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The Jordan curve theorem in the Khalimsky plane

Ezzeddine Bouassida

Abstract. The connectivity in Alexandroff topological spaces is equivalent to the path connectivity. This fact gets some specific properties to $\mathbb{Z}^2$, equipped with the Khalimsky topology. This allows a sufficiently precise description of the curves in $\mathbb{Z}^2$ and permit to prove a digital Jordan curve theorem in $\mathbb{Z}^2$.

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1. Introduction

The computer sciences, the medical imagery, the robotic sciences and other applied sciences, make more and more useful the study of discrete sets. There is an approach where the graph theory takes place, it permits to check some results as the Jordan curve theorem: *If* Γ *is an* n*-connected closed curve in* $\mathbb{Z}^2$*, then* $\mathbb{Z}^2 \setminus \Gamma$ *has two and only two* $\overline{n}$*-connectivity components* $(n + \overline{n} = 12, n = 4, 8)$ (see [8]).

This result is a kind of generalization of the classical Jordan curve theorem in $\mathbb{R}^2$ stating that: *If* Γ *is a simple closed curve in* $\mathbb{R}^2$*, then* $\mathbb{R}^2 \setminus \Gamma$ *has two and only two connectivity components.*

A generalization of this theorem to the discrete sets needs to define topologies on this kind of sets. On $\mathbb{Z}^2$, more than one is developed (Khalimsky, Marcus, Wyse), see [5, 9]. Nowadays, the Khalimsky's one is one of the most important concepts of the digital topology. It is well known that a criterion of the convenience of a topology on $\mathbb{Z}^2$ is the validity of an analogue of the Jordan curve theorem.

In [5] Kopperman, Khalimsky and Meyer stated a generalization in $\mathbb{Z}^2$ equipped with the Khalimsky topology.

Our purpose in this note is to present a new proof of the Khalimsky's Jordan curve theorem using the specificity of the Khalimsky's plane as an Alexandroff topological space and the specific properties of connectivity on these spaces.

First of all, we present the result about the connectivity in Alexandroff spaces useful for our purpose: It is proved that in an Alexandroff topological space that the connectivity is equivalent to the C.O.T.S-arc-connectivity which is equivalent to the C.O.T.S-path-connectivity [1, 3, 9]. The C.O.T.S (connected ordered topological space) is described in [5, 8]. Here our C.O.T.S is an interval of $\mathbb{Z}$ equipped with the Khalimsky topology. Following this, we present the Khalimsky plane and we describe the specialization order, our aim is to understand the behaviour of the line-complex of an arc. This leads to a geometric description of arcs and simple closed curves. The specificity of the Khalimsky topology gives us some properties of the points in $(\mathbb{Z}^2, \kappa)$ and there adjacency sets. This allows a sufficiently precise description of the arcs in $(\mathbb{Z}^2, \kappa)$ and permit to prove the digital Jordan curve theorem.

2. Alexandroff Topological spaces

Definition 2.1. *Let (X, τ) be a topological space, (X, τ) is said to be an Alexandroff topological space , or shortly an A-space if any intersection of elements of τ is an element of τ.*

It is well known that a topological space (X, τ) is an A-space if and only if every point $x \in X$ has a smallest open neighborhood denoted by $N(x)$. The set $\boldsymbol{B} = \{N(x); x \in X\}$ is a base for the topology τ .

Recall that, given an Alexandroff topology τ on a set X, the specialization preorder $(\preceq)$ on X, associated to τ is defined by:

$$\forall (x, y) \in X \times X, \quad (x \preceq y \Leftrightarrow y \in N(x) \Leftrightarrow x \in \overline{\{y\}})$$

Where $\overline{\{y\}}$ is the closure of the point $\{y\}$.

It is proved in [2] (see [1, 5]) that:

Theorem 2.2. *On a given set X the specialization preorder determines a one to one correspondence between the T_0-Alexandroff Topologies and the partial orders.*

In what follows, we present some terminologies and definitions necessary for understanding the results of the next section.

