APPLIED GENERAL TOPOLOGY

Well-posedness, bornologies, and the structure of metric spaces

GERALD BEER AND MANUEL SEGURA*

ABSTRACT. Given a continuous nonnegative functional λ that makes sense defined on an arbitrary metric space $\langle X, d \rangle$, one may consider those spaces in which each sequence $\langle x_n \rangle$ for which $\lim_{n \to \infty} \lambda(x_n) = 0$ clusters. The compact metric spaces, the complete metric spaces, the cofinally complete metric spaces, and the UC-spaces all arise in this way. Starting with a general continuous nonnegative functional λ defined on $\langle X, d \rangle$, we study the bornology $\mathcal{B}_\lambda$ of all subsets A of X on which $\lim_{n \to \infty} \lambda(a_n) = 0 \Rightarrow \langle a_n \rangle$ clusters, treating the possibility $X \in \mathcal{B}_\lambda$ as a special case. We characterize those bornologies that can be expressed as $\mathcal{B}_\lambda$ for some λ, as well as those that can be so induced by a uniformly continuous λ.

1. INTRODUCTION

In a first course in analysis, one is introduced to two important classes of metric spaces as those in which certain sequences have cluster points: a metric space $\langle X, d \rangle$ is called *compact* if each sequence $\langle x_n \rangle$ in X has a cluster point, whereas $\langle X, d \rangle$ is called *complete* if each Cauchy sequence in X has a cluster point. A Cauchy sequence is one of course for which there exists for each $\varepsilon > 0$ a residual set of indices whose terms are pairwise ε-close. If we replace residual by cofinal in the definition, we get a so-called cofinally Cauchy sequence and the metric spaces X in which each cofinally Cauchy sequence has a cluster point are called *cofinally complete* [10, 13, 18, 24, 36]. These are a well-studied class of spaces lying between the compact spaces and the complete ones. Notably,

these are the metric spaces that are uniformly paracompact [13, 23, 24, 35] and also those on which each continuous function with values in a metric space is uniformly locally bounded [10]. Lying between the compact spaces and the cofinally complete spaces is the class of *UC-spaces*, also known as *Atsuji spaces*, which are those metric spaces on which each continuous function with values in a metric space is uniformly continuous [1, 5, 6, 7, 8, 27, 34, 38]. These are also called the *Lebesgue spaces* [32], as they are those metric spaces $\langle X, d \rangle$ for which each open cover has a Lebesgue number [1, 8]. These UC-spaces, too, can be characterized sequentially, as observed by Toader [37]: $\langle X, d \rangle$ is a UC space if and only if each pseudo-Cauchy sequence in X with distinct terms clusters, where $\langle x_n \rangle$ is called *pseudo-Cauchy* [8, p. 59] if for each $\varepsilon > 0$ and $n \in \mathbb{N}$, there exists $k > j > n$ with $d(x_j, x_k) < \varepsilon$.

It seems worthwhile to study in some organized way classes of metric spaces on which prescribed sequences have cluster points. One program could be to look at other modifications of the definition of Cauchy sequence, but this approach is limited in scope and is not our purpose here. Instead, given some continuous nonnegative extended real-valued functional λ that makes sense defined on an arbitrary metric space, we look at the class of "λ-spaces", i.e., the class of metric spaces $\langle X, d \rangle$ such that each sequence $\langle x_n \rangle$ in X with $\lim_{n \to \infty} \lambda(x_n) = 0$ has a cluster point. In terms of the language of optimization theory, a space is in this class if either $\inf\{\lambda(x) : x \in X\} > 0$ or the functional λ is *Tychonoff well-posed in the generalized sense* [20, 31]. All of the classes mentioned in the first paragraph fall within this framework. For the compact spaces, the zero functional does the job. For the UC-spaces, the measure of isolation functional $I(x) = d(x, X \backslash \{x\})$ is characteristic [1, 8, 27]. For the cofinally complete spaces, it is the measure of local compactness functional [10, 13] defined by

$$\nu(x) = \begin{cases} \sup\{\alpha > 0 : \mathrm{cl}(S_\alpha(x)) \text{ is compact} \} & \text{if } x \text{ is a point of local compactness} \\ 0 & \text{otherwise} \end{cases}.$$

For the complete metric spaces, and paralleling the cofinally complete spaces as we will see in Section 5 *infra*, it is it is the measure of local completeness functional defined by

$$\beta(x) = \begin{cases} \sup\{\alpha > 0 : \mathrm{cl}(S_\alpha(x)) \text{ is complete} \} & \text{if } x \text{ has a complete neighborhood} \\ 0 & \text{otherwise} \end{cases}.$$

In each case discussed above, unless identically equal to ∞, the functional is uniformly continuous; but we do not restrict ourselves in this way, nor do we insist that our metric spaces be complete.

We find it advantageous to first study more primitively the "λ-subsets" of an arbitrary metric space $\langle X, d \rangle$: those nonempty subsets A such that each sequence $\langle a_n \rangle$ within satisfying $\lambda(a_n) \to 0$ clusters. In general these form a bornology with closed base. As a major result, we characterize those bornologies that arise in this way.

2. Preliminaries

All metric spaces are assumed to contain at least two points. We denote the closure, set of limit points and interior of a subset A of a metric space $\langle X, d \rangle$ by $\mathrm{cl}(A)$, A' and $\mathrm{int}(A)$, respectively. We denote the power set of A by $\mathcal{P}(A)$ and the nonempty subsets of A by $\mathcal{P}_0(A)$. We denote the set of all closed and nonempty subsets of X by $\mathcal{C}_0(X)$, and the set of all closed subsets by $\mathcal{C}(X)$. We call $A \in \mathcal{P}_0(X)$ *uniformly discrete* if $\exists \varepsilon > 0$ such that whenever a_1, a_2 are in A and $a_1 \neq a_2$, then $d(a_1, a_2) \geq \varepsilon$. If $\langle Y, \rho \rangle$ is a second metric space, we denote the continuous functions from X to Y by $\mathcal{C}(X, Y)$.

If $x_0 \in X$ and $\varepsilon > 0$, we write $S_\varepsilon(x_0)$ for the open ε-ball with center x_0. If A is a nonempty subset of X, we write $d(x_0, A)$ for the distance from x_0 to A, and if $A = \varnothing$ we agree that $d(x_0, A) = \infty$. With $d(x, A)$ now defined, we denote for $\varepsilon > 0$ the *ε-enlargement* of $A \in \mathcal{P}(X)$ by $S_\varepsilon(A)$, i.e.,

$$S_\varepsilon(A) := \{x \in X : d(x, A) < \varepsilon\} = \bigcup_{x \in A} S_\varepsilon(x).$$

If $A \in \mathcal{P}_0(X)$ and $B \in \mathcal{P}(X)$, we define the *gap* between them by

$$D_d(A, B) := \inf \{d(a, B) : a \in A\}.$$

We can define the *Hausdorff distance* [8, 28] between two nonempty subsets A and B in terms of enlargements:

$$H_d(A, B) := \inf \{\varepsilon > 0 : A \subseteq S_\varepsilon(B) \text{ and } B \subseteq S_\varepsilon(A)\}.$$

Hausdorff distance so defined is an extended real-valued pseudometric on $\mathcal{P}_0(X)$ which when restricted to the nonempty bounded sets is finite valued, and which when restricted to $\mathcal{C}_0(X)$ is an extended real-valued metric. Hausdorff distance restricted to $\mathcal{C}_0(X)$ preserves the following properties of the underlying space (see, e.g., [8, Thm 3.2.4]).

Proposition 2.1. *Let $\langle X, d \rangle$ be a metric space. The following are true:*

(1) *$\langle \mathcal{C}_0(X), H_d \rangle$ is complete if and only if $\langle X, d \rangle$ is complete;*
(2) *$\langle \mathcal{C}_0(X), H_d \rangle$ is totally bounded if and only if $\langle X, d \rangle$ is totally bounded;*
(3) *$\langle \mathcal{C}_0(X), H_d \rangle$ is compact if and only if $\langle X, d \rangle$ is compact.*

A weaker form of convergence for sequences of closed sets than convergence with respect to Hausdorff distance is *Kuratowski convergence* [8, 29, 28]. Given a sequence $\langle A_n \rangle$ in $\mathcal{C}_0(X)$, we define $\mathrm{Li}\, A_n := \{x \in X : \forall \varepsilon > 0, \ S_\varepsilon(x) \cap A_n \neq \varnothing \text{ residually}\}$ and $\mathrm{Ls}\, A_n := \{x \in X : \forall \varepsilon > 0, \ S_\varepsilon(x) \cap A_n \neq \varnothing \text{ cofinally}\}$. We say $\langle A_n \rangle$ is *Kuratowski convergent* to A and write $K\text{-}\lim A_n = A$ if $A = \mathrm{Li}\, A_n = \mathrm{Ls}\, A_n$.

The following facts are well-known.

Proposition 2.2. *Let $\langle A_n \rangle$ be a sequence in $\mathcal{C}_0(X)$. Then the following are true:*

(1) *$Li\ A_n$ and $Ls\ A_n$ are both closed (but perhaps empty);*

(2) *$Li\ A_n \subseteq Ls\ A_n$;*

(3) *If $A_n = \{a_n\}$, then $Li\ A_n = \{lim\ a_n\}$ if $\ lim\ a_n$ exists and $Li\ A_n = \varnothing$ if not;*

(4) *If $A_n = \{a_n\}$, then $Ls\ A_n = \{x : x \text{ is a cluster point of } \langle a_n \rangle\} = \bigcap_{n \in \mathbb{N}}^{\infty} cl(\{a_k : \ k \geq n\})$;*

(5) *$\lim H_d(A_n, A) = 0 \Rightarrow K\text{-}\lim A_n = A$;*

(6) *If $\langle A_n \rangle$ is decreasing, then $K\text{-}\lim A_n = \bigcap_{n=1}^{\infty} A_n$.*

We can also define the *Hausdorff measure of noncompactness* [4] of a nonempty subset A in terms of enlargements:

$$\alpha(A) = \inf \{\varepsilon > 0 : A \subseteq S_\varepsilon(F), \text{ where } F \text{ is a nonempty finite subset of } X\}.$$

Clearly, $\alpha(A) = \infty$ if and only if A is unbounded. The functional α behaves as follows:

(1) If $A \subseteq B$, then $\alpha(A) \leq \alpha(B)$;

(2) $\alpha(cl(A)) = \alpha(A)$;

(3) $\alpha(A) = 0$ if and only if A is totally bounded;

(4) $\alpha(A \cup B) = \max \{\alpha(A), \alpha(B)\}$;

(5) If $\lim H_d(A_n, A) = 0$ then $\lim \alpha(A_n) = \alpha(A)$.

A famous theorem concerning the Hausdorff measure of noncompactness is *Kuratowski's Theorem* [4, 29], proved in a novel way below; but first, we state and prove a useful lemma:

Lemma 2.3. *Let $\langle X, d \rangle$ be a metric space. Suppose $\langle A_n \rangle$ is a decreasing sequence in $\mathcal{C}_0(X)$ which is not H_d-Cauchy. Then $\exists\ n_1 < n_2 < n_3 < \cdots$ and $x_{n_k} \in A_{n_k}$ such that $\{x_{n_k} : k \in \mathbb{N}\}$ is uniformly discrete.*

Proof. Let $\langle A_n \rangle$ be a decreasing sequence in $\mathcal{C}_0(X)$ that is not H_d-Cauchy. Then $\exists \varepsilon > 0$ such that $\forall n_0 \in \mathbb{N}$, $\exists m > n \geqslant n_0$ such that $H_d(A_m, A_n) > \varepsilon$. Choose m_1, m_2 with $m_2 > m_1 \geqslant 1$ such that $H_d(A_{m_1}, A_{m_2}) > \varepsilon$. Then let $i_1 > m_2$, and choose m_3, m_4 with $m_4 > m_3 \geqslant i_1$ such that $H_d(A_{m_3}, A_{m_4}) > \varepsilon$. Then let $i_2 > m_4$, and choose m_5, m_6 with $m_6 > m_5 \geqslant i_2$ such that $H_d(A_{m_5}, A_{m_6}) > \varepsilon$. Continuing, we have $m_1 < m_2 < m_3 < \cdots$ such that $H_d(A_{m_{2j-1}}, A_{m_{2j}}) > \varepsilon$ where $j \geqslant 1$. Now for $i \geqslant 1$, pick $x_{n_i} \in A_{m_{2i-1}}$ with $d(x_{n_i}, A_{m_{2i}}) > \varepsilon$. Then for $i < k$,

$$d(x_{n_i}, x_{n_k}) > d(x_{n_i}, A_{m_{2k-1}}) \geqslant d(x_{n_i}, A_{m_{2k-2}}) \geqslant d(x_{n_i}, A_{m_{2i}}) > \varepsilon.$$

We have shown $\langle x_{n_i} \rangle$ has distinct terms and is a uniformly discrete sequence. $\square$

Here is our novel proof (in one direction) of Kuratowski's Theorem based on completeness of $\langle \mathcal{C}_0(X), H_d \rangle$.

