem* 13 *of Ekici* [4].

Theorem 3.4. *Let $\{X_\alpha : \alpha \in \Lambda\}$ be a cl-open cover of X. If for each α $f_\alpha = f|X_\alpha$ is almost cl-supercontinuous, then f is almost cl-supercontinuous.*

Proof. Let V be a regular open subset of Y. Then $f^{-1}(V) = \cup\{f_\alpha^{-1}(V) : \alpha \in \Lambda\}$. Since each f_α is almost cl-supercontinuous, each $f_\alpha^{-1}(V)$ is cl-open in X_α and hence in X. Thus $f^{-1}(V)$ being the union of cl-open sets is cl-open and so f is almost cl-supercontinuous. □

Remark 3.5. *Since every clopen set is cl-open, Theorem* 3.4 *is an improvement of Theorem* 11 *of Ekici* [4].

Our next result gives a sufficient condition for the preservation of almost cl-supercontinuity under the shrinking of range. First we formulate the concept of a δ-embedded set which seems to be of considerable significance in itself.

Definition 3.6. *A subset S of a space X is said to be δ-embedded in X if every regular open set in S is the intersection of a regular open set in X with S or equivalently every regular closed set in S is the intersection of a regular closed set in X with S.*

Theorem 3.7. *Let $f : X \to Y$ be an almost cl-supercontinuous function. If $f(X)$ is δ-embedded in Y, then $f : X \to f(X)$ is almost cl-supercontinuous.*

Proof. Let V_1 be a regular open set in $f(X)$. Since $f(X)$ is δ-embedded in Y, there exists a regular open set V in Y such that $V_1 = V \cap f(X)$. Again, since f is almost cl-supercontinuous, $f^{-1}(V)$ is cl-open in X. Now $f^{-1}(V_1) = f^{-1}(V \cap f(X)) = f^{-1}(V) \cap f^{-1}(f(X)) = f^{-1}(V)$ and so $f : X \to f(X)$ is almost cl-supercontinuous. □

Remark 3.8. *In contrast to Theorem 3.7, it is easily verified that almost cl-supercontinuity is preserved under the expansion of range. The following lemma due to Singal and Singal [24] will be used in the sequel.*

Lemma 3.9 ([24])**.** *Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of spaces and let $X = \prod X_\alpha$ be the product space. If $x = (x_\alpha) \in X$ and V is a regular open set containing x, then there exists a basic regular open set $\prod V_\alpha$ such that $x \in \prod V_\alpha \subset V$, where V_α is a regular open set in X_α for each $\alpha \in \Lambda$ and $V_\alpha = X_\alpha$ for all except finitely many $\alpha_1, \alpha_2 \ldots \alpha_n \in \Lambda$.*

Our next result shows that a mapping into a product space is almost cl-supercontinuous if and only if its composition with each projection map onto a co-ordinate space is almost cl-supercontinuous.

Theorem 3.10. *Let $\{f_\alpha : X \to X_\alpha : \alpha \in \Lambda\}$ be a family of functions and let $f : X \to \prod_{\alpha\varepsilon\Lambda} X_\alpha$ be defined by $f(x) = (f_\alpha(x))$ for each $x \in X$. Then f is almost cl-supercontinuous if and only if each $f_\alpha : X \to X_\alpha$ is almost cl-supercontinuous.*

Proof. Let $f : X \to \prod_{\alpha\varepsilon\Lambda} X_\alpha$ be almost cl-supercontinuous. Since projection maps are δ-continuous, then in view of Theorem 3.2 (c) the composition $f_\alpha = p_\alpha \circ f$, where p_α denotes the projection of $\prod_{\alpha\varepsilon\Lambda} X_\alpha$ onto α^{th}-coordinate space X_α, is almost cl-supercontinuous for each α.