Recall that a topological space (X, τ) is a T_0-space if for each pair of distinct points of X, there exist a neighborhood of one of them not containing the other.

A topological space, (X, τ), is said to be connected if it is a non empty set and the only subsets which are both open and closed are the empty set $\varnothing$ and X. A subset A of a topological space (X, τ) is called connected if it is connected as a topological space with the induced topology, equivalently: A is non empty and for all non empty open subsets U and V of X, we have:

$$(U \cap A \neq \varnothing, V \cap A \neq \varnothing) \Rightarrow U \cap V \cap A \neq \varnothing.$$

A connectivity component (or shortly a component) of a topological space is a connected subset which is maximal with respect to the inclusion. A component is always closed.

In an A-space (X, τ), a subset of distinct points $\{x, y\}$, is connected if and only if either $x \in N(y)$ or $y \in N(x)$. We say that x and y are adjacent or y is adjacent to x.

Let x and y be two points in a topological space (X, τ), a path (resp. an arc) linking x and y is a couple (I, γ) where $I = [a, b]$ is a compact interval of $\mathbb{R}$ equipped with the usual topology and γ a continuous map (resp. an homeomorphism) of I onto X such that $\gamma(a) = x$ and $\gamma(b) = y$.

A topological space (X, τ) is said to be path (resp. arc) connected if for any two points x and y in (X, τ), there exists a path (resp. an arc) linking x and y.

When (X, τ) is a point set, the path (resp. arc) connectivity is generalized, by the C.O.T.S connectivity, see [2, 9]. A connected ordered topological space (C.O.T.S) is a connected topological space L with the property: for any x_1, x_2, x_3 distinct points in L, there is an i such that x_j and x_k lie in different components of $L \setminus \{x_i\}$ where $\{i, j, k\} = \{1, 2, 3\}$.

A C-path (resp. C-arc) in the topological space (X, τ) is a couple (I, γ) where I is a C.O.T.S and γ is a continuous map (resp. an homeomorphism) from I onto X. The space (X, τ) (resp. a nonempty subset A of X) is said to be C.O.T.S path connected (C.P.C) (resp. C.O.T.S arc connected (C.A.C)) if for any two points x and y in (X, τ) (resp. in A) there exists a C-path (resp. C-arc) in X (resp. in A) joining x and y. We recall from [1, 3, 5, 9], the next result.

Theorem 2.3. *Let (X, τ) be an Alexandroff topological space and $A \subseteq X$ a subset. Then the following conditions are equivalent.*

(i) *A is path-wise connected (P.C).*
(ii) *A is C.O.T.S-path-wise connected (C.P.C).*
(iii) *A is C.O.T.S-arc-wise connected (C.A.C).*
(iv) *A is connected.*

The following proposition, (see [7, 8]), establishes the relationship between continuity in A-spaces and preorders. As a consequence, the rich theory of preordered sets can be put to work here.

Proposition 2.4. *Let X and Y be two T_0-Alexandroff topological spaces and $f : X \to Y$ be an application, the following conditions are equivalent.*

(i) *f is a continuous map.*
(ii) *f is an increasing map for the specialization pre-orders of X and Y.*

3. The Khalimsky plane

Let $\boldsymbol{B} = \{\{2n+1\}, \{2n-1, 2n, 2n+1\}, n \in \mathbb{Z}\}$ be a subset of $\mathfrak{P}(\mathbb{Z})$; $\boldsymbol{B}$ is a basis of a topology κ on $\mathbb{Z}$. The topological space $(\mathbb{Z}, \kappa)$ is a T_0-Alexandroff topological space called the Khalimsky or the digital line. In $(\mathbb{Z}, \kappa)$, the point set $\{x\}$ is open (resp. closed) if and only if x is odd (resp. even), $N(2n+1) = \{2n+1\}, N(2n) = \{2n-1, 2n, 2n+1\}$. A half line is open if and only if its end point is open. The specialization order is as follows:

$$... - 3 \succeq -2 \preceq -1 \succeq 0 \preceq 1 \succeq 2 \preceq 3...$$

A pair $\{x, y\}, x \neq y$, is a connected subset of $(\mathbb{Z}, \kappa)$ if and only if $y = x+1$.