Theorem 2.4 (**Kuratowski's Theorem on Completeness**). *Let $\langle X, d \rangle$ be a metric space. Then $\langle X, d \rangle$ is complete if and only if whenever $\langle A_n \rangle$ is a decreasing sequence in $\mathcal{C}_0(X)$ with $\lim \alpha(A_n) = 0$, then $A := \bigcap_{n \in \mathbb{N}} A_n \neq \varnothing$.*

Proof. Suppose $\langle X, d \rangle$ is complete and $\langle A_n \rangle$ is a decreasing sequence in $\mathcal{C}_0(X)$ with $\lim \alpha(A_n) = 0$. Suppose $\langle A_n \rangle$ is not H_d-Cauchy. Then $\exists n_1 < n_2 < n_3 < ...$ and $x_{n_k} \in A_{n_k}$ such that $\langle x_{n_k} \rangle$ is a uniformly discrete sequence with distinct terms. Then $\exists \varepsilon > 0$ such that $d(x_{n_i}, x_{n_j}) > \varepsilon$ where $i \neq j$. Hence $\forall n \in \mathbb{N}$, $\alpha(A_n) \geq \frac{\varepsilon}{2} \Rightarrow \lim \alpha(A_n) \geq \frac{\varepsilon}{2}$, which is a contradiction. Thus $\langle A_n \rangle$ must be H_d-Cauchy. Since $\langle X, d \rangle$ is complete, by Proposition 2.1(1) $\langle \mathcal{C}_0(X), H_d \rangle$ is complete, so $\exists B \in \mathcal{C}_0(X)$ with $B = H_d - \lim \langle A_n \rangle$. Since $\lim H_d(A_n, B) = 0$, by Proposition 2.2(5) $B = K - \lim A_n$, . Since $\langle A_n \rangle$ is decreasing in $\mathcal{C}_0(X)$, by Proposition 2.2(6) K-$\lim A_n = \bigcap_{n=1}^{\infty} A_n$. Hence $B = A \Rightarrow A \neq \varnothing$.

Conversely, suppose whenever $\langle A_n \rangle$ is a decreasing sequence in $\mathcal{C}_0(X)$ with $\lim \alpha(A_n) = 0$, then $A \neq \varnothing$. Then if $\langle x_n \rangle$ is a Cauchy sequence, we have $\bigcap_{n \in \mathbb{N}}^{\infty} \mathrm{cl}\{x_k : k \geq n\}$ nonempty, so by Proposition 2.2(4), $\langle x_k \rangle$ has a cluster point. Hence $\langle X, d \rangle$ is complete. $\square$

Let $f : X \to [0, \infty]$. Then by saying f is *lower semi-continuous* at a point $x_0 \in X$, we mean whenever $\alpha < f(x_0)$, $\alpha \in \mathbb{R}$, then $\exists \delta > 0$ such that $\forall x \in S_\delta(x_0)$, $f(x) > \alpha$. By saying f is *upper semi-continuous* at a point $x_0 \in X$, we mean whenever $\alpha > f(x_0)$, $\alpha \in \mathbb{R}$, then $\exists \delta > 0$ such that $\forall x \in S_\delta(x_0)$, $f(x) < \alpha$. Note that $f(x_0) = \infty \Rightarrow f$ is upper semi-continuous at x_0, and $f(x_0) = 0 \Rightarrow f$ is lower semi-continuous at x_0. If f is both upper and lower semi-continuous at a point $x_0 \in X$, then we say f is *continuous* at x_0.

3. NOTES ON NONNEGATIVE CONTINUOUS FUNCTIONALS

We first discuss a framework in which many important nonnegative continuous functionals arise.

Let P be an hereditary property of open subsets of $\langle X, d \rangle$: if V, W are open sets where $W \subseteq V$ then $P(V) \Rightarrow P(W)$. Define $\lambda_P : X \to [0, \infty]$ by

$$\lambda_P(x) = \begin{cases} \sup\{\alpha > 0 : P(S_\alpha(x))\} & \text{if } \exists \alpha > 0 \text{ where } P(S_\alpha(x)); \\ 0 & \text{otherwise.} \end{cases}$$

Example 3.1. Consider $P(V) := V$ contains at most one point. Then the resulting λ_P is the measure of isolation functional $\lambda_P(x) = I(x) := d(x, X \setminus \{x\})$.

Example 3.2. Next consider $P(V) := V \cap E = \varnothing$ where $E \subseteq X$. Then the induced λ_P gives the distance from a variable point of X to the set E.

Example 3.3. Now consider the case when $P(V) := \mathrm{cl}(V)$ is compact. Then the resulting λ_P is the measure of local compactness functional ν giving the supremum of the radii of the closed balls with center x that are compact.

Example 3.4. A final example of an hereditary property P is $P(V) := V$ is countable.

Of particular importance is the *kernel* of the metric space $\langle X, d \rangle$ with respect to a continuous function $\lambda : X \to [0, \infty]$, which we define as $\text{Ker}(\lambda) := \{x \in X : \lambda(x) = 0\}$. If we consider the resulting λ_P from Example 3.1, then $\text{Ker}(\lambda_P) = X'$, the set of limit points of X. For λ_P from Example 3.2, we get $\text{Ker}(\lambda_P) = \text{cl}(E)$. For λ_P from Example 3.3, $\text{Ker}(\lambda_P)$ equals the points of non-local compactness of X. Finally, if we consider the corresponding λ_P for Example 3.4, then $\text{Ker}(\lambda_P)$ equals the set of condensation points of X.

Proposition 3.5. *Let P be an hereditary property of open sets in $\langle X, d \rangle$. If $\lambda_P(x_0) = \infty$ for some $x_0 \in X$, then $\lambda_P(x) = \infty$ for all $x \in X$. Otherwise if λ_P is finite valued, then λ_P is 1-Lipschitz.*

Proof. Suppose $\lambda_P(x_0) = \infty$ for some $x_0 \in X$. Let $x \in X$ where $x_0 \neq x$, and let $\alpha > 0$ be arbitrary. Since $\sup\{\mu > 0 : P(S_\mu(x_0))\} = \infty$, $\exists \alpha_0 > 0$ such that $P(S_{\alpha_0}(x_0))$ and $S_\alpha(x) \subseteq S_{\alpha_0}(x_0)$, so that $P(S_\alpha(x))$. This shows that $\lambda_P(x) = \infty$ for all $x \in X$. Otherwise, λ_P is finite valued. If λ_P fails to be 1-Lipschitz, there exist $x, w \in X$ with $\lambda_P(x) > \lambda_P(w) + d(x, w)$. Take an $\alpha > 0$ where $\lambda_P(x) > \alpha > \lambda_P(w) + d(x, w)$, so that $P(S_\alpha(x))$. Then $S_{\alpha - d(x,w)}(w) \subseteq S_\alpha(x)$, so $P(S_{\alpha - d(x,w)}(w))$. However, $\alpha - d(x, w) > \lambda_P(w)$, which is a contradiction. Hence, λ_P is 1-Lipschitz. $\qquad\square$

We next introduce the induced set functional $\overline{\lambda} : \mathcal{P}_0(X) \to [0, \infty]$ that we will use to characterize λ-spaces in Section 5:

$$\overline{\lambda}(A) := \sup\{\lambda(a) : a \in A\},$$

where $\lambda : X \to [0, \infty]$ is a continuous functional. The following proposition lists obvious properties of the set functional $\overline{\lambda}$.

Proposition 3.6. *Let $\langle X, d \rangle$ be a metric space and let $\overline{\lambda} : \mathcal{P}_0(X) \to [0, \infty]$ be as defined above. Then the following are true for nonempty subsets A, B:*

(1) $\overline{\lambda}(A \cup B) = \max\{\overline{\lambda}(A), \ \overline{\lambda}(B)\}$;

(2) $\overline{\lambda}(cl(A)) = \overline{\lambda}(A)$;

(3) $\overline{\lambda}(A) = 0$ *if and only if* $A \subseteq Ker(\lambda)$.

It is now useful to introduce a strengthening of uniform continuity of a function restricted to a subset of X as considered in [14, 15].

Definition 3.7. *Let $\langle X, d \rangle$ and $\langle Y, \rho \rangle$ be metric spaces and let A be a subset of X. We say that a function $f : X \to Y$ is strongly uniformly continuous on A if $\forall \varepsilon > 0 \ \exists \delta > 0$ such that if $d(x, w) < \delta$ and $\{x, w\} \cap A \neq \varnothing$, then $\rho(f(x), f(w)) < \varepsilon$.*

Note that strong uniform continuity on $A = \{x_0\}$ means simply that f is continuous at x_0. Strong uniform continuity on $A = X$ is uniform continuity. A continuous function on X is strongly uniformly continuous on each nonempty

compact subset, not merely uniformly continuous when restricted to such a subset.

Lemma 3.8. *Let $\lambda : X \to [0, \infty]$ be continuous. If λ is finite-valued and strongly uniformly continuous on $A \in \mathcal{P}_0(X)$ then $\overline{\lambda}$ is H_d-continuous at A.*

Proof. We show that $\overline{\lambda}$ is lower and upper semi-continuous at A, respectively. For lower semi-continuity, we have nothing to show if $\overline{\lambda}(A) = 0$. Otherwise, fix $\alpha_0 > 0$ and suppose $\alpha_0 < \overline{\lambda}(A)$. Then $\exists a_0 \in A$ such that $\lambda(a_0) > \alpha_0 + \varepsilon_0$, where $\varepsilon_0 > 0$. Choose by strong uniform continuity of λ on A $\delta_0 > 0$ such that if $a \in A, x \in X$ and $d(a, x) < \delta_0$, then $|\lambda(a) - \lambda(x)| < \varepsilon_0$. Now suppose $H_d(A, B) < \delta_0$; choose $b \in B$ such that $d(a_0, b) < \delta_0$. Then $|\lambda(a_0) - \lambda(b)| < \varepsilon_0 \Rightarrow \lambda(b) > \alpha_0 \Rightarrow \overline{\lambda}(B) > \alpha_0$.

For upper semi-continuity, we have nothing to show if $\overline{\lambda}(A) = \infty$. Otherwise, fix $\alpha_1 > 0$ with $\overline{\lambda}(A) < \alpha_1$. Fix $\varepsilon_1 > 0$ so that $\forall a \in A, \lambda(a) < \alpha_1 - \frac{\varepsilon_1}{2}$. Let $\delta_1 > 0$ be such that if $a \in A, x \in X$ and $d(a, x) < \delta_1$ then $|\lambda(a) - \lambda(x)| < \frac{\varepsilon_1}{3}$. Suppose $H_d(A, B) < \delta_1$ and let $b \in B$ be arbitrary. Choose $a \in A$ such that $d(a, b) < \delta_1$. Then

$$|\lambda(a) - \lambda(b)| < \tfrac{\varepsilon_1}{3} \Rightarrow \tfrac{-\varepsilon_1}{3} < \lambda(a) - \lambda(b) < \alpha_1 - \tfrac{\varepsilon_1}{2} - \lambda(b)$$

$$\Rightarrow \lambda(b) < \alpha_1 - \tfrac{\varepsilon_1}{6}.$$

Since $b \in B$ was arbitrary, $\overline{\lambda}(B) < \alpha_1$. $\square$

The next counterexample shows that when λ is not strongly uniformly continuous on A, it is not guaranteed that the $\overline{\lambda}$ functional is H_d-continuous at A.

Example 3.9. Let $X = [0, \infty) \times [0, \infty)$ and define $\lambda : X \to [0, \infty)$ by $\lambda(x, y) = xy$. Let $A = \{(0, y) : y \in [0, \infty)\}$. Obviously, λ is not strongly uniformly continuous on A, since one can take $\varepsilon = 1$ and for any $\delta > 0$, if $n > \frac{2}{\delta}$ we have $d((0, n), (\frac{2}{n}, n)) < \delta$, but $|\lambda(0, n) - \lambda(\frac{2}{n}, n)| = 2 > \varepsilon$. If we let $A_n = \{\frac{1}{n}\} \times [0, \infty)$, then $\langle A_n \rangle \overset{H_d}{\to} A$. But for all n, $\overline{\lambda}(A_n) = \infty$ while $\overline{\lambda}(A) = 0$, showing $\overline{\lambda}$ is not H_d-continuous at A.