Conversely, suppose that each $f_\alpha : X \to X_\alpha$ is almost cl-supercontinuous. To show that the function f is almost cl-supercontinuous, it is sufficient to show that $f^{-1}(V)$ is cl-open for each regular open set V in the product space $\prod_{\alpha\varepsilon\Lambda} X_\alpha$. In view of Lemma 3.9, it is clear that each regular open set V in the product space $\prod X_\alpha$ is the union of basic regular open sets of the form $\prod V_\alpha$ where each V_α is regular open in X_α and $V_\alpha = X_\alpha$ for each α except finitely many indices $\alpha_1, \alpha_2 \ldots \alpha_n$. Thus each basic regular open set in $\prod X_\alpha$ is the finite intersection of sub-basic regular open sets of the form $V_\beta \times \prod_{\alpha \neq \beta} X_\alpha$, where V_β is a regular open set in X_β. Since arbitrary unions and finite intersections of cl-open sets is cl-open, it suffices to prove that $f^{-1}(S)$ is cl-open for every subbasic regular open set S in the product space $\prod_{\alpha\varepsilon\Lambda} X_\alpha$. Let $V_\beta \times \prod_{\alpha \neq \beta} X_\alpha$ be a subbasic regular open set in $\prod_{\alpha\varepsilon\Lambda} X_\alpha$. Then

$f^{-1}(V_\beta \times \prod_{\alpha\neq\beta} X_\alpha) = f^{-1}(p_\beta{}^{-1}(V_\beta)) = f_\beta^{-1}(V_\beta)$ is cl-open in X. Hence f is almost cl-supercontinuous. □

Definition 3.11 ([7]). *A space X is said to be almost zero dimensional at $x \in X$ if for every regular open set V containing x there exists a clopen set U containing x such that $U \subset V$. The space X is said to be almost zero dimensional if it is almost zero dimensional at each $x \in X$.*

Theorem 3.12 ([7]). *A space X is almost zero dimensional if and only if each regular open set in X is cl-open.*

Theorem 3.13. *Let $f : X \to Y$ be a function and $g : X \to X \times Y$, defined by $g(x) = (x, f(x))$ for each $x \in X$, be the graph function. Then g is almost cl-supercontinuous if and only if f is almost cl-supercontinuous and X is almost zero dimensional.*

Proof. Let $g : X \to X \times Y$ be almost cl-supercontinuous. Then in view of Theorem 3.2 (c) it is immediate that the composition $f = p_y \circ g$ is almost cl-supercontinuous, where p_y is the projection from $X \times Y$ onto Y (see also [4,Theorem12]). To prove that X is almost zero dimensional, let U be a reguar open set in X and let $x \in U$. Then $U \times Y$ is a regular open set containing $g(x)$. Since g is almost cl-supercontinuous, there exists a clopen set W containing x such that $g(W) \subset U \times Y$. Thus $x \in W \subset U$, which shows that U is a cl-open and so the space X is almost zero dimensional.

To prove sufficiency, let $x \in X$ and let W be a regular open set containing $g(x)$. By Lemma 3.9 there exist regular open sets $U \subset X$ and $V \subset Y$ such that $(x, f(x)) \in U \times V \subset W$. Since X is almost zero dimensional, there exists a clopen set G_1 in X containing x such that $x \in G_1 \subset U$. Since f is almost cl-supercontinuous, there exists a clopen set G_2 in X containing x such that $f(G_2) \subset V$. Let $G = G_1 \cap G_2$. Then G is a clopen set containing x and $g(G) \subset U \times V \subset W$. This proves that g is almost cl-supercontinuous. □

4. Separation axioms

Definitions 4.1. *A space X is said to be*

(i) *ultra Hausdorff* [27] *if for each pair of distinct points x and y in X there exist disjoint clopen sets U and V containing x and y, respectively.*
(ii) *ultra T_1 (≡clopen T_1 [4]) if for each pair of distinct points x and y in X there exist clopen sets U and V containing x and y, respectively such that $y \notin U$ and $x \notin V$.*
(iii) *ultra T_0-space if for each pair of distinct points x and y in X there exists a clopen set U containing one of the points x and y but not the other.*

Proposition 4.2. *For a topological space X the following statements are equivalent.*

(a) *X is an ultra Hausdorff space.*
(b) *X is an ultra T_1-space.*
(c) *X is an ultra T_0-space.*

Proof. Clearly (a)⇒(b)⇒(c). To prove (c)⇒(a), let X be an ultra T_0-space and let x, y be any two distinct points in X. Then there exists a clopen set U containing one of the points x and y but not the other. To be precise assume that $x \in U$ Then U and $X \setminus U$ are disjoint clopen sets containing x and y, respectively and so X is an ultra Hausdroff space. □