In this note the Khalimsky plane $(\mathbb{Z}^2, \kappa)$ is the cartesian product $(\mathbb{Z}, \kappa) \times (\mathbb{Z}, \kappa)$ equipped with the product topology. Let (x, y) be a point in $(\mathbb{Z}^2, \kappa)$, (x, y) is an open (resp. closed) point if both x and y are odd (resp. even), (x, y) is said to be a pure point. Otherwise (x, y) is said to be a mixed point, open-closed (resp. closed-open) if x is odd (resp. even) and y is even (resp. odd). The smallest neighborhood $N(x, y)$ of the point (x, y) of $(\mathbb{Z}^2, \kappa)$ is:

- $N(x, y) = \{(x, y)\}$ when (x, y) is an open point.
- $N(x, y) = \{(x, y-1), (x, y), (x, y+1)\}$ when (x, y) is an open-closed point.
- $N(x, y) = \{(x-1, y), (x, y), (x+1, y)\}$ when (x, y) is a closed-open point.
- $N(x, y) = \{(x-1, y-1), (x-1, y), (x-1, y+1), (x, y-1), (x, y), (x, y+1), (x+1, y-1), (x+1, y), (x+1, y+1)\}$
  $= \{(a, b) \in \mathbb{Z}^2, \|(a, b) - (x, y)\|_\infty \leq 1\}$ when (x, y) is a closed point.

We denote by $A(x, y)$ the adjacency set of (x, y) and we have:

- If (x, y) is a pure point

$$A(x, y) = \{(x-1, y-1), (x-1, y), (x-1, y+1), (x, y-1),$$
$$(x, y+1), (x+1, y-1), (x+1, y), (x+1, y+1)\}$$

- If (x, y) is a mixed point

$$A(x, y) = \{(x-1, y), (x, y+1), (x+1, y), (x, y-1)\}$$

The specialization order $\preceq$ on $\mathbb{Z}^2$ is as follow:
$\forall (x, y) \in \mathbb{Z}^2, \quad (x, y) \preceq (x, y)$

If (x, y) is a closed point, we have $(x, y) \prec (a, b)$ for all (a, b) such that $\|(a, b) - (x, y)\|_\infty = 1$.

If (x, y) is a open point, we have $(a, b) \prec (x, y)$ for all (a, b) such that $\|(a, b) - (x, y)\|_\infty = 1$.

If (x, y) is a open-closed point, $(x, y) \prec (x, y+1)$, $(x, y) \prec (x, y-1)$, $(x, y) \succ (x-1, y)$ and $(x, y) \succ (x+1, y)$.

If (x, y) is a closed-open point, $(x, y) \prec (x-1, y)$, $(x, y) \prec (x+1, y)$, $(x, y) \succ (x, y+1)$ and $(x, y) \succ (x, y-1)$.

The connectivity graph of $(\mathbb{Z}^2, \kappa)$ has all the points of $\mathbb{Z}^2$ as vertices. Two vertices (x, y) and (a, b) are connected if and only if $\|(a, b) - (x, y)\|_\infty = 1$ and (x, y) and (a, b) are not simultaneously mixed points. We note that two edges can not cross.

The following remark is useful to imagine the geometric behaviour of various lines in $\mathbb{Z}^2$.

Remark 3.1. If $(p+q)$ is even, the connectivity graph is invariant under the action of the translation $T_{(p,q)}$.

If p and q are even, then, the Hass diagram (graph of the order) of $(\mathbb{Z}^2, \preceq)$ is invariant under the action $T_{(p,q)}$.

If p and q are odd and $(p+q)$ is even, then $T_{(p,q)}$ transforms $(\mathbb{Z}^2, \preceq)$ in $(\mathbb{Z}^2, \succeq)$.

The connectivity graph and the Hass diagram on $\mathbb{Z}^2$ ar invariant under a rotation by $\frac{\pi}{2}$.