4. λ-Subsets

Definition 4.1. *Let $\langle X, d \rangle$ be a metric space, and $\lambda : X \to [0, \infty]$ be continuous. We say $A \in \mathcal{P}_0(X)$ is a λ-subset of X if whenever $\langle a_n \rangle$ is a sequence in A and $\lambda(a_n) \to 0$, then $\langle a_n \rangle$ has a cluster point in X. When X is itself a λ-subset, then $\langle X, d \rangle$ is called a λ-space.*

We denote the family of λ-subsets by $\mathcal{B}_\lambda$. Note that $\mathcal{B}_\lambda$ is not altered by replacing λ by $\min\{\lambda, 1\}$, if one is bothered by functionals that naturally assume values of ∞. We now provide some examples.

Example 4.2. If $\langle X, d \rangle$ is any metric space, and $\lambda(x) \equiv 1$, then $\mathcal{B}_\lambda = \mathcal{P}_0(X)$.

Example 4.3. If $\langle X, d \rangle$ is a metric space, then the family of nonempty subsets with compact closure $\mathcal{K}_0(X)$ is $\mathcal{B}_\lambda$ for the zero functional λ on X.

Example 4.4. If $\langle X, d \rangle$ is an unbounded metric space and $x_0 \in X$, then the family of nonempty d-bounded subsets $\mathcal{B}_d(X)$ is $\mathcal{B}_\lambda$ for the continuous functional on X defined by

$$\lambda(x) = \frac{1}{1 + d(x, x_0)}$$

Notice here that while $\inf \lambda(X) = 0$, we have $\mathrm{Ker}(\lambda) = \varnothing$. We shall see presently that $\mathcal{B}_\lambda$ for all such λ-functionals arises in this way (see Theorem 4.17 *infra*).

Example 4.5. The λ-subsets of a metric space corresponding to the measure of isolation functional $I(x) = d(x, X \backslash \{x\})$ are called the *UC-subsets*, as studied in [15]. The λ-subsets of a metric space corresponding to the measure of local compactness functional ν are called the *cofinally complete subsets*, as studied in [13].

Definition 4.6. *Let X be a topological space. We call a family of nonempty subsets $\mathcal{A}$ of X a bornology* [9, 14, 22, 30] *provided*

(1) $\bigcup \mathcal{A} = X$;
(2) $\{A_1, A_2, A_3, ..., A_n\} \subseteq \mathcal{A} \Rightarrow \bigcup_{i=1}^{n} A_i \in \mathcal{A}$;
(3) $A \in \mathcal{A}$ *and* $\varnothing \neq B \subseteq A \Rightarrow B \in \mathcal{A}$.

We will of course be focusing on bornologies in a metric space $\langle X, d \rangle$. The largest bornology is $\mathcal{P}_0(X)$ and the smallest is the set of nonempty finite subsets $\mathcal{F}_0(X)$. The bornologies $\mathcal{K}_0(X)$ and $\mathcal{B}_d(X)$ lie between these extremes. Of importance in the sequel are *functional bornologies*, that is, bornologies arising as the family of subsets on which a real-valued function with domain X is bounded. The proof of the next proposition is left to the reader, and it implies that $\mathcal{K}_0(X)$ is the smallest possible $\mathcal{B}_\lambda$.

Proposition 4.7. *Let $\lambda : X \to [0, \infty]$ be continuous. Then $\mathcal{B}_\lambda$ forms a bornology containing the nonempty compact subsets.*

By a *base* for a bornology, we mean a subfamily that is cofinal in the bornology with respect to inclusion. For example, a countable base for the metrically bounded subsets of $\langle X, d \rangle$ consists of all balls with a fixed center and integral radius. The next result says that $\mathcal{B}_\lambda$ has a closed base, that is, a base that consists of closed sets.

Proposition 4.8. *Let $\lambda : X \to [0, \infty]$ be continuous, and let A be a λ-set. Then $cl(A)$ is also a λ-set.*

Proof. Let $\langle x_n \rangle$ be a sequence in $cl(A)$ where $\lambda(x_n) \to 0$. We may assume $\forall n \in \mathbb{N}$ that $\lambda(x_n) < \infty$. By the continuity of λ, $\exists$ a sequence $\langle a_n \rangle$ in A where $\forall n \in \mathbb{N}$, $d(x_n, a_n) < \frac{1}{n}$ and $\lambda(a_n) < \lambda(x_n) + \frac{1}{n}$. Then since $\langle a_n \rangle$ has a cluster point, $\langle x_n \rangle$ must have one also. $\qquad \square$

The following elementary proposition was not noticed for either the bornology of UC-subsets or the bornology of cofinally complete subsets. It will be used to characterize those bornologies that are $\mathcal{B}_\lambda$ for some $\lambda \in \mathcal{C}(X, [0, \infty))$.

Proposition 4.9. *Let $\langle X, d \rangle$ be a metric space and let $\lambda : X \to [0, \infty]$ be continuous. Suppose B is a nonempty closed subset of X. Then B is a λ-subset if and only if $B \cap \mathrm{Ker}(\lambda)$ is compact, and whenever A is a nonempty closed subset of B with $A \cap \mathrm{Ker}(\lambda) = \varnothing$, then $\inf \lambda(A) > 0$.*

Proof. Suppose first that B is a λ-set. Then each sequence in $B \cap \mathrm{Ker}(\lambda)$ is a minimizing sequence and since $B \cap \mathrm{Ker}(\lambda)$ is closed, the sequence clusters to a point of $B \cap \mathrm{Ker}(\lambda)$. Suppose next that $A \in \mathcal{C}_0(X) \cap \mathcal{P}_0(B)$ does not intersect $\mathrm{Ker}(\lambda)$, yet $\inf \lambda(A) = 0$. Then λ has a minimizing sequence in A that clusters to a point of A which by continuity also must be in $\mathrm{Ker}(\lambda)$, contradicting $A \cap \mathrm{Ker}(\lambda) = \varnothing$.

Conversely, suppose B satisfies the two conditions, and $\langle b_n \rangle$ is a sequence in B with $\lambda(b_n) \to 0$ but that does not cluster. By the assumed compactness of $B \cap \mathrm{Ker}(\lambda)$, and by passing to a subsequence, we may assume that $\forall n$, $b_n \notin \mathrm{Ker}(\lambda)$. But then with $A = \{b_n : n \in \mathbb{N}\}$, the second condition is violated. $\square$

Proposition 4.10. *Let $\lambda : X \to [0, \infty]$ be continuous. Then a λ-set A is compact if and only if $\forall \varepsilon > 0$, $B_\varepsilon := \{a \in A : \lambda(a) \geq \varepsilon\}$ is compact.*

Proof. Let A be compact λ-set. Since λ is a continuous function, $\{x : \lambda(x) \geq \varepsilon\}$ is closed. Since $B_\varepsilon = A \cap \{x : \lambda(x) \geq \varepsilon\}$, B_ε is compact.

Conversely, suppose $\langle a_n \rangle$ is an arbitrary sequence in A. If $\lambda(a_n) \to 0$, then the sequence clusters because A is a λ-set. Otherwise, $\exists \varepsilon > 0$ and an infinite subset $\mathbb{N}_1$ of $\mathbb{N}$ such that $\forall n \in \mathbb{N}_1$, $\lambda(a_n) \geq \varepsilon$. Hence $\langle a_n \rangle_{n \in \mathbb{N}_1}$ is a sequence in the compact set B_ε. Thus, the sequence $\langle a_n \rangle$ clusters, and A is compact. $\square$

Our next proposition involves λ-subsets and strong uniform continuity.

Proposition 4.11. *Let $\lambda : X \to [0, \infty)$ be continuous.*

(1) *If A is a λ-subset, λ is strongly uniformly continuous on A, and $\langle x_n \rangle$ is a sequence in X with $\lim d(x_n, A) = 0$ and $\lim \lambda(x_n) = 0$, then $\langle x_n \rangle$ clusters.*

(2) *Strong uniform continuity of λ on each member of $\mathcal{B}_\lambda$ coincides with global uniform continuity.*

Proof. We prove statement (2), leaving (1) to the reader. Suppose λ fails to be globally uniformly continuous. Then for some $\varepsilon > 0$, there exist sequences $\langle x_n \rangle$ and $\langle w_n \rangle$ in X such that for each n, $d(x_n, w_n) < \frac{1}{n}$ yet $f(x_n) + \varepsilon < f(w_n)$. While $B := \{w_n : n \in \mathbb{N}\}$ is in $\mathcal{B}_\lambda$, λ is not strongly uniformly continuous on B. $\square$

Example 4.12. For a counterexample to Proposition 4.11(1), let us revisit the metric space X and the functional λ of Example 3.9. Then $A := \{(x, y) : xy = 1\} \cup \{(x, y) : x = y \text{ and } x \leq 1\}$, as shown in Figure 1, is a λ-set. If

$x_n = (0, n)$, then $\lim d(x_n, A) = 0$ and $\lim \lambda(x_n) = 0$, but the sequence $\langle x_n \rangle$ does not cluster.
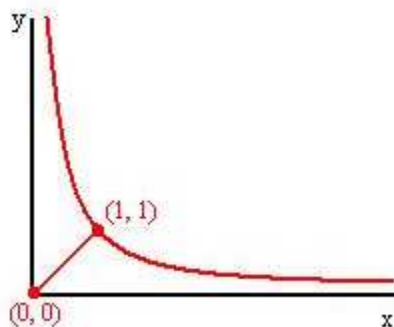

FIGURE 1

The next result is anticipated by a decomposition theorem for spaces on which a continuous function that is Tychonoff well-posed in the generalized sense is defined [31, Prop 10.1.7]. It is also anticipated by particular decomposition theorems in the special cases of the bornology of UC-subsets and the bornology of cofinally complete subsets [13, 15] (see previously for UC spaces and cofinally complete spaces [8, 10, 23]).

Theorem 4.13. *Let $\langle X, d \rangle$ be a metric space and suppose $\lambda \in \mathcal{C}(X, [0, \infty])$. Then $A \in \mathcal{P}_0(X)$ is a λ-subset if and only if $cl(A) \cap Ker(\lambda)$ is compact and $\forall \delta > 0$, $\exists \varepsilon > 0$ such that $a \in A \setminus S_\delta(cl(A) \cap Ker(\lambda)) \Rightarrow \lambda(a) > \varepsilon$.*

Proof. First, suppose $\mathrm{cl}(A) \cap \mathrm{Ker}(\lambda)$ is not compact, and therefore nonempty. Choose a sequence $\langle a_n \rangle$ in $\mathrm{cl}(A) \cap \mathrm{Ker}(\lambda)$ with no cluster point. Then $\lambda(a_n) \to 0$, but $\langle a_n \rangle$ has no cluster point $\Rightarrow \mathrm{cl}(A)$ is not a λ-set $\Rightarrow A$ is not a λ-set. Suppose now that for some $\delta > 0$ that $\inf\{\lambda(a) : a \in A \setminus S_\delta(\mathrm{cl}(A) \cap \mathrm{Ker}(\lambda))\} = 0$. Select $a_n \in A \setminus S_\delta(\mathrm{cl}(A) \cap \mathrm{Ker}(\lambda))$ with $\lambda(a_n) < \frac{1}{n}$. There can be no possible cluster point p for $\langle a_n \rangle$ as by continuity $\lambda(p) = 0$ must hold, while $d(p, \mathrm{cl}(A) \cap \mathrm{Ker}(\lambda)) \geq \delta$. Again, A is not a λ-set.

Conversely, suppose $\mathrm{cl}(A) \cap \mathrm{Ker}(\lambda)$ is compact, and $\forall \delta > 0$, $\exists \varepsilon_\delta > 0$ such that $a \in A \setminus S_\delta(\mathrm{cl}(A) \cap \mathrm{Ker}(\lambda)) \Rightarrow \lambda(a) > \varepsilon_\delta$. Let $\langle a_n \rangle$ be a sequence in A where $\lambda(a_n) \to 0$. If $\mathrm{cl}(A) \cap \mathrm{Ker}(\lambda) = \varnothing$, then $A = A \setminus S_\delta(\mathrm{cl}(A) \cap \mathrm{Ker}(\lambda))$ for each δ. So then given $\delta > 0$, $\forall n \ \lambda(a_n) > \varepsilon_\delta$, which is a contradiction. We conclude that $\mathrm{cl}(A) \cap \mathrm{Ker}(\lambda) \neq \varnothing$. Then given $\delta > 0$, $\lambda(a_n) \leq \varepsilon_\delta$ eventually $\Rightarrow a_n \in S_\delta(\mathrm{cl}(A) \cap \mathrm{Ker}(\lambda))$ eventually $\Rightarrow \langle a_n \rangle$ has a cluster point by the compactness of $\mathrm{cl}(A) \cap \mathrm{Ker}(\lambda)$. $\qquad \square$

We now address a basic question: what are necessary and sufficient conditions on a bornology $\mathcal{B}$ in a metric space $\langle X, d \rangle$ such that $\mathcal{B} = \mathcal{B}_\lambda$ for some

$\lambda \in \mathcal{C}(X, [0, \infty))$? The key tools in answering this question are Proposition 4.9 and an important lemma of S.-T. Hu ([25, Thm 13.2] or [26, p. 189]), proved using the Urysohn Lemma.