Definitions 4.3. *A topological space X is said to be*

(i) *δT_1-space* ($\equiv$ *r-T_1 space* [4]) *if for each pair of distinct points x and y in X there exist regular open sets U and V containing x and y, respectively such that $y \notin U$ and $x \notin V$.*

(ii) *δT_0-space if for each pair of distinct points x and y in X there exists a regular open set containing one of the points x and y but not the other.*

$$\begin{array}{ccccc} \text{Hausdorff space} & \Rightarrow & \delta T_1\text{-}space & \Rightarrow & \delta T_0\text{-}space \\ & & \Downarrow & & \Downarrow \\ & & T_1\text{-}space & \Rightarrow & T_0\text{-}space \end{array}$$

Example 4.4. *The real line with co-finite topology is a T_1-space which is not δT_0 and so not a δT_1-space.*

It is shown in [26] that if $f : X \to Y$ is a cl-supercontinuous injection into a T_0-space Y, then X is an ultra-Hausdorff space. In contrast, for an almost cl-supercontinuous injection we have the following.

Theorem 4.5. *Let $f : X \to Y$ be an almost cl-supercontinuous injection. If Y is a δT_0-space, then X is an ultra-Hausdorff space.*

Proof. Let x_1 and x_2 be two distinct points in X. Then $f(x_1) \neq f(x_2)$. Since Y is a δT_0-space, there exists a regular open set V containing one of the points $f(x_1)$ or $f(x_2)$ but not the other. To be precise, assume that $f(x_1) \in V$. Since f is an almost cl-supercontinuous function, there exists a clopen set U containing x_1 such that $f(U) \subset V$. Then U and $X \setminus U$ are disjoint clopen sets containing x_1 and x_2 respectively and so X is ultra-Hausdorff. □

Remark 4.6. *The above theorem generalizes Theorems* 20 *and* 22 *of Ekici* [4].

Further, Ekici ([4,Theorem23]) proved that the equalizer of two almost cl-supercontinuous functions into a Hausdorff space is closed. Here we obtain the following stronger version.

Theorem 4.7. *Let $f, g : X \to Y$ be almost cl-supercontinuous functions into a Hausdorff space Y. Then the equalizer $E = \{x \in X : f(x) = g(x)\}$ of the functions f and g is a cl-closed subset of X.*

Proof. To prove that E is cl-closed, we shall show that $X \setminus E$ is cl-open. To this end, let $x \in X \setminus E$. Then $f(x) \neq g(x)$. Since Y is Hausdorff, there exist disjoint open sets U_1 and V_1 containing $f(x)$ and $g(x)$, respectively. Then $U = (\bar{U}_1)^0$ and $V = (\bar{V}_1)^0$ are disjoint regular open sets containing $f(x)$ and $g(x)$, respectively. Since f and g are almost cl-supercontinuous functions, there exist clopen sets G_1 and G_2 containing x such that $f(G_1) \subset U$ and $g(G_2) \subset V$.

Then $G = G_1 \cap G_2$ is a clopen set containing x. Since U and V are disjoint, clearly $G \subset X \setminus E$ and so $X \setminus E$ is cl-open. □

The following theorem represents an strengthening of Theorem 24 of Ekici [4].

Theorem 4.8. *Let $f : X \to Y$ be an almost cl-supercontinuous function into a Hausdorff space Y. Then the set $A = \{(x_1, x_2) \in X \times X : f(x_1) = f(x_2)\}$ is a cl-closed subset of $X \times X$.*

Proof. Let $(x, y) \notin A$. Then $f(x) \neq f(y)$. Since Y is Hausdorff, there exist disjoint open sets U_1 and V_1 containing $f(x)$ and $f(y)$, respectively. Then $U = (\bar{U}_1)^0$ and $V = (\bar{V}_1)^0$ are disjoint regular open sets containing $f(x)$ and $f(y)$, respectively. Since f is almost cl-supercontinuous, there exist clopen sets G_1 and G_2 containing x and y, respectively such that $f(G_1) \subset U$ and $f(G_2) \subset V$. Then $G_1 \times G_2$ is a clopen subset of $X \times X$ containing (x, y) and $(G_1 \times G_2) \cap A = \phi$. Hence $G_1 \times G_2 \subset (X \times X) \setminus A$ and so $(X \times X) \setminus A$ is cl-open being the union of clopen sets. Thus A is a cl-closed subset of $X \times X$. □