4. Paths, arcs and curves in the Khalimsky plane

The spaces $(\mathbb{Z}, \kappa)$ and $(\mathbb{Z}^2, \kappa)$ are equipped with the Khalimsky topology. An interval $I = [a, b] = \{a, a+1, a+2, ..., a+n = b\}$ is a C.O.T.S. Thus give the following:

Definition 4.1. *A path* (*resp. an arc*) *in* $(\mathbb{Z}^2, \kappa)$ *is a couple* (I, γ), *where* $I = [a, b] = \{a, a+1, a+2, ..., a+n = b\}$ *is an interval in* $(\mathbb{Z}, \kappa)$ *and* γ *is a continuous map* (*resp. an homeomorphism*) *from* I *into* $(\mathbb{Z}^2, \kappa)$.

It is obvious that the image $\gamma(I) = \Gamma$ is connected. Denoting by $\hat{\Gamma}$ the line-complex of γ, $\hat{\Gamma}$ is the broken line in $\mathbb{R}^2$ having the points of Γ as vertices and completed with the edges linking the consecutive vertices of Γ.

Among the several interesting consequences of Proposition (2.4), one can check the following:

Proposition 4.2. *Let* $\hat{\Gamma}$ *be the line-complex of an arc* $(I = [a, b] = \{a, a+1, a+2, ..., a+n = b\}, \gamma)$ *in* $(\mathbb{Z}^2, \kappa)$ *and denote by* $\gamma_a, \gamma_{a+1}, ...\gamma_{a+n}$ *the vertices of* $\hat{\Gamma}$. *Then the following hold:*

- (i) *For all* $i \in \{a, a+1, a+2, ..., a+n = b\}$, *we have, either* $\gamma_{i-1} \succeq \gamma_i \preceq \gamma_{i+1}$ *or* $\gamma_{i-1} \preceq \gamma_i \succeq \gamma_{i+1}$.
- (ii) *If the vertex* γ_i *is a mixed point of* $(\mathbb{Z}^2, \kappa)$, *then* $\hat{\Gamma}$, *remains a straight line at this point.*
- (iii) *If the vertex* γ_i *is a pure point of* $(\mathbb{Z}^2, \kappa)$, *then* $\hat{\Gamma}$, *can rotate by* $\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{4}$ *or* 2π (*that means* $\hat{\Gamma}$ *can't have an acute angle* $\frac{\pi}{4}$).

Proof. The map γ is one-to-one, thus Γ don't have double points.

(i) Without loss of generality, we can suppose that $a \in 2\mathbb{Z}$, so $a \preceq a+1 \succeq a+2 \preceq a+3 \succeq a+4....$ Since the map γ is continuous, it is strictly increasing. Hence: $\gamma_a \preceq \gamma_{a+1} \succeq \gamma_{a+2} \preceq \gamma_{a+3} \succeq ....$

(ii) Let γ_i be a mixed point (closed-open for example) in Γ. According to the parity of i, we have, $i-1 \succeq i \preceq i+1$ or $i-1 \preceq i \succeq i+1$. So we must have $\gamma_{i-1} \succeq \gamma_i \preceq \gamma_{i+1}$ or $\gamma_{i-1} \preceq \gamma_i \succeq \gamma_{i+1}$. In the first case, the three points must have the same second coordinate and in the second case, they must have the same first coordinate. This implies that $\hat{\Gamma}$ still straight.

The same argument can be used when γ_i is open-closed

(iii) Let γ_i be a pure point in Γ, then γ_i is comparable (smaller or greater) to P for all P in $\{P \in \mathbb{Z}^2, \|P - \gamma_i\|_\infty = 1\}$. Thus $\gamma_{i+1} \in \{P \in \mathbb{Z}^2, \|P - \gamma_i\|_\infty = 1\}$.