Lemma 4.14 (Hu's Lemma). *Let* $\mathcal{B} \neq \mathcal{P}_0(X)$ *be a bornology on a normal topological space* X *having a countable base* $\{B_n : n \in \mathbb{N}\}$ *such that* $\forall n \in \mathbb{N},\ cl(B_n) \subseteq int(B_{n+1})$. *Then there exists an unbounded* $f \in \mathcal{C}(X, [0, \infty))$ *such that*

$$\mathcal{B} = \{A : f(A) \text{ is a bounded set of reals}\}.$$

It is easy to see that the conditions of the lemma are satisfied if and only if (1) $\forall B \in \mathcal{B},\ B \neq X$; (2) $\mathcal{B}$ has a countable base; (3) $\mathcal{B}$ has an open base; and (4) $\mathcal{B}$ has a closed base.

To obtain our characterization, we break our λ-functionals into two classes: those for which $\mathrm{Ker}(\lambda) = \varnothing$, and those for which $\mathrm{Ker}(\lambda) \neq \varnothing$. We need an immediate consequence of Theorem 4.13 to deal with the first situation that we record as a lemma.

Lemma 4.15. *Let* $\lambda \in \mathcal{C}(X, [0, \infty))$ *have no minimum value, yet* $\inf \lambda(X) = 0$. *Then* $\mathcal{B}_\lambda = \{A \in \mathcal{P}_0(X) : \inf \lambda(A) > 0\}$.

Theorem 4.16. *Let* $\mathcal{B}$ *be a bornology on* $\langle X, d \rangle$. *Then* $\mathcal{B} = \mathcal{B}_\lambda$ *for some* $\lambda \in \mathcal{C}(X, [0, \infty))$ *with* $\mathrm{Ker}(\lambda) = \varnothing$ *if and only if* $\mathcal{B}$ *has a countable base* $\{B_n : n \in \mathbb{N}\}$ *such that* $\forall n \in \mathbb{N},\ cl(B_n) \subseteq int(B_{n+1})$.

Proof. For sufficiency, if $X \in \mathcal{B}$, we can put $\lambda(x) \equiv 1$. Otherwise, applying Hu's Lemma to generate an unbounded $f \in \mathcal{C}(X, [0, \infty))$, put $\lambda(x) := (1 + f(x))^{-1}$. Noting that λ is bounded away from zero on a subset of X if and only if f is bounded above on the subset, we see by Lemma 4.15 that λ does the job.

For necessity, if $\mathcal{B} = \mathcal{B}_\lambda$ where $\mathrm{Ker}(\lambda) = \varnothing$, then by Lemma 4.15, $B \in \mathcal{B} \Leftrightarrow \inf \lambda(B) > 0$. By the continuity of λ, $\{\lambda^{-1}([\frac{1}{n}, \infty)) : n \in \mathbb{N}\}$ is the desired countable base. $\square$

Theorem 4.17. *Let* $\mathcal{B}$ *be a bornology on* $\langle X, d \rangle$. *The following conditions are equivalent:*

(1) $\mathcal{B} = \mathcal{B}_\lambda$ *for some* $\lambda \in \mathcal{C}(X, [0, \infty))$ *with* $\mathrm{Ker}(\lambda) = \varnothing$;
(2) $\mathcal{B} = \mathcal{B}_\rho(X)$ *for some metric* ρ *equivalent to* d.

Proof. $(2) \Rightarrow (1)$. If ρ is a bounded metric, take $\lambda(x) \equiv 1$. Otherwise, we invoke Theorem 4.16 for $\mathcal{B}_\rho(X)$, putting $B_n := \{x : \rho(x, x_0) \leq n\}$ where $x_0 \in X$ is fixed.

$(1) \Rightarrow (2)$. If $\mathcal{B}_\lambda = \mathcal{P}_0(X)$, take $\rho = \min\{1, d\}$. Otherwise, with $B_n = \lambda^{-1}([\frac{1}{n}, \infty)) \neq X$, apply Hu's Lemma to once again generate an unbounded f. The metric

$$\rho(x, w) := \min\{1, d(x, w)\} + |f(x) - f(w)|$$

satisfies $\mathcal{B}_\lambda = \mathcal{B}_\rho(X)$ and is equivalent to d. $\square$

We now come to the harder part.

Theorem 4.18. *Let $\mathcal{B}$ be a bornology on $\langle X, d \rangle$. The following conditions are equivalent:*

    (1) $\mathcal{B} = \mathcal{B}_\lambda$ *for some* $\lambda \in \mathcal{C}(X, [0, \infty))$ *with* $Ker(\lambda) \neq \varnothing$;

    (2) $\mathcal{B}$ *has a closed base, and* $\exists C \in \mathcal{C}_0(X)$ *with open neighborhoods* $\{V_n : n \in \mathbb{N}\}$ *satisfying* $\cap_{n=1}^\infty V_n = C$ *and* $\forall n \in \mathbb{N}$, $cl(V_{n+1}) \subseteq V_n$ *such that* $\forall B \in \mathcal{C}_0(X)$, $B \in \mathcal{B} \Leftrightarrow B \cap C$ *is compact, and whenever A is a nonempty closed subset of B disjoint from C, then for some n, $A \cap V_n = \varnothing$.*

Proof. $(1) \Rightarrow (2)$. By Proposition 4.8, $\mathcal{B}$ has a closed base, and by Proposition 4.9, we can take $C = \mathrm{Ker}(\lambda)$ and $V_n = \lambda^{-1}([0, \frac{1}{n}))$.

$(2) \Rightarrow (1)$. We consider several cases for the set C. First if $C = X$, then a nonempty closed set B is in $\mathcal{B}$ if and only if B is compact, and since the bornology has a closed base, it is the bornology $\mathcal{K}_0(X)$ of nonempty subsets with compact closure and with $\lambda(x) \equiv 0$, we get $\mathcal{B} = \mathcal{B}_\lambda$. A second possibility is that $C = V_n \subset X$ for some n. Since $\{C, X \backslash C\}$ forms a nontrivial separation of X, the function λ assigning 0 to each point of C and 1 to each point of $X \backslash C$ is continuous. We intend to show that $\mathcal{B} = \mathcal{B}_\lambda$.

Since both bornologies have closed bases, it suffices to show closed members of one belong to the other. If $B \in \mathcal{B} \cap \mathcal{C}_0(X)$, then any minimizing sequence in B lies eventually in C, and since $B \cap C$ is compact, it clusters. This shows $B \in \mathcal{B}_\lambda$. For the reverse inclusion, if $B \in \mathcal{B}_\lambda$ is closed, then $B \cap \mathrm{Ker}(\lambda)$ is compact, that is, $B \cap C$ is compact. Also if A is a closed subset of B disjoint from C, then $A \cap V_n = \varnothing$ without any consideration of λ.

In the remaining case we may assume without loss of generality that $\forall n \in \mathbb{N}, C \subset V_n \subset X$. We now apply Hu's Lemma to the metric subspace $X \backslash C$ with respect to the bornology having the closed base $\{X \backslash V_n : n \in \mathbb{N}\}$. We produce an unbounded continuous $f : X \backslash C \to [0, \infty)$ such that $\forall A \in \mathcal{P}_0(X \backslash C)$, $f(A)$ is bounded if and only if for some n, $A \subseteq X \backslash V_n$. We next define our function λ by

$$\lambda(x) = \begin{cases} 0 & \text{if } x \in C \\ \frac{1}{1+f(x)} & \text{otherwise} \end{cases}.$$

Evidently λ is continuous restricted to the open set $X \backslash C$. Given $\varepsilon \in (0, 1)$, choose $n \in \mathbb{N}$ with $\{x \in X \backslash C : f(x) \leq \frac{1-\varepsilon}{\varepsilon}\} \subseteq X \backslash V_n$. It follows that $\forall x \in V_n$, we have $\lambda(x) < \varepsilon$, establishing global continuity of λ.

Again we must show that $\mathcal{B} \cap \mathcal{C}_0(X) = \mathcal{B}_\lambda \cap \mathcal{C}_0(X)$. For a closed set B, $B \cap \mathrm{Ker}(\lambda)$ is compact if and only if $B \cap C$ is compact because by construction $\mathrm{Ker}(\lambda) = C$. If $B \in \mathcal{C}_0(X)$ and A is a nonempty closed subset with $A \cap C = A \cap \mathrm{Ker}(\lambda) = \varnothing$ then $\exists n$ with $A \cap V_n = \varnothing \Leftrightarrow \exists n$ with $A \subseteq X \backslash V_n \Leftrightarrow f$ is bounded above on $A \Leftrightarrow \inf \lambda(A) > 0$. The result now follows from Proposition 4.9. $\qquad\square$

We next show that that there are bornologies with closed base that fail to be a bornology of λ-subsets.

Example 4.19. Consider $\mathbb{R}$ with the zero-one metric and and let $\mathcal{B}$ be the bornology of countable nonempty subsets. Since $\mathbb{R}$ is uncountable, $\mathcal{B}$ fails to have a countable base. By Theorem 4.16, it remains to show that $\mathcal{B}$ cannot be $\mathcal{B}_\lambda$ for any λ with nonempty kernel. We show that condition (2) of Theorem 4.18 cannot hold. Suppose to the contrary that such a C with neighborhoods $\{V_n : n \in \mathbb{N}\}$ existed. Since the intersection of C with each countable set must be compact, we conclude C is finite. For each n, put $B_n := X \backslash V_n$. Clearly, $B_n \cap C$ is compact as it is empty. Also each (closed) subset of B_n is trivially disjoint from V_n. By condition (2) of Theorem 4.18, B_n must be countable, and since $X \backslash C = \cup_{n=1}^{\infty} B_n$, it too must be countable. This is a contradiction, and so the bornology of countable subsets cannot be a bornology of λ-subsets.

Here is a natural follow-up question: when is a bornology $\mathcal{B}$ a bornology of λ-subsets for some uniformly continuous $\lambda : X \to [0, \infty)$? In our analysis, strong uniform continuity of a function on members of a bornology plays a key role. We first obtain an analog of Hu's Lemma, which is implicit in the proof of [14, Thm. 3.18].

Lemma 4.20. *Suppose $\mathcal{B}$ is a bornology on a metric space $\langle X, d \rangle$ that does not contain X. Suppose $\mathcal{B}$ has a countable base $\{B_n : n \in \mathbb{N}\}$ such that $\forall n \in \mathbb{N}, \exists \delta_n > 0$ with $S_{\delta_n}(B_n) \subseteq B_{n+1}$. Then there exists an unbounded $f \in \mathcal{C}(X, [0, \infty))$ such that f is strongly uniformly continuous on each B_n and such that*

$$\mathcal{B} = \{A : f(A) \text{ is a bounded set of reals}\}.$$

Proof. For each $n \in \mathbb{N}$ let $f_n : X \to [0, 1]$ be the uniformly continuous function defined by $f_n(x) = \min\{1, \frac{1}{\delta_n} d(x, B_n)\}$. The values of f_n all lie in $[0, 1]$, and $f_n(B_n) = \{0\}$ and $f_n(X \backslash B_{n+1}) = \{1\}$. Put $f = f_1 + f_2 + f_3 + \cdots$. First note that the restriction of f to each B_n agrees with $f_1 + f_2 + f_3 + \cdots + f_{n-1}$ so that

(1) $\forall n$, f restricted to B_n is uniformly continuous;

(2) $\forall n$, $f(B_n) \subseteq [0, n-1]$.

By (1) f is strongly uniformly continuous on each B_n because f is uniformly continuous restricted to B_{n+1} and this larger set contains an enlargement of B_n. By (2) $\forall n$, $f(B_n)$ is bounded, so f restricted to each member of $\mathcal{B}$ is bounded because $\{B_n : n \in \mathbb{N}\}$ is a base. Finally, if $f(A)$ is bounded, then for some n, $A \subseteq B_n$ because $x \notin B_{n+1} \Rightarrow f(x) \geq n$. $\qquad \square$

We note that the function f in the Lemma 4.20 is strongly uniformly continuous on each member of $\mathcal{B}$. More generally, the sets on which a continuous real function g is strongly uniformly continuous always form a bornology containing the UC-subsets; in fact, the UC-subsets form the largest common bornology as g runs over $\mathcal{C}(X, \mathbb{R})$ [15]. We also note that if δ_n can be chosen independent of

n in the statement of Lemma 4.20, one can construct a uniformly continuous function f, but the proof is a little more delicate [9, Thm. 4.2].