Definitions 4.9. *A space X is said to be*

(i) *almost regular* [22] *if for each regularly closed set F and each $x \notin F$ there exist disjoint open sets U and V containing x and F, respectively.*
(ii) *mildly normal* [25] *if for every pair of disjoint regular closed sets A and B there exist disjoint open sets U and V containing A and B, respectively.*

The following theorem shows that the hypothesis that "X is regular" in Theorem 27 of Ekici [4] is superfluous and hence can be omitted.

Theorem 4.10. *Let $f : X \to Y$ be an almost cl-supercontinuous open bijection. Then Y is an almost regular space.*

Proof. Let F be a regular closed subset of Y and let y be a point outside F. Then $f^{-1}(y) \cap f^{-1}(F) = \phi$ and $f^{-1}(y)$ is a singleton. Since f is almost cl-supercontinuous, $f^{-1}(F)$ is a cl-closed subset of X. Hence $X \setminus f^{-1}(F)$ is a cl-open subset of X containing $f^{-1}(y)$. So there exists a clopen set G containing $f^{-1}(y)$ such that $G \subset X \setminus f^{-1}(F)$. Then G and $X \setminus G$ are disjoint clopen sets containing $f^{-1}(y)$ and $f^{-1}(F)$, respectively. Since f is an open bijection, $f(G)$ and $f(X \setminus G)$ are disjoint open sets containing y and F, respectively. So Y is an almost regular space. □

Definitions 4.11. *A space X is said to be weakly Δ-normal* [2] *(weakly θ-normal* [8], [10]*) if each pair of disjoint δ-closed (θ-closed) are contained in disjoint open sets.*

The following theorem represents a significant improvement of Theorem 28 of Ekici [4].

Theorem 4.12. *Let $f : X \to Y$ be an almost cl-supercontinuous open bijection defined on a weakly θ-normal space X. Then Y is mildly normal.*

Proof. Let A and B be disjoint regular closed subsets of Y. Since f is almost cl-supercontinuous, $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint cl-closed subsets of X. Since every cl-closed set is θ-closed and since X is weakly θ-normal, there exist disjoint open sets U and V containing $f^{-1}(A)$ and $f^{-1}(B)$, respectively. Since f is an open bijection, $f(U)$ and $f(V)$ are disjoint open sets containing A and B, respectively and hence Y is mildly normal. □

Corollary 4.13. *Let $f : X \to Y$ be an almost cl-supercontinuous open bijection defined on a weakly Δ-normal space X. Then Y is mildly normal.*

Corollary 4.14 (Ekici [4, Theorem 28])**.** *If f is an almost cl-supercontinuous open bijection from a normal space X onto a space Y, then Y is mildly normal.*

Proof. Every normal space is a weakly θ-normal space. □

5. Connectedness

Ekici [4] calls a space X almost connected if X can not be written as a disjoint union of two nonempty regular open sets.

We observe that a space is connected if and only if it can not be expressed as a disjoint union of two nonempty clopen sets and hence it can not be written as the disjoint union of two nonempty regular open sets. Thus the notion of almost connectedness introduced by Ekici is precisely connectedness.

Moreover, the hypothesis of Theorem 30 of Ekici [4] is too strong and can be considerably weakened, since connectedness is preserved under functions satisfying fairly mild continuity conditions. The known such weakest variant of continuity is slight continuity [6]. A function $f : X \to Y$ is said to be slightly continuous if $f^{-1}(V)$ is open in X for every clopen subset V of Y. Thus connectedness is preserved under each of the following variants of continuity listed in the following diagram, each of which is weaker than continuity except δ-continuity (which is independent of continuity).

almost cl-supercontinuous
⇓
continuous — δ-continuous [17]
⇓ ⇓
almost continuous [24]
⇓
θ-continuous [5]
⇓ ⇓
quasi-θ-continuous [20] — weakly continuous [14]
⇓ ⇓
faintly continuous [15]
⇓
D_δ-continuous [9]
⇓
z-continuous [23]
⇓
slightly-continuous [6]

Definition 5.1 ([27], [1])**.** *A space X is said to be hyperconnected if every nonempty open set in X is dense in X.*

Ekici [4] showed that an almost cl-supercontinuous image of a connected space is hyperconnected. In contrast, our next result shows that cl-supercontinuous image of a connected space is indiscrete.