This ables $\hat{\Gamma}$ to turn at P by the angle $\alpha = 0$ or $\frac{\pi}{4}$ or $\frac{\pi}{2}$ or $\frac{3\pi}{4}$ or π or $\frac{5\pi}{4}$ or $\frac{3\pi}{2}$ or $\frac{7\pi}{4}$ or 2π. Assume $\alpha = \pi$, then we have $\gamma_{i-1} = \gamma_{i+1}$ which is forbidden. Now if $\alpha = \frac{3\pi}{4}$ or $\frac{5\pi}{4}$, then the points γ_{i-1} and γ_{i+1} are adjacent and we get $\gamma_{i+1} \preceq \gamma_{i-1}$ or $\gamma_{i+1} \succeq \gamma_{i-1}$. The map γ^{-1} is continuous, so $(\gamma_{i+1} \preceq \gamma_{i-1} \Rightarrow i+1 \preceq i-1)$ and $(\gamma_{i+1} \succeq \gamma_{i-1} \Rightarrow i+1 \succeq i-1)$ which is impossible. □

It follows readily from the previous Proposition, that we have the following remarks.

Remark 4.3. Let $(I = [a, b], \gamma)$ be an arc, $\gamma(I) = \Gamma$ and $\hat{\Gamma}$ its line-complex.

(i) When we follow $\hat{\Gamma}$, we don't meet an acute angle. We can turn only on the pure vertices.

These restrictions disappear if $(I = [a, b], \gamma)$ is only a path. To see this, consider the map $\gamma : \{0, 1, 2, 3\} \to \mathbb{Z}^2, \gamma(0) = (0, 0)$, $\gamma(1) = (0, 1) = \gamma(2), \gamma(3) = (1, 1)$. One can check easily that γ is continuous and $\hat{\Gamma}$ rotate at $(0, 1)$ which is a mixed point.

(ii) If P_i $(a + 1 \leq i \leq a + n - 1)$ is a vertex in $\hat{\Gamma}$, then P_i has two adjacent points in Γ.

(iii) It is forbidden to have two mixed points in consecutive positions in $\hat{\Gamma}$.

5. The Jordan curve theorem

This section is mainly concerned with the Jordan curve theorem. The purpose of this section is to prove the Jordan curve theorem. We start with the following definitions.

Definition 5.1. *A simple closed curve (S.C.C) in* $(\mathbb{Z}^2, \kappa)$ *is the image of an interval* $I = [a, a+n] = \{a, a+1, ..., a+n\}$ *by a continuous map* γ *in* $(\mathbb{Z}^2, \kappa)$ *such that* $\gamma(a) = \gamma(a+n)$ *and any connected subset of* Γ *is the image of an arc.*

Remark 5.2. Our definition of the closed simple curve meets the Kiselmann's one for the Khalimsky Jordan curve which is a homeomorphic image of a Khalimsky circle $\mathbb{Z}/m\mathbb{Z}$, where m is an even integer ≥ 4.

As an initial step toward understanding the structure of (S.C.C), we check the following.

Proposition 5.3. *If* Γ *is a simple closed curve (S.C.C) in* $(\mathbb{Z}^2, \kappa)$*, then* $Card(\Gamma)$ *is an even integer* ≥ 4.

Proof. Assume that $\Gamma = \{P_1, P_2, P_3\}$, $\hat{\Gamma}$ is a triangle in the connectivity graph of $(\mathbb{Z}^2, \kappa)$. Thus, two of the angles of $\hat{\Gamma}$ are $\frac{\pi}{4}$ which is in contradiction with the definition of an arc. Thus $Card(\Gamma) \geq 4$.

Assume now that $\Gamma = \{P_1, P_2, P_3, P_4\}$. If one of the P_i, $(1 \leq i \leq 4)$ is a mixed point, then, $\hat{\Gamma}$ can not rotate at this P_i. To be closed, Γ needs to be a triangle with acute angles. Thus $\{P_1, P_2, P_3, P_4\}$ are pure points. The line-complex $\hat{\Gamma}$ of Γ is a square , two of its vertices are closed points and the others are open points and we have the following order:

$$...P_1 \preceq P_2 \succeq P_3 \preceq P_4 \succeq P_1 \preceq P_2 \succeq P_3 \preceq P_4...$$

or

$$...P_1 \succeq P_2 \preceq P_3 \succeq P_4 \preceq P_1 \succeq P_2 \preceq P_3 \succeq P_4...$$