We will need the following fact about strong uniform continuity.

Proposition 4.21. *Suppose $g : \langle X, d \rangle \to (0, \infty)$ is strongly uniformly continuous on a nonempty subset A of X, and g is bounded away from zero in some enlargement of A. Then $\lambda(x) := \frac{1}{g(x)}$ is strongly uniformly continuous on A.*

Proof. Suppose $\forall x \in S_\delta(A)$, we have $g(x) \geq \alpha > 0$. Given $\varepsilon > 0$, $\exists \delta_\varepsilon \in (0, \delta)$ such that if $a \in A$ and $x \in X$ and $d(a, x) < \delta_\varepsilon$, then $|g(x) - g(a)| < \varepsilon \alpha^2$. We compute

$$|\lambda(x) - \lambda(a)| = \left| \frac{1}{g(x)} - \frac{1}{g(a)} \right| = \frac{|g(a) - g(x)|}{|g(x)g(a)|},$$

and since $\{a, x\} \subseteq S_{\delta_\varepsilon}(A) \subseteq S_\delta(A)$, we further have

$$\frac{|g(a) - g(x)|}{|g(x)g(a)|} < \frac{\varepsilon \alpha^2}{|g(x)g(a)|} \leq \frac{\varepsilon \alpha^2}{\alpha^2} = \varepsilon,$$

and this yields $|\lambda(x) - \lambda(a)| < \varepsilon$. $\qquad\square$

Theorem 4.22. *Let $\mathcal{B}$ be a bornology on $\langle X, d \rangle$. The following conditions are equivalent:*

   (1) *$\mathcal{B} = \mathcal{B}_\lambda$ for some uniformly continuous $\lambda : X \to [0, \infty)$ with $Ker(\lambda) = \varnothing$;*
   (2) *$\mathcal{B}$ has a countable base $\{B_n : n \in \mathbb{N}\}$ such that $\forall n \in \mathbb{N}$, $\exists \delta_n > 0$ with $S_{\delta_n}(B_n) \subseteq B_{n+1}$.*

Proof. (1) $\Rightarrow$ (2). If $\inf \lambda(X) > 0$, then $X \in \mathcal{B}$ and we can put $B_n := X$ for each $n \in \mathbb{N}$. Otherwise, put $B_n = \lambda^{-1}([\frac{1}{n}, \infty)) \in \mathcal{B}$; choose by uniform continuity of λ a positive δ_n such that

$$d(x, w) < \delta_n \Rightarrow |f(x) - f(w)| < \frac{1}{n} - \frac{1}{n+1}.$$

Then we have $\forall n \in \mathbb{N}$, $S_{\delta_n}(B_n) \subseteq B_{n+1}$.

(2) $\Rightarrow$ (1) The case $X \in \mathcal{B}$, that is $\mathcal{B} = \mathcal{P}_0(X)$, is of course trivial. Otherwise, we take f as guaranteed by Lemma 4.20 and as expected put $\lambda(x) = (1 + f(x))^{-1}$. We use Proposition 4.11(2) to establish uniform continuity. Fix $n \in \mathbb{N}$. We know $g(x) := 1 + f(x)$ is strongly uniformly continuous on B_n and that g is bounded below by 1 on all of X. Taking the reciprocal, by Proposition 4.21, we see that λ is strongly uniformly continuous on each B_n and thus on each $B \in \mathcal{B}$, as required. $\qquad\square$

As expected, the bornologies that fulfill the conditions of Theorem 4.22 are *metric boundedness structures* [9], that is, they are of the form $\mathcal{B}_\rho$ for certain ρ equivalent to d. In turns out that the metrics ρ are those for which the identity

id $: \langle X, d \rangle \to \langle X, \rho \rangle$ is strongly uniformly continuous on each ρ-bounded subset. We leave this as an exercise to the interested reader, following the proof of Theorem 4.17 (see also [14]).

Theorem 4.23. *Let $\mathcal{B}$ be a bornology on $\langle X, d \rangle$. The following conditions are equivalent:*

(1) $\mathcal{B} = \mathcal{B}_\lambda$ *for some uniformly continuous* $\lambda : X \to [0, \infty)$ *with* $Ker(\lambda) \neq \varnothing$;

(2) $\mathcal{B}$ *has a closed base, and* $\exists C \in \mathcal{C}_0(X)$ *with open neighborhoods* $\{V_n : n \in \mathbb{N}\}$ *satisfying* $\cap_{n=1}^\infty V_n = C$ *and* $\forall n \in \mathbb{N}$, $\exists \delta_n > 0$ *with* $S_{\delta_n}(V_{n+1}) \subseteq V_n$ *such that* $\forall B \in \mathcal{C}_0(X)$, $B \in \mathcal{B} \Leftrightarrow B \cap C$ *is compact, and whenever* A *is a nonempty closed subset of* B *disjoint from* C, *then for some* n, $A \cap V_n = \varnothing$.

Proof. $(1) \Rightarrow (2)$. Let λ satisfy condition (1), and put $C = \text{Ker}(\lambda)$. If $C = X, \forall n \in \mathbb{N}$, put $V_n = X$. Otherwise, $\exists k \in \mathbb{N}$ and $x \in X$ with $\lambda(x) > \frac{1}{k}$. In this case $\forall n \in \mathbb{N}$, put $V_n := \{x \in X : \lambda(x) < \frac{1}{n+k}\}$. By uniform continuity of λ, $\exists \delta_n > 0$ with

$$d(x, w) < \delta_n \Rightarrow |\lambda(x) - \lambda(w)| < \frac{1}{n+k} - \frac{1}{n+k+1}$$

which means that $S_{\delta_n}(V_{n+1}) \subseteq V_n$. By Proposition 4.8 and Proposition 4.9, $\mathcal{B}_\lambda$ satisfies the conditions on a bornology $\mathcal{B}$ listed in (2).

$(2) \Rightarrow (1)$. We handle this implication by modifying the proof of $(2) \Rightarrow (1)$ in Theorem 4.18. The case $C = X$ is handled in exactly the same manner. In the case that $C = V_n \subset X$ for some n, we define a uniformly continuous function λ on X by $\lambda(x) = \min\{\frac{1}{\delta_n} d(x, C), 1\}$. Since $S_{\delta_n}(C) \subseteq V_n$, we see that λ maps each point of $X \backslash C$ to 1 and each point of C to 0. Verification that $\mathcal{B} = \mathcal{B}_\lambda$ proceeds exactly as in the proof of Theorem 4.18.

In the remaining case we can assume for each $n \in \mathbb{N}$ that $C \subset V_n \subset X$. By condition (2), $\forall n \in \mathbb{N}$, we have

$$S_{\delta_n}(X \backslash V_n) \subseteq X \backslash V_{n+1}.$$

We now apply Lemma 4.20 to the space $X \backslash C$ equipped with the bornology with base $\{X \backslash V_n : n \in \mathbb{N}\}$ to produce an unbounded $f : X \backslash C \to [0, \infty)$ that is strongly uniformly continuous on each set $X \backslash V_n$ and such that $f(A)$ is bounded if and only if A is a subset of some $X \backslash V_n$. We now define $\lambda : X \to [0, \infty)$ by

$$\lambda(x) = \begin{cases} 0 & \text{if } x \in C \\ \frac{1}{1+f(x)} & \text{otherwise} \end{cases}.$$

The proof of Theorem 4.22 shows that the restriction of λ to $X \backslash C$ is uniformly continuous, so if λ fails to be globally uniformly continuous, $\exists \varepsilon > 0$ such that

$\forall k \in \mathbb{N}$, $\exists c_k \in C$ and $x_k \in X \backslash C$ such that $d(c_k, x_k) < \frac{1}{k}$ while $\lambda(x_k) > \varepsilon$. Now as λ is bounded below by ε on $\{x_k : k \in \mathbb{N}\}$, f is bounded above so restricted. It follows that for some $n_0 \in \mathbb{N}$, we have $\{x_k : k \in \mathbb{N}\} \cap V_{n_0} = \varnothing$. But choosing $\frac{1}{k} < \delta_{n_0}$, by condition (2)

$$d(c_k, x_k) < \frac{1}{k} \Rightarrow c_k \in X \backslash V_{n_0+1}.$$

This is a contradiction because $X \backslash V_{n_0+1} \cap C = \varnothing$. This contradiction establishes global uniform continuity, and agreement of the bornologies is argued as before. □

To end this section, we note that convergence in Hausdorff distance need not preserve λ-sets, even when the λ-functional is uniformly continuous.

Example 4.24. Let $\lambda : \mathbb{R}^2 \to [0, \infty)$, where $\lambda(x, y) = y$, and for each positive integer n put $A_n := \{(x, 0) : x \in [0, n]\} \cup \{(x, y) : y = \frac{1}{n}(x - n), \ x \in [n, n + 1]\} \cup \{(x, y) : y = \frac{1}{n}, \ x \geq n + 1\}$, as shown in Figure 2. Then λ is uniformly continuous and $\langle A_n \rangle$ is a sequence of closed λ-sets converging in Hausdorff distance to A, where $A := \{(x, y) : y = 0, x \geq 0\}$. But A is not a λ-set.

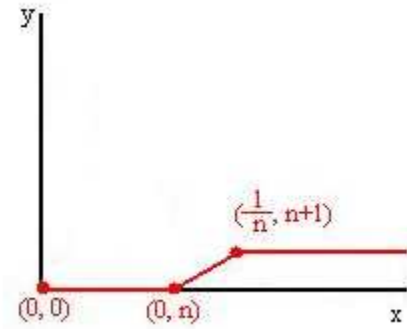FIGURE 2

5. λ-SPACES

Given a continuous nonnegative function λ on a metric space $\langle X, d \rangle$, recall that X is called a λ-space provided each sequence $\langle x_n \rangle$ in X with $\lim \lambda(x_n) = 0$ has a cluster point. As noted in the introduction, if λ is defined appropriately, the λ-spaces include the compact metric spaces, the UC-spaces and the cofinally complete metric spaces. We now show that they include the complete metric spaces.

Proposition 5.1. *Let $\langle X, d \rangle$ be a metric space, and let $P(V)$ mean $cl(V)$ is a complete subspace equipped with the metric d. Put $\beta := \lambda_P$, so that*

$$\beta(x) = \begin{cases} \sup\{\alpha > 0 : cl(S_\alpha(x)) \text{ is complete}\} & \text{if } \exists \alpha > 0 \text{ with } cl(S_\alpha(x)) \text{ complete}; \\ 0 & \text{otherwise.} \end{cases}$$

Then $\langle X, d \rangle$ is a complete metric space if and only if $\langle X, d \rangle$ is a β-space.

Proof. Proving this is straightforward. First suppose $\langle X, d \rangle$ is complete, so $\forall x \in X$, $\beta(x) = \infty$. Each sequence $\langle x_n \rangle$ with $\lim \beta(x_n) = 0$ has a cluster point as this is true vacuously. Hence $\langle X, d \rangle$ is a β-space.

To see the converse, suppose $\langle X, d \rangle$ is a β-space and $\langle x_n \rangle$ is a Cauchy sequence. There are two possibilities: (1) $\lim \beta(x_n) = 0$, and (2) $\limsup \beta(x_n) > 0$. If $\lim \beta(x_n) = 0$, then there exists a cluster point by the definition of a β-space. Otherwise $\exists \varepsilon > 0$ and and infinite subset $\mathbb{N}_1$ of $\mathbb{N}$ such that $\forall n \in \mathbb{N}_1$, $\beta(x_n) > \varepsilon$. Choose $k \in \mathbb{N}$ such that if $n > m > k$, then $d(x_n, x_m) < \varepsilon$. If $n_1 \in \mathbb{N}_1$ and $n_1 > k$, then $\{x : d(x, x_{n_1}) \leq \varepsilon\}$ contains a tail of $\langle x_n \rangle$ that is also Cauchy. Since $\beta(x_{n_1}) > \varepsilon$, $\{x : d(x, x_{n_1}) \leq \varepsilon\}$ is complete, which implies the tail has a cluster point, so $\langle x_n \rangle$ has a cluster point also. Hence $\langle X, d \rangle$ is complete. $\square$

Proposition 4.9 and Theorem 4.13 provide characterizations of λ-spaces, which we now list.

Theorem 5.2. *Let $\langle X, d \rangle$ be a metric space, and let $\lambda : X \to [0, \infty]$ be continuous. The following are equivalent:*

(1) *$\langle X, d \rangle$ is a λ-space;*

(2) *$Ker(\lambda)$ is compact, and if $A \in \mathcal{C}_0(X)$ with $A \cap Ker(\lambda) = \varnothing$, then $\inf \lambda(A) > 0$;*

(3) *$Ker(\lambda)$ is compact, and $\forall \delta > 0, \exists \varepsilon > 0$ such that $d(x, Ker(\lambda)) > \delta \Rightarrow \lambda(x) > \varepsilon$.*

Although all λ-spaces must have a compact kernel, it is easy to produce examples showing that this alone is not sufficient (see, e.g., [31, Ex. 10.1.3]). The following proposition shows how normal pathology is in this regard.