Theorem 5.2. *Let $f : X \to Y$ be a cl-supercontinuous function from a connected space X onto a space Y. Then Y is an indiscrete space.*

Proof. Suppose that Y is not indiscrete and let $V \neq Y$ be an open set in Y. Since f is cl-supercontinuous, by [26, Theorem 2.2] $f^{-1}(V)$ is a nonempty proper cl-open subset of X. So there exists a nonempty proper clopen subset of X, contradicting the fact that X is connected. □

Thus there exists no cl-supercontinuous function from a connected space onto a non indiscrete space. In contrast it is shown in [26, Theorem 4.9] that there exist no non constant cl-supercontinuous function from a connected space into a T_0-space.

6. Clopen almost closed graphs

Definition 6.1 ([4])**.** *The graph $G(f)$ of a function $f : X \to Y$ is said to be clopen almost closed if for each $(x, y) \notin G(f)$ there exists a clopen set U of x and a regular open set V containing y such that $(U \times V) \cap G(f) = \phi$.*

The following theorem represents an improved version of Theorem 35 of Ekici [4] which was essentially proved by him. However, for the convenience of the reader we include its proof.

Theorem 6.2. *Let $f : X \to Y$ be an injection such that its graph $G(f)$ is clopen almost closed. Then X is ultra Hausdorff.*

Proof. Let $x, y \in X, x \neq y$. Since f is an injection, $(x, f(y)) \notin G(f)$. In view of almost closedness of the graph $G(f)$, there exist a clopen set U of x and a regular open set V containing $f(y)$ such that $(U \times V) \cap G(f) = \phi$. Then $f(U) \cap V = \phi$ and hence $U \cap f^{-1}(V) = \phi$. Therefore $y \notin U$. Then U and $X \setminus U$ are disjoint clopen sets containing x and y, respectively. Hence X is ultra Hausdorff. □

Finally, we point out that in the hypothesis of [4, Theorem 41, Part 3], it is sufficient to assume X to be countably compact instead of compact.

The next result is an strengthening of Theorem 39 of Ekici [4] which was essentially proved by him. However, for the sake of completeness and continuity of presentation, we include its proof.

Theorem 6.3. *Let $f : X \to Y$ be a function such that the graph $G(f)$ of f is clopen almost closed in $X \times Y$. Then $f^{-1}(K)$ is cl-closed in X for every N-closed subset K of Y.*

Proof. Let K be an N-closed subset of Y. To prove that $f^{-1}(K)$ is cl-closed, we shall show that $X \setminus f^{-1}(K)$ is cl-open. To this end, let $x \in X \setminus f^{-1}(K)$. Then for each $y \in K, (x, y) \notin G(f)$. So there exists a clopen set U_y containing x and a regular open set V_y containing y such that $(U_y \times V_y) \cap G(f) = \phi$ and hence $f(U_y) \cap V_y = \phi$. The collection $\{V_y : y \in K\}$ is a cover of K by regular open sets in Y. So there exist finitely many $y_1, \ldots, y_n \in K$ such that $K \subset \bigcup_{i=1}^{n} Y_{y_i}$. Let $U = \bigcap_{i=1}^{n} U_{y_i}$. Then U is a clopen set containing x such that $f(U) \cap K = \phi$. Hence $U \subset X \setminus f^{-1}(K)$ and so U is cl-open being the union of clopen sets. □

References

[1] N. Ajmal and J. K. Kohli, *Properties of hyperconnected spaces, their mapping into Hausdorff space and embedding into hyperconnected spaces*, Acta Math. Hungar. **60**, no. 1-2 (1992), 41–49.

[2] A. K. Das, *Δ-normal spaces and factorizations of normality*, preprint.

[3] J. Dontchev, M. Ganster and I. Reilly, *More on almost s-continuity*, Indian J. Math. **41** (1999), 139–146.