Assume that $\Gamma = \{P_1, P_2, P_3, P_4, P_5\}$. The fact that any subset of Γ is the image of an arc yields:

$$...P_1 \preceq P_2 \succeq P_3 \preceq P_4 \succeq P_5 \preceq P_1 \succeq P_2 \preceq P_3...$$

or

$$...P_1 \succeq P_2 \preceq P_3 \succeq P_4 \preceq P_5 \succeq P_1 \preceq P_2 \succeq P_3...$$

which is incoherent in the both cases. This incoherence disappears when $Card(\Gamma)$ is even. □

Let Γ be a (S.C.C) in $(\mathbb{Z}^2, \kappa)$, we denote by A the subset of $\mathbb{Z}^2$ interior to $\hat{\Gamma}$ and B the subset of $\mathbb{Z}^2$ exterior to $\hat{\Gamma}$. Our aim is to prove that A and B are connectivity components of $\mathbb{Z}^2 \setminus \Gamma$.

We start our search with the case where $Card(\Gamma) \leq 8$.

Proposition 5.4. *Let Γ be a (S.C.C) in $(\mathbb{Z}^2, \kappa)$ and $Card(\Gamma) \leq 8$. Denote by A the subset of $\mathbb{Z}^2$ interior to $\hat{\Gamma}$ and B the subset of $\mathbb{Z}^2$ exterior to $\hat{\Gamma}$. Then A and B are two connectivity components of $\mathbb{Z}^2 \setminus \Gamma$.*

Proof. If $Card(\Gamma) = 4$, the unique curve (modulo the geometric translations and the rotation mentioned in Remark(0.3.1)) is the square where the four vertices are pure points and A is a mixed point P. The smaller neighborhood of P is $N(P) = \{P_1, P, P_3\}$ where P_1 and P_3 are the two open vertices in Γ. The set $\{P_2, P, P_4\}$ is closed in $(\mathbb{Z}^2, \kappa)$, thus A is a component of $\mathbb{Z}^2 \setminus \Gamma$.

An example of such Γ is

$$\Gamma = \{P_1 = (-1, 1), P_2 = (0, 2), P_3 = (1, 1), P_4 = (0, 0)\},$$

here $A = \{P\} = \{(0, 1)\}$.

If M and N are two pure points in B, we can avoid $(\Gamma \cup A)$ by the line-complex of a path J_M^N where all its vertices are pure points.

If M is a mixed point in B, M has an adjacent pure point M_1 in B otherwise $M = P$. Let N be a pure point in B ($N \neq M$), we can avoid $(\Gamma \cup A)$ by the line-complex of a path $J_{M_1}^N$ where all the vertices are pure points. We add to $J_{M_1}^N$ the point M and we have $J_M^N \subset B$ a path joining M and N, so B is (C.P.C). Hence B is a connectivity component of $\mathbb{Z}^2 \setminus \Gamma$.

Assume $Card(\Gamma) = 6$, $\hat{\Gamma}$ can not turn at a mixed point, can not have an acute angle and each point of Γ has two and only two adjacent points in Γ. These constraints get off the possibility of closed simple curve Γ with $Card(\Gamma) = 6$.

Modulo the geometric translation and the rotation mentioned in Remark(0.3.1)), there is four type of closed simple curves Γ with $Card(\Gamma) = 8$. It is easy to verify directly that in the four cases A and B are connectivity components of $\mathbb{Z}^2 \setminus \Gamma$. □

To prove the general case we need the following lemmas:

Lemma 5.5. *Let Γ be a (S.C.C) in $(\mathbb{Z}^2, \kappa)$. Let $\hat{\Gamma}$, A, and B defined as before, we have:*

- (i) *If $P \in A$, then $A(P) \subset A \cup \Gamma$.*
- (ii) *If $P \in B$, then $A(P) \subset B \cup \Gamma$.*
- (iii) *If P is a pure point in A and $A \neq \{P\}$, then P has an adjacent mixed point in A.*
- (iv) *If P is a mixed point in A and $A \neq \{P\}$, then P has an adjacent pure point in A.*

Proof. (i) Let P be a mixed point in A, $A(P) = \{P_1, P_2, P_3, P_4\}$, where $P_i, 1 \leq i \leq 4$ are pure points. P_1 and P_3 have the same second coordinate with P, and P_2 and P_4 have the same first coordinate with P. If $P_i \in B$ then $\hat{\Gamma}$ must run between P_i and P, which is impossible, so $A(P) \subset A \cup \Gamma$.