Proposition 5.3. *Let $\langle X, d \rangle$ be a noncompact metric space and let C be an arbitrary compact subset. Then there exists $\lambda \in \mathcal{C}(X, [0, \infty))$ with $Ker(\lambda) = C$ for which X is not a λ-space.*

Proof. Pick distinct points $x_1, x_2, x_3, \ldots$ in $X \backslash C$ such that $\langle x_n \rangle$ has no cluster point. Note that $A := \{x_n : n \in \mathbb{N}\}$ is a closed discrete set. If $C = \varnothing$, choose by the Tietze Extension Theorem [21, p. 149] $f \in \mathcal{C}(X, [0, \infty))$ satisfying $f(x_n) = n$, and clearly $\lambda(x) = (1 + f(x))^{-1}$ does the job. When C is nonempty, by the Tietze Extension Theorem, there is a nonnegative continuous function λ_1 on X mapping C to 0 such that $\forall n$, $\lambda_1(x_n) = \frac{1}{n}$. The desired λ is defined by $\lambda(x) = \lambda_1(x) + d(x, A \cup C)$. $\square$

The last result of course shows that whenever C is a nonempty compact subset of a metric space $\langle X, d \rangle$, then there is a function having C as its set of minimizers that fails to be Tychonoff well-posed in the generalized sense.

The next result characterizes λ-spaces in terms of a general Cantor-type theorem. As its validity is known in the most important special cases (see [6, 10]), it comes as no surprise.

Theorem 5.4. *Let $\lambda : \langle X, d \rangle \to [0, \infty]$ be a continuous function. Then $\langle X, d \rangle$ is a λ-space if and only if whenever $\langle A_n \rangle$ is a decreasing sequence in $\mathcal{C}_0(X)$ with $\overline{\lambda}(A_n) \to 0$ then $\bigcap_{n \in \mathbb{N}} A_n$ is nonempty.*

Proof. Suppose $\langle X, d \rangle$ is a λ-space and $\langle A_n \rangle$ is decreasing in $\mathcal{C}_0(X)$ with $\overline{\lambda}(A_n) \to 0$. For each $n \in \mathbb{N}$, pick $x_n \in A_n$ arbitrarily. We have

$$0 \le \lambda(x_n) \le \sup\{\lambda(a) : a \in A_n\}.$$

As $\overline{\lambda}(A_n) \to 0$, we have $\lambda(x_n) \to 0$, so $\langle x_n \rangle$ must have a cluster point, say p. Then given $\varepsilon > 0$ and $n_0 \in \mathbb{N}$, $\exists k \ge n_0$ such that

$$d(x_k, p) < \varepsilon \Rightarrow x_k \in S_\varepsilon(p) \Rightarrow p \in \mathrm{cl}(\{x_j : j \ge n_0\}) \subseteq \mathrm{cl}\left(\bigcup_{j=n_0}^{\infty} A_j \right) \subseteq A_{n_0},$$

because $\langle A_n \rangle$ is a decreasing sequence and A_{n_0} is closed. Hence, $p \in \cap_{n \in \mathbb{N}} A_n$.

Conversely, let $\langle y_n \rangle$ be a sequence in $\langle X, d \rangle$ where $\lim \lambda(y_n) = 0$. For each $n \in \mathbb{N}$, put $A_n := \mathrm{cl}(\{y_k : k \ge n\})$. Fix $\varepsilon > 0$; $\exists n_0 \in \mathbb{N}$ such that $n \ge n_0 \Rightarrow \lambda(y_n) < \varepsilon$. As a result, $\forall n \ge n_0$, $\sup\{\lambda(a) : a \in A_n\} \le \varepsilon \Rightarrow \lim \overline{\lambda}(A_n) = 0$. Hence $\bigcap_{n=1}^{\infty} \mathrm{cl}(\{y_k : k \ge n\}) \ne \varnothing$, and $\langle y_n \rangle$ has a cluster point. $\square$

Lemma 5.5. *Let $\langle X, d \rangle$ be a λ-space. Suppose $\langle A_n \rangle$ is a decreasing sequence in $\mathcal{C}_0(X)$ with $\lim \overline{\lambda}(A_n) = 0$. Then $A := \bigcap_{n \in \mathbb{N}} A_n$ is nonempty and compact and $\lim H_d(A_n, A) = 0$.*

Proof. The set A is nonempty by Theorem 5.4. Choose an arbitrary sequence $x_1, x_2, x_3, \ldots$ in A. Since $\overline{\lambda}$ is monotone and $\lim \overline{\lambda}(A_n) = 0$, we have $\overline{\lambda}(A) = 0$. Hence $\forall n \in \mathbb{N}$, $\lambda(x_n) = 0 \Rightarrow \langle x_n \rangle$ has a cluster point in A because A is closed. Thus, A is compact.

Now we show $\lim H_d(A_n, A) = 0$. Suppose this does not hold; then $\exists \varepsilon > 0$ such that $\forall n_0 \in \mathbb{N}$, $\exists k \ge n_0$ with $H_d(A_k, A) > \varepsilon$. Since $A_{n_0} \supseteq A_k$, clearly $A_{n_0} \nsubseteq S_\varepsilon(A)$. Pick $\forall n \in \mathbb{N}$ $x_n \in A_n \setminus S_\varepsilon(A)$. Since $\lim \lambda(x_n) = 0$, $\langle x_n \rangle$ must have a cluster point, say p. Hence

$$p \in \bigcap_{k \in \mathbb{N}} \mathrm{cl}(\{x_n \; : \; n \ge k\}) \subseteq \bigcap_{k \in \mathbb{N}} A_k = A.$$

But $\forall n \in \mathbb{N}$, $d(x_n, p) \ge d(x_n, A) \ge \varepsilon$, which is a contradiction. Thus, $\langle A_n \rangle$ converges to A in Hausdorff distance. $\square$

Theorem 5.6. *If $\langle X, d \rangle$ is complete, then the following statements are equivalent:*

(1) $\langle X, d \rangle$ *is a λ-space;*

(2) *the measure of noncompactness functional α is continuous with respect to $\overline{\lambda}$ on $\mathcal{C}_0(X) : \forall \varepsilon > 0, \; \exists \delta > 0$ such that $A \in \mathcal{C}_0(X)$ and $\overline{\lambda}(A) < \delta \Rightarrow \alpha(A) < \varepsilon$.*

Proof. $(2) \Rightarrow (1)$. Let $\langle A_n \rangle$ be a decreasing sequence in $\mathcal{C}_0(X)$ with $\lim \overline{\lambda}(A_n) = 0$. Fix $\varepsilon > 0$; $\exists \delta > 0$ such that $\overline{\lambda}(A_n) < \delta \Rightarrow \alpha(A_n) < \varepsilon$. Since $\lim \overline{\lambda}(A_n) = 0$, we have $\lim \alpha(A_n) = 0$. Since X is complete, by Kuratowski's Theorem, $\bigcap_{n \in \mathbb{N}} A_n \neq \varnothing$. Hence, by Theorem 5.4, X is a λ-space.

$(1) \Rightarrow (2)$. Assume (1) holds but (2) fails, i.e., $\exists \varepsilon > 0$ such that given $n \in \mathbb{N}$, $\exists B_n \in \mathcal{C}_0(X)$ with $\overline{\lambda}(B_n) \leq \frac{1}{n}$ but $\alpha(B_n) \geq \varepsilon$. Let $A_n := \{x \; : \; \lambda(x) \leq \frac{1}{n}\}$ and put $A := \bigcap_{n \in \mathbb{N}} A_n$ which by Lemma 5.5 is nonempty and compact and $\lim H_d(A_n, A) = 0$. Since $A_n \supseteq B_n$, by continuity of α with respect to Hausdorff distance, $\forall n \in \mathbb{N}, \; \alpha(A_n) \geq \varepsilon \Rightarrow \alpha(A) \geq \varepsilon$. But $\alpha(A) = 0$ as A is compact; thus we have a contradiction. $\square$

Given an hereditary property P of open subsets of a metrizable space X, the induced functional λ_P depends on the nature of the balls of the particular metric chosen. With one choice, we might obtain a λ_P-space but with another, not so.

Example 5.7. Let $X = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{4 - \frac{1}{n} : n \in \mathbb{N}\}$ as a topological subspace of $\mathbb{R}$, and let $P(V)$ be the property that V contains at most one point. For a particular compatible metric d, the associated functional λ_P^d is of course the measure of isolation functional. When d is the Euclidean metric, the resulting space is not a λ_P-space, as $\lambda_P^d(4 - \frac{1}{n}) = \frac{1}{n^2 + n}$ while $\langle 4 - \frac{1}{n} \rangle$ fails to cluster in X. On the other hand the mapping $g : X \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} 2n & \text{if } x = 4 - \frac{1}{n} \text{ for some } n \\ x & \text{otherwise} \end{cases}$$

is a topological embedding, and this yields a metric ρ on X defined by $\rho(x, w) = |g(x) - g(w)|$ for which $\langle X, \rho \rangle$ is a λ_P-space.

The next result, in the special case of UC-spaces, appears in the first John Rainwater paper [34], a pseudonym used by mathematicians associated with the University of Washington. In the special case of cofinally complete spaces, it is due to S. Romaguera [36].

Theorem 5.8. *Let X be a metrizable topological space, and let P be an hereditary property of open sets. The following conditions are equivalent:*

(1) *X has a compatible metric d such that $\langle X, d \rangle$ is a λ_P-space;*

(2) *$Ker(\lambda_P)$ is compact.*

Proof. If d is a compatible metric, let us write for the purposes of this proof $S_\alpha^d(x)$ for the open d-ball with center x and radius α, and λ_P^d for the induced functional. Note that the set $\{x \in X : \lambda_P(x) = 0\}$ is well-defined, i.e., it does not depend on the particular metric chosen, for if ρ is another compatible metric, then at each x,

$$\forall \alpha > 0, \ \neg P(S_\alpha^d(x)) \text{ if and only if } \forall \alpha > 0, \ \neg P(S_\alpha^\rho(x)).$$

Let us denote this well-defined set by $\mathrm{Ker}(\lambda_P)$. With this in mind, it follows from Theorem 5.2 that (2) is necessary for (1). For the sufficiency of (2) for (1), we use this technical fact about open covers: if X is metrizable and $\{\Omega_k : k \in \mathbb{N}\}$ is a family of open covers of X, then there exists a compatible metric d for X such that $\forall k \in \mathbb{N}, \ \{S_{1/k}^d(x) : x \in X\}$ refines Ω_k [21, p. 196].

It is possible that while compact, $\mathrm{Ker}(\lambda_P)$ is empty. Then each $x \in X$ has an open neighborhood V_x such that $P(V_x)$. By the just-stated refinement result, there exists a compatible metric d such that $\{S_1^d(x) : x \in X\}$ refines $\{V_x : x \in X\}$. Since P is hereditary, $\forall x, \lambda_P^d(x) = \sup\{\alpha > 0 : P(S_\alpha^d(x))\} \geq 1$, and so $\langle X, d \rangle$ is a λ_P-space. Otherwise, $\mathrm{Ker}(\lambda_P)$ is nonempty and compact and so there is a countable family of open neighborhoods $\{W_k : k \in \mathbb{N}\}$ of $\mathrm{Ker}(\lambda_P)$ such that whenever V is open and $\mathrm{Ker}(\lambda_P) \subseteq V$, $\exists k \in \mathbb{N}$ with $W_k \subseteq V$. Again, for each $x \notin \mathrm{Ker}(\lambda_P)$, let V_x be an open neighborhood of x with $P(V_x)$. For each $k \in \mathbb{N}$, define an open cover Ω_k of X as follows:

$$\Omega_k := \{V_x : x \notin W_k\} \cup \{W_k\}.$$

Choose a compatible metric d such that for each k, $\{S_{1/k}^d(x) : x \in X\}$ refines Ω_k. Now let $\langle x_n \rangle$ satisfy $\lim_{n \to \infty} \lambda_P^d(x_n) = 0$. For each k, W_k contains a tail of $\langle x_n \rangle$, specifically $x_n \in W_k$ when $\lambda_P^d(x_n) < \frac{1}{k}$. Since $\{W_k : k \in \mathbb{N}\}$ forms a base for the neighborhoods of $\mathrm{Ker}(\lambda_P)$, $\forall \varepsilon > 0, \ \exists n_\varepsilon \in \mathbb{N} \ \forall n \geq n_\varepsilon, \ x_n \in S_\varepsilon^d(\mathrm{Ker}(\lambda_P))$. Since $\mathrm{Ker}(\lambda_P)$ is compact, $\langle x_n \rangle$ has a cluster point and $\langle X, d \rangle$ is a λ_P-space in this second case, too. $\qquad\square$

With respect to product spaces equipped with the box metric, if we consider again an hereditary property of open sets P, we can write a formula for λ_P if the property P "factors", as it does in the case of the measure of isolation functional and the measure of local compactness functional.