[4] E. Ekici, *Generalizations of perfectly continuous, regular set connected and clopen functions*, Acta Math. Hungar. **107**, no. 3 (2005), 193–205.

[5] S. Fomin, *Extensions of topological spaces*, Annals of Math. **44** (1943), 471–480.

[6] R. C. Jain, *The role of regularly open sets in general topology*, Ph.D. thesis, Meerut Univ., Institute of Advanced Studies, Meerut, India (1980).

[7] J. K. Kohli, *Localization of topological propreties and certain generalizations of zero dimensionality*, preprint.

[8] J. K. Kohli and A. K. Das, *New normality axioms and decompositions of normality*, Glasnik Mat. **37**, no. 57 (2003), 105–114.

[9] J. K. Kohli and D. Singh, *Between weak continuity and set connectedness*, Studii Si Cercetari Stintifice Seria Mathematica **15** (2005), 55–65.

[10] J. K. Kohli and D. Singh, *Weak normality properties and factorizations of normality*, Acta Math. Hungar. **110**, no. 1–2 (2006), 67–80.

[11] J. K. Kohli, D. Singh and R. Kumar, *Generalizations of Z-supercontinuous functions and D_δ-supercontinuous functions*, Applied General Topology, to appear.

[12] J. K. Kohli and R. Kumar, *Z-supercontinuous functions*, Indian J. Pure Appl. Math. **33**, no. 7 (2002), 1097–1108.

[13] N. Levine, *Strong continuity in topological spaces*, Amer. Math. Monthly **67** (1960), 269.

[14] N. Levine, *A decomposition of continuity in topological spaces*, Amer. Math. Monthly **68** (1961), 44–46.

[15] P. E. Long and L. L. Herrington, *T_θ-topology and faintly continuous functions*, Kyungpook Math. J. **22** (1982), 7–14.

[16] B. M. Munshi and D. S. Bassan, *Super-continuous mappings*, Indian J. Pure Appl. Math. **13** (1982), 229–236.

[17] T. Noiri, *On δ-continuous functions*, J. Korean Math. Soc. **16** (1980), 161–166.

[18] T. Noiri, *Supercontinuity and some strong forms of continuity*, Indian J. Pure. Appl. Math. **15**, no. 3 (1984), 241–250.

[19] T. Noiri, *Strong forms of continuity in topological spaces*, Suppl. Rendiconti Circ. Mat. Palermo, II **12** (1986), 107–113.

[20] T. Noiri and V. Popa, *Weak forms of faint continuity*, Bull. Math. de la Soc. Math. de la Roumanic **34**, no. 82 (1990), 263–270.

[21] I. L. Reilly and M. K. Vamanamurthy, *On super-continuous mappings*, Indian J.Pure. Appl. Math. **14**, no. 6 (1983), 767–772.

[22] M. K. Singal and S. P. Arya, *On almost regular spaces*, Glasnik Mat. **4** (1969), 89–99.

[23] M. K. Singal and S. B. Nimse, *Z-continuous mappings*, The Mathematics Student **66**, no. 1-4 (1997), 193–210.
[24] M. K. Singal and A. R. Singal, *Almost continuous mappings*, Yokohama Math. Jour. **16** (1968), 63–73.
[25] M. K. Singal and A. R. Singal, *Mildly normal spaces*, Kyungpook Math. J. **3** (1973), 27–31.
[26] D. Singh, *cl-supercontinuous functions*, Applied General Topology **8**, no. 2 (2007), 293–300.
[27] R. Staum, *The algebra of bounded continuous functions into nonarchimedean field*, Pacific J. Math. **50** (1974), 169–185.
[28] L. A. Steen and J. A. Seeback, Jr., *Counter Examples in Topology,* Springer Verlag, New York, 1978.
[29] N. V. Velicko, *H-closed topological spaces*, Amer. Math. Soc. Transl. **78**, no. 2 (1968), 103–118.

Received September 2007

Accepted April 2008

J. K. Kohli (jk_kohli@yahoo.com)
Department of Mathematics, Hindu College, University of Delhi, Delhi 110 007, India

D. Singh (dstopology@rediffmail.com)
Department of Mathematics, Sri Aurobindo College, University of Delhi-South Campus, Delhi 110 017, India