Let P be a pure point in A,

$$A(P) = \{P_1, P_2, P_3, P_4, M_1, M_2, M_3, M_4\}$$

where M_i is a mixed point and P_i is a pure point $(i = 1, 2, 3, 4)$. M_1 and M_3 have the same second coordinate with P, M_2 and M_4 have the same first coordinate with P, P_1 and P_2 have the same second coordinate with M_2, P_3 and P_4 have the same second coordinate with M_4.

If $M_i \in B$, then $\hat{\Gamma}$ must run between M_i and P, which is impossible. Thus $M_i \in A \cup \Gamma$.

If $P_i \in B$, then $\hat{\Gamma}$ links M_i and M_{i+1}, which is impossible too, thus $A(P) \subset A \cup \Gamma$.

(ii) The same Proof as in (i).

(iii) If P is a pure point in A, then $A(P) \subset A \cup \Gamma$, and $Card(A(P)) = 8$. The assumption $A(P) \subset \Gamma$ leads to $\Gamma = A(P)$ and $A = \{P\}$, a contradiction. Thus we get $A(P) \cap A \neq \varnothing$. If one of the M_i's belong to $A(P) \cap A$, we obtain the desired conclusion. If one of the P_i's belong to $A(P) \cap A$, the two mixed points M_i and M_{i+1} are in A too, otherwise $\hat{\Gamma}$ turns in a mixed point, thus P has an adjacent mixed point in A.

(iv) Let P be a mixed point in A, then $A(P) = \{P_1, P_2, P_3, P_4\}$. If $A(P) \subset \Gamma$, then $\Gamma = A(P)$ and $A = \{P\}$ which contradict the hypothesis $A \neq \{P\}$, so $A(P) \cap A \neq \varnothing$. □

Lemma 5.6. *Let Γ be a (S.C.C) in $(\mathbb{Z}^2, \kappa)$ such that $Card(\Gamma) > 8$, and let γ_1 and γ_2 be two successive points in Γ, $(\|\gamma_1 - \gamma_2\|_\infty = 1)$, then $A(\gamma_1) \cap A(\gamma_2) \cap A \neq \varnothing$ and $A(\gamma_1) \cap A(\gamma_2) \cap B \neq \varnothing$.*

Proof. Modulo the geometric translation and the rotation mentioned in remark(0.3.1), four cases may occur:

First case: γ_1 and γ_2 are successive points in a straight line parallel to one of the coordinates axes on $\hat{\Gamma}$, then one of the two points is pure (suppose it γ_1), and the other is mixed. We denote $A(\gamma_1) = \{P_1, P_2, P_3, P_4, M_1, M_2, M_3, M_4\}$

and $A(\gamma_2) = \{P'_1, P'_2, P'_3, P'_4\}$, $A(\gamma_1) \cap A(\gamma_2) \cap A = \{P'_3 = P_2\}$ or $\{P'_1 = P_1\}$, $A(\gamma_1) \cap A(\gamma_2) \cap B = \{P'_1 = P_1\}$ or $\{P'_3 = P_2\}$.

Second case:γ_1 and γ_2 are successive points in a straight diagonal line on $\hat{\Gamma}$, then the two points are pure. We denote $A(\gamma_1) = \{P_1, P_2, P_3, P_4, M_1, M_2, M_3, M_4\}$ and $A(\gamma_2) = \{P'_1, P'_2, P'_3, P'_4, M'_1, M'_2, M'_3, M'_4\}$, $A(\gamma_1) \cap A(\gamma_2) \cap A = \{M'_4 = M_3\}$ or $\{M'_2 = M_1\}$, $A(\gamma_1) \cap A(\gamma_2) \cap B = \{M'_2 = M_1\}$ or $\{M'_4 = M_3\}$.