Proposition 5.9. *Let P_1, P_2 be hereditary properties of open sets in X_1, X_2 respectively, and P be a property of open sets in $X_1 \times X_2$ such that $P(U \times V)$ if and only if both $P_1(U)$ and $P_2(V)$. Then $\lambda_P : X_1 \times X_2 \to [0, \infty]$ can be expressed by $\lambda_P(x, y) = \min\{\lambda_{P_1}(x), \ \lambda_{P_2}(y)\}$.*

Proof. Let $x \in X_1$ and $y \in X_2$. Suppose $\alpha < \min\{\lambda_{P_1}(x), \ \lambda_{P_2}(y)\}$. Then $P_1(S_\alpha(x)) \wedge P_2(S_\alpha(y)) \Rightarrow P(S_\alpha(x, y))$, and so $\lambda_P(x, y) \geq \alpha$. As a result, $\lambda_P(x, y) \geq \min\{\lambda_{P_1}(x), \ \lambda_{P_2}(y)\}$. Suppose $\beta < \lambda_P(x, y)$. Then $P(S_\beta(x, y)) \Rightarrow$

$P_1(S_\beta(x)) \wedge P_2(S_\beta(y)) \Rightarrow \lambda_{P_1}(x) \geq \beta \wedge \lambda_{P_2}(y) \geq \beta$, so $\min\{\lambda_{P_1}(x),\ \lambda_{P_2}(y)\} \geq \beta$. Hence $\min\{\lambda_{P_1}(x),\ \lambda_{P_2}(y)\} \geq \lambda_P(x,y)$. $\square$

Proposition 5.10. *Suppose* $\lambda(x_1, x_2) = \min\{\lambda_1(x_1), \lambda_2(x_2)\}$ *where* λ_1 *and* λ_2 *are continuous, nonnegative extended real-valued functions on* X_1 *and* X_2, *respectively. Then* λ *is continuous and nonnegative, and*

$$Ker(\lambda) = [Ker(\lambda_1) \times X_2] \cup [X_1 \times Ker(\lambda_2)].$$

The next result is hinted at by a result of Hohti [23, Thm. 2.2.1] for cofinally complete metric spaces.

Theorem 5.11. *Let* $\langle X_1, d_1 \rangle$ *and* $\langle X_2, d_2 \rangle$ *be metric spaces, where* $\lambda_1 : X_1 \rightarrow [0, \infty)$ *and* $\lambda_2 : X_2 \rightarrow [0, \infty)$ *are continuous. Consider the metric space* $\langle X_1 \times X_2, d \rangle$, *where* d *is the box metric, and*

$$\lambda(x_1, x_2) = \min\{\lambda_1(x_1),\ \lambda_2(x_2)\}.$$

The following are equivalent:

(1) $X_1 \times X_2$ *is a* λ*-space;*
(2) X_1 *is a* λ_1*-space,* X_2 *is a* λ_2*-space, and additionally both (i)* $Ker(\lambda_1) \neq \varnothing \Rightarrow X_2$ *is compact, and (ii)* $Ker(\lambda_2) \neq \varnothing \Rightarrow X_1$ *is compact.*

Proof. (1)$\Rightarrow$(2): To show that X_1 is a λ_1-space, let $\langle a_n \rangle$ be a sequence in X_1 where $\lambda_1(a_n) \rightarrow 0$. Consider $\langle (a_n, c) \rangle$ as a sequence in $X_1 \times X_2$, where $c \in X_2$ is fixed arbitrarily. Then $\lambda(a_n, c) = \min\{\lambda_1(a_n),\ \lambda_2(c)\} \rightarrow 0$ because $\lambda(a_n) \rightarrow 0$. As a result, $\langle (a_n, c) \rangle$ must have a cluster point (p_1, c). Hence, $\langle a_n \rangle$ clusters. In a similar manner, it can be shown that X_2 is a λ_2-space.

Suppose now $Ker(\lambda_1) \neq \varnothing$. Then $\exists x \in X_1$ with $\lambda_1(x) = 0$. Let $\langle b_n \rangle$ be an arbitrary sequence in X_2. We can then let $\langle (x, b_n) \rangle$ be a sequence in $X_1 \times X_2$. Then $\lambda(x, b_n) = \min\{\lambda_1(x),\ \lambda_2(b_n)\} \rightarrow 0$ so $\langle (x, b_n) \rangle$ has a cluster point (x, p_3). Hence $\langle b_n \rangle$ clusters $\Rightarrow X_2$ compact. Similarly, it can be shown that if $Ker(\lambda_2) \neq \varnothing$, then X_1 is compact.

(2)$\Rightarrow$(1): To show $X_1 \times X_2$ is a λ-space, let $\langle (a_n, b_n) \rangle$ be a sequence in $X_1 \times X_2$ with $\lambda(a_n, b_n) \rightarrow 0$. Consider the case where there exists a subsequence of $\langle a_n \rangle$, say $\langle a_{n_1} \rangle_{n_1 \in \mathbb{N}_1}$ with $\mathbb{N}_1 \subseteq \mathbb{N}$, where $\lambda_1(a_{n_1}) \rightarrow 0$. Then $\exists \mathbb{N}_2 \subseteq \mathbb{N}_1$ where $\langle a_{n_2} \rangle_{n_2 \in \mathbb{N}_2}$ converges to a point of $Ker(\lambda_1)$. Since $\langle b_n \rangle$ is in X_2, which must be compact, then $\exists \mathbb{N}_3 \subseteq \mathbb{N}_2$ such that $\langle b_{n_3} \rangle_{n_3 \in \mathbb{N}_3}$ converges and $\langle a_{n_3} \rangle_{n_3 \in \mathbb{N}_3}$ converges. Hence, $\langle (a_{n_3}, b_{n_3}) \rangle_{n_3 \in \mathbb{N}_3}$ converges, which implies $\langle (a_n, b_n) \rangle$ clusters. In the case where there exists a subsequence of $\langle b_n \rangle$, say $\langle b_{n_1} \rangle_{n_1 \in \mathbb{N}_1}$ with $\mathbb{N}_1 \subseteq \mathbb{N}$, where $\lambda_2(b_{n_1}) \rightarrow 0$, it can be similarly shown that $\langle (a_n, b_n) \rangle$ clusters, and this is left to the reader. $\square$

Remark 5.12. Proposition 5.10 gives an alternate justification that conditions (2*i*) and (2*ii*) are necessary in Theorem 5.11.

Example 5.13. In the case that $\lambda_1 = \lambda_2 =$ the measure of local completeness functional, when both X_1 and X_2 are complete, it is clear that $Ker(\lambda_1) =$

$\text{Ker}(\lambda_1) = \varnothing$, so that $X_1 \times X_2$ is complete if and only if X_1 and X_2 are complete, as we all know.

Example 5.14. In the case that $\lambda_1 = \lambda_2 =$ the measure of isolation functional, condition (2) becomes X_1 and X_2 are both UC-spaces, and if either space has limit points, the other must be compact.

What is most interesting about this result emerges after we take a closer look at statement (2) of Theorem 5.11 from the perspective of mathematical logic. Formally, statement (2) is of the form

$$P \wedge (Q \Rightarrow S) \wedge (R \Rightarrow T),$$

which is logically equivalent to

$$[(P \wedge \neg Q) \vee (P \wedge S)] \wedge [(P \wedge \neg R) \vee (P \wedge T)].$$

Since conjunction is distributive over disjunction, the following four-part disjunction is equivalent to (2):

$$\inf\{\lambda_1(x) : x \in X_1\} > 0 \ \text{ and } \ \inf\{\lambda_2(x) : x \in X_2\} > 0,$$

$$\text{or}$$

$$X_2 \text{ is an } \lambda_2\text{-space, } X_1 \text{ is compact, and } \inf\{\lambda_1(x) : x \in X_1\} > 0,$$

$$\text{or}$$

$$X_1 \text{ is an } \lambda_1\text{-space, } X_2 \text{ is compact, and } \inf\{\lambda_2(x) : x \in X_2\} > 0,$$

$$\text{or}$$

$$\text{both } X_1 \text{ and } X_2 \text{ are compact.}$$

Thus, all factor spaces that would yield a product space that is a λ-space, where λ is as defined in Theorem 5.11, must fall into one of these four categories.

Example 5.15. In the case that $\lambda_1 = \lambda_2 =$ the measure of local compactness functional, when X_1 (resp. X_2) is compact, then automatically $\lambda_1(x) \equiv \infty$ (resp. $\lambda_2(x) \equiv \infty$). Thus, the final three statements of the four just listed can be condensed down to one statement: either X_1 or X_2 is compact, while the other is cofinally complete. The disjunction of this statement with the first, which in this context says that both X_1 and X_2 are uniformly locally compact, can be seen to be equivalent to Hohti's formulation [23].

6. λ-Subsets and Bornological Convergence

Over the last few years, there has been intense interest in bornological convergence of nets of sets in a metric space [12, 14, 15, 16, 30]. This was first described for nets of closed sets by Borwein and Vanderwerff [17] as follows.

Definition 6.1. *Let $\mathcal{B}$ be a bornology in metric space $\langle X, d \rangle$. We declare a net $\langle A_i \rangle_{i \in I}$ of closed subsets of X $\mathcal{B}$-convergent to a closed subset A of X if for each $B \in \mathcal{B}$ and each $\varepsilon > 0$, we have eventually both*

$$A_i \cap B \subseteq S_\varepsilon(A) \ \text{ and } \ A \cap B \subseteq S_\varepsilon(A_i).$$

Notice that convergence to the empty set means that eventually the net lies outside each set in the bornology. When $\mathcal{B} = \mathcal{P}_0(X)$, we obtain restricting our attention to $\mathcal{C}_0(X)$ convergence in Hausdorff distance because $X \in \mathcal{P}_0(X)$. When $\mathcal{B}$ is the bornology of nonempty bounded subsets, we obtain *Attouch-Wets convergence* [2, 3, 8], also called *bounded-Hausdorff convergence* [33]. When $\mathcal{B}$ is the bornology of nonempty subsets with compact closure, we obtain convergence with respect to the *Fell topology* [8, Theorem 5.1.6], also called the *topology of closed convergence* [28], which for sequences of closed sets reduces to classical Kuratowski convergence [8, Theorem 5.2.10]. Recently it has been shown that convergence of linear transformations with respect to standard topologies of uniform convergence can be understood as bornological convergence of their associated graphs [11].

Each of the bornological convergences just listed above are topological; in fact, the first two are compatible with metrizable topologies on $\mathcal{C}(X)$. As shown in [12], those bornologies for which $\mathcal{B}$-convergence is topological on $\mathcal{C}(X)$ are those that are shielded from closed sets, according to the following definition.

Definition 6.2. *Let $\mathcal{B}$ be a bornology on a metric space $\langle X, d \rangle$. We say that $B_1 \in \mathcal{B}$ is a shield for $B \in \mathcal{B}$ provided $B \subseteq B_1$ and whenever $C \in \mathcal{C}_0(X)$ is disjoint from B_1, we have $D_d(B, C) > 0$. We say $\mathcal{B}$ is shielded from closed sets provided each B in $\mathcal{B}$ has a shield in the bornology.*

In terms of open sets, $\mathcal{B}$ is shielded from closed sets if and only if given $B \in \mathcal{B} \; \exists B_1 \in \mathcal{B}$ such that $B \subseteq B_1$ and each neighborhood of B_1 contains some ε-enlargement of B. Hence, a bornology having the property that $B \in \mathcal{B} \Rightarrow \exists \varepsilon > 0$ with $S_\varepsilon(B) \in \mathcal{B}$ is obviously shielded from closed sets. So is a bornology having a base of compact sets, as then for each $B \in \mathcal{B}$, the compact set $\mathrm{cl}(B)$ serves as shield for B. More generally, whenever $\mathcal{B}$ is shielded from closed sets, then $\forall B \in \mathcal{B}, \; \mathrm{cl}(B) \in \mathcal{B}$. A wealth of additional information about this concept can be found in [12].