Third case: The line-complex $\hat{\Gamma}$ rotate at γ_1 by $\frac{\pi}{4}$, in this case γ_2 is a pure point and we have: $A(\gamma_1) \cap A(\gamma_2) \cap A = \{M'_4 = M_3\}$ or $\{M'_2 = M_1\}$, $A(\gamma_1) \cap A(\gamma_2) \cap B = \{M'_2 = M_1\}$ or $\{M'_4 = M_3\}$.

Fourth case: The line-complex $\hat{\Gamma}$ rotate at γ_1 by $\frac{\pi}{2}$, in this case γ_2 is a mixed point and we have: $A(\gamma_1) \cap A(\gamma_2) \cap A = \{P'_4 = P_4\}$ or $\{P'_2 = P_1\}$, $A(\gamma_1) \cap A(\gamma_2) \cap B = \{P'_2 = P_1\}$ or $\{P'_4 = P_4\}$.

Thus $A(\gamma_1) \cap A(\gamma_2) \cap A \neq \varnothing$ and $A(\gamma_1) \cap A(\gamma_2) \cap B \neq \varnothing$. $\square$

As a consequence of the previous lemma, we get:

Corollary 5.7. *Let Γ be as in the previous lemma and let P and Q two points in Γ, then, there exists a path in A linking $A(P)$ to $A(Q)$.*

Proposition 5.8. *Let Γ be a (S.C.C) in $(\mathbb{Z}^2, \kappa)$, and let $\hat{\Gamma}$, A, and B defined as before. If U and V are two points of A, then, there exists a path J_U^V linking U and V.*

Proof. Let $U \in A$, If $A(U) \cap \Gamma \neq \varnothing$, we choose $\gamma(U) \in A(U) \cap \Gamma$. Otherwise, consider a point $U_1 \in A(U)$. If $A(U_1) \cap \Gamma \neq \varnothing$, we choose $\gamma(U) \in A(U_1) \cap \Gamma$, otherwise we consider $A(U_2)$ where $U_2 \in A(U_1)$, etc...After finite steps, we obtain $\gamma(U) \in \Gamma$ and a path in A, $J_U^{\gamma(U)}$ linking U and $\gamma(U)$.

Let $V \in A$, $(U \neq V)$, and consider $J_V^{\gamma(V)}$.

Now, using the previous corollary, we obtain a path in A, $J_{\gamma(V)}^{\gamma(U)}$, linking $A(\gamma(U))$ and $A(\gamma(V))$. The needed path is:

$$J_U^V = \{J_U^{\gamma(U)} \setminus \gamma(U)\} \cup J_{\gamma(U)}^{\gamma(V)} \cup \{J_{\gamma(V)}^V \setminus \gamma(V)\}.$$

$\square$

Corollary 5.9. *Let Γ be a (S.C.C) in $(\mathbb{Z}^2, \kappa)$, and let $\hat{\Gamma}$, A, and B defined as in the previous proposition. If U and V are two points of B, there exists a path J_U^V linking U and V.*

We close this section by the following results.

Theorem 5.10. *Let Γ be a (S.C.C) in $(\mathbb{Z}^2, \kappa)$, then Γ share $\mathbb{Z}^2$ in two components, both of them is (C.P.C).*

Corollary 5.11 (Khalimsky Jordan curve theorem)**.** *Let Γ be a (S.C.C) in $((\mathbb{Z}^2, \kappa)$, then $\mathbb{Z}^2 \setminus \Gamma$ has exactly two and only two connectivity components.*

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Ezzeddine Bouassida (ezzeddine_bouassida@yahoo.fr)
Faculty of Scienses of Sfax, Departement of Mathematics. 3018 Sfax, P.D.Box:802, Tunisia.