Theorem 6.3. *Let $\lambda \in \mathcal{C}(X, [0, \infty))$ be strongly uniformly continuous on some $B \in \mathcal{B}_\lambda$. Then B has a shield in $\mathcal{B}_\lambda$.*

Proof. Without loss of generality, we may assume X is not a λ-space and B is a closed λ-set. By strong uniform continuity of λ on B, $\forall n \in \mathbb{N}, \exists \delta_n \in (0, \frac{1}{n})$ such that $\forall b \in B, \forall x \in X, d(x, b) < \delta_n \Rightarrow |\lambda(b) - \lambda(x)| < \frac{1}{n}$. We may also assume that $\langle \delta_n \rangle$ is decreasing. Let $b \in B \setminus \mathrm{Ker}(\lambda)$. There exists a smallest $n_b \in \mathbb{N}$ such that $\frac{1}{n_b} < \lambda(b)$. If $x \in X$ satisfies $d(x, b) < \delta_{2n_b}$, then

$$\frac{1}{2n_b} < \lambda(x) < \lambda(b) + \frac{1}{2n_b}.$$

Also note that $\lambda(b) \le \frac{1}{n_b - 1}$, whenever $n_b \ne 1$. Set $\delta(b) = \delta_{2n_b}$. We claim

$$B_1 := (\mathrm{Ker}(\lambda) \cap B) \cup \bigcup_{b \in B \setminus \mathrm{Ker}(\lambda)} S_{\delta(b)}(b)$$

is a shield for B which lies in $\mathcal{B}_\lambda$.

We first show B_1 is λ-set. Let $\langle x_k \rangle$ be a sequence in B_1 with $\lambda(x_k) \to 0$. If infinitely many terms of $\langle x_k \rangle$ are contained in $\mathrm{Ker}(\lambda) \cap B$, then $\langle x_k \rangle$ must cluster by the compactness of $\mathrm{Ker}(\lambda) \cap B$. Otherwise, by passing to a subsequence we can assume $\forall k \in \mathbb{N}$, $x_k \in \bigcup_{b \in B \setminus \mathrm{Ker}(\lambda)} S_{\delta(b)}(b)$ and $\lambda(x_k) < \frac{1}{2}$. Pick $b_k \in B \setminus \mathrm{Ker}(\lambda)$ with $x_k \in S_{\delta(b_k)}(b_k)$. Fix k and let's for the moment write $n := n_{b_k}$. We know that $\frac{1}{2n} < \lambda(x_k)$, so $n \geq 2$ and $\lambda(b_k) \leq \frac{1}{n-1}$. Note also $\frac{1}{n-1} \leq \frac{2}{n}$, so

$$\lambda(b_k) \leq \frac{2}{n} = 4 \cdot \frac{1}{2n} < 4\lambda(x_k).$$

Hence $\lambda(b_k) \to 0$, so $\langle b_k \rangle$ has a cluster point p. Thus, p is a cluster point of $\langle x_k \rangle$ because $\delta(b_k) \to 0$.

Now we must show whenever $C \in \mathcal{C}_0(X)$ with $C \cap B_1 = \varnothing$, then $D_d(C, B) > 0$. By the compactness of $\mathrm{Ker}(\lambda) \cap B$, we find $\mu > 0$ such that $D_d(C, \mathrm{Ker}(\lambda) \cap B) > 2\mu$. Put $T_1 := B \cap S_\mu(\mathrm{Ker}(\lambda) \cap B)$ and $T_2 := B \setminus S_\mu(\mathrm{Ker}(\lambda) \cap B)$, so that $T_1 \cup T_2 = B$. Then $D_d(C, T_1) \geq \mu > 0$. By Theorem 4.13, there exists $\varepsilon > 0$ such that $\forall b \in T_2$, $\lambda(b) > \varepsilon$. Let $k \in \mathbb{N}$ satisfy $\frac{1}{k} < \varepsilon$. If $b \in T_2$, then $\lambda(b) > \frac{1}{k}$ so $\delta_{2k} \leq \delta(b)$. Hence,

$$\bigcup_{b \in T_2} S_{\delta_{2k}}(b) \subseteq \bigcup_{b \in T_2} S_{\delta(b)}(b) \subseteq B_1.$$

As a result, $C \cap \bigcup_{b \in T_2} S_{\delta_{2k}}(b) = \varnothing$. Then $D_d(C, T_2) \geq \delta_{2k} > 0$. Thus, $D_d(C, B) = D_d(C, T_1 \cup T_2) = \min\{D_d(C, T_1), \ D_d(C, T_2)\} > 0$. $\qquad \square$

Corollary 6.4. *Let $\lambda : X \to [0, \infty)$ be uniformly continuous. Then $\mathcal{B}_\lambda$ is shielded from closed sets.*

Example 6.5. Consider for a counterexample $[0, \infty) \times [0, \infty)$ equipped with the usual metric. If $\lambda : [0, \infty) \times [0, \infty) \to [0, \infty)$, where $\lambda(x, y) = xy$, then $B := \{(x, y) : xy = 1\} \cup \{(x, y) : x = y \text{ and } x \leq 1\}$ is a λ-set. Suppose B_1 were a shield for B. As a result of B_1 being a λ-set, $\exists n \in \mathbb{N}$ such that $B_1 \cap \{(x, 0) : x \geq 0\} \subseteq [0, n] \times \{0\}$. Then $C := [2n, \infty) \times \{0\}$ is closed and disjoint from B_1, but $D_d(C, B) = 0$. This is a contradiction. Note of course that λ is not strongly uniformly continuous on B.

Bornological convergence of a net $\langle A_i \rangle_{i \in I}$ of closed sets to a closed set A as determined by a bornology $\mathcal{B}$ can obviously be broken into two conditions, the first of which is called *upper $\mathcal{B}$-convergence*, and the second *lower $\mathcal{B}$-convergence* [30]:

(i) $\forall B \in \mathcal{B}, \forall \varepsilon > 0$ eventually $A_i \cap B \subseteq S_\varepsilon(A)$, and

(ii) $\forall B \in \mathcal{B}, \forall \varepsilon > 0$ eventually $A \cap B \subseteq S_\varepsilon(A_i)$.

As bornologies are hereditary, evidently, (ii) is in general equivalent to

(ii$'$) $\forall B \in \mathcal{B}, B \subseteq A \Rightarrow \forall \varepsilon > 0, \ B \subseteq S_\varepsilon(A_i)$ eventually.

As shown in [12], when the two-sided convergence is topological, condition (i) can be replaced by the following condition:

(i′) $\forall B \in \mathcal{B}$, if $D_d(B, A) > 0$, then eventually $D_d(B, A_i) > 0$.

From condition (i′), the topology $\mathcal{T}_{\mathcal{B}}^+$ of upper $\mathcal{B}$-convergence is generated by all sets of the form $\{A \in \mathcal{C}(X) : D_d(A, B) > 0\}$ ($B \in \mathcal{B}$), called the upper $\mathcal{B}$-*proximal topology* in the literature [19]. The topology $\mathcal{T}_{\mathcal{B}}^-$ of lower $\mathcal{B}$-convergence is not so transparent. In the case that $\mathcal{B}$ is the bornology of cofinally complete subsets, this was executed in [13]. Here we show that the description obtained for $\mathcal{T}_{\mathcal{B}_\lambda}^-$ when λ is the measure of local compactness extends naturally to the case when λ is a general uniformly continuous nonnegative functional. Our proof here is based on condition (ii′) rather than on condition (ii) as it was in the particular case addressed in [13] and seems simpler to us.

To describe a set of generators for the topology, we employ notation used in [13]: if V is a nonempty open subset of X, put $V^- := \{A \in \mathcal{C}(X) : A \cap V \neq \varnothing\}$, and if $\mathcal{W}$ is a family of nonempty open subsets of X, put

$$\mathcal{W}^{--} := \{A \in \mathcal{C}(X) : \exists \varepsilon > 0 \; \forall W \in \mathcal{W}, \; \exists a_W \in A \text{ with } S_\varepsilon(a_W) \subseteq W\}.$$

Note that for a nonempty open subset V, $\{V\}^{--} = V^-$.

Theorem 6.6. *Let λ be a nonnegative uniformly continuous real-valued function on a metric space $\langle X, d \rangle$. Then the topology $\mathcal{T}_{\mathcal{B}_\lambda}^-$ of lower $\mathcal{B}_\lambda$-convergence on the closed subsets of X is generated by all sets of the form V^- where V is a nonempty open subset of X plus all sets of the form $\mathcal{W}^{--}$ where $\mathcal{W}$ is a family of nonempty open sets with $\inf \{\lambda(x) : x \in \cup\mathcal{W}\} > 0$.*

Proof. First suppose $\langle A_i \rangle_{i \in I}$ is a net in $\mathcal{C}(X)$ that is lower $\mathcal{B}_\lambda$-convergent to A. Suppose $A \in V^-$ where V is open. Pick $a \in A$ and $\varepsilon > 0$ with $S_\varepsilon(a) \subseteq V$. Since $\{a\} \subseteq A$, applying condition (ii′) with $B = \{a\} \in \mathcal{B}_\lambda$ gives eventually $A_i \cap S_\varepsilon(a) \neq \varnothing$, so eventually $A_i \cap V \neq \varnothing$. Next suppose $A \in \mathcal{W}^{--}$ where $\inf\{\lambda(x) : x \in \cup\mathcal{W}\} = \mu > 0$. Choose $\alpha > 0$ such that $\forall W \in \mathcal{W}, \; \exists a_W \in A$ with $S_\alpha(a_W) \subseteq W$. Since $B = \{a_W : W \in \mathcal{W}\} \in \mathcal{B}_\lambda$ and $B \subseteq A$, by (ii′) $\exists i_0 \in I \; \forall i \succeq i_0, \; B \subseteq S_{\frac{\alpha}{2}}(A_i)$. Fix $i \succeq i_0$; $\forall W \in \mathcal{W}, S_{\frac{\alpha}{2}}(a_W) \cap A_i \neq \varnothing$, and we conclude

$$A_i \in \{S_\alpha(a_W) : W \in \mathcal{W}\}^{--} \subseteq \mathcal{W}^{--}.$$

For the converse, suppose $\langle A_i \rangle_{i \in I}$ converges to A in the topology with the prescribed set of generators. Let B_1 be a fixed λ-set with $B_1 \subseteq A$ and let $\varepsilon > 0$ be arbitrary. Put $B := \mathrm{cl}(B_1) \subseteq A$; it suffices to show that eventually $B \subseteq S_\varepsilon(A_i)$. We first consider two extreme cases for B: (1) B is compact, and (2) $\inf \{\lambda(b) : b \in B\} = \mu > 0$.

In case (1), by compactness $\exists \{b_1, b_2, b_3, \ldots, b_n\}$ in B such that $B \subseteq \cup_{j=1}^n S_{\frac{\varepsilon}{2}}(b_j)$. As $\{b_1, b_2, b_3, \ldots, b_n\} \subseteq A, \forall j \leq n$ we have $A \in S_{\frac{\varepsilon}{2}}(b_j)^-$, and so eventually $A_i \in \cap_{j=1}^n S_{\frac{\varepsilon}{2}}(b_j)^-$. It follows that $\{b_1, b_2, b_3, \ldots, b_n\} \subseteq S_{\frac{\varepsilon}{2}}(A_i)$ eventually and so $B \subseteq S_\varepsilon(A_i)$ eventually. In case (2) by uniform continuity there exists $\delta \in (0, \varepsilon)$ such that whenever $b \in B$ and $x \in X$ with $d(x, b) < \delta$, then $\lambda(x) > \frac{\mu}{2}$. With $\mathcal{W} = \{S_\delta(b) : b \in B\}$, we have $A \in \mathcal{W}^{--}$, so $A_i \in \mathcal{W}^{--}$ eventually, and when this occurs, $B \subseteq S_\delta(A_i) \subseteq S_\varepsilon(A_i)$.

For B which does not fit into either case (1) or (2), in view of Theorem 4.13 we have $B \cap \mathrm{Ker}(\lambda) \neq \varnothing$, and for some $\varepsilon > 0$ we have $B \backslash S_{\frac{\varepsilon}{2}}(B \cap \mathrm{Ker}\,(\lambda)) \neq \varnothing$. By the two extreme cases just considered, eventually both

(i) $B \cap \mathrm{Ker}(\lambda) \subseteq S_{\frac{\varepsilon}{2}}(A_i)$, and

(ii) $B \backslash S_{\frac{\varepsilon}{2}}(B \cap \mathrm{Ker}(\lambda)) \subseteq S_{\frac{\varepsilon}{2}}(A_i)$,

and for all such i, we have $B \subseteq S_\varepsilon(A_i)$, as required. $\qquad\square$

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Gerald Beer (gbeer@cslanet.calstatela.edu)
Department of Mathematics, California State University Los Angeles, 5151 State University Drive, Los Angeles, California 90032, USA

Manuel Segura (msegura4@calstatela.edu)
Department of Mathematics, California State University Los Angeles, 5151 State University Drive, Los Angeles, California 90032, USA