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On σ-starcompact spaces

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ABSTRACT. A space X is *σ-starcompact* if for every open cover $\mathcal{U}$ of X, there exists a σ-compact subset C of X such that $St(C,\mathcal{U}) = X$. We investigate the relations between σ-starcompact spaces and other related spaces, and also study topological properties of σ-starcompact spaces.

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1. INTRODUCTION

By a space, we mean a topological space. Let us recall that a space X is *countably compact* if every countable open cover of X has a finite subcover. Fleischman [3] defined a space X to be *starcompact* if for every open cover $\mathcal{U}$ of X, there exists a finite subset F of X such that $St(F,\mathcal{U}) = X$, where $St(F,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap F \neq \varnothing\}$, and he proved that every countably compact space is starcompact. Conversely, van Douwen-Reed-Roscoe-Tree [1] proved that every Hausdorff starcompact space is countably compact, but this does not hold for T_1-space (see [7]). As generalizations of starcompactness, the following classes of spaces were given:

Definition 1.1 ([1, 6])**.** A space X is *star-Lindelöf* if for every open cover $\mathcal{U}$ of X, there exists a countable subset F of X such that $St(F,\mathcal{U}) = X$.

Definition 1.2. A space X is *σ-starcompact* if for every open cover $\mathcal{U}$ of X, there exists a σ-compact subset C of X such that $St(C,\mathcal{U}) = X$.

Definition 1.3 ([3, 6, 8])**.** A space X is *$\mathcal{L}$-starcompact* if for every open cover $\mathcal{U}$ of X, there exists a Lindelöf subset L of X such that $St(L,\mathcal{U}) = X$.

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In [1], a star-Lindelöf space is called strong star-Lindelöf, in [3], $\mathcal{L}$-starcompactness is called sLc property.

From the above definitions, we have the following diagram:

$$\text{star-Lindelöf} \Rightarrow \sigma\text{-starcompact} \Rightarrow \mathcal{L}\text{-starcompact}.$$

In the following section, we give examples showing that the converses in the above Diagram do not hold.

Thorough this paper, the symbol $\beta(X)$ means the Čech-Stone compactification of a Tychonoff space X. The cardinality of a set A is denoted by $|A|$. Let ω be the first infinite cardinal, ω_1 the first uncountable cardinal and $\mathfrak{c}$ the cardinality of the set of all real numbers. As usual, a cardinal is the initial ordinal ordinals. For each ordinals α, β with $\alpha < \beta$, we write $(\alpha, \beta) = \{\gamma : \alpha < \gamma < \beta\}$, $(\alpha, \beta] = \{\gamma : \alpha < \gamma \leq \beta\}$ and $[\alpha, \beta] = \{\gamma : \alpha \leq \gamma \leq \beta\}$. Every cardinal is often viewed as a space with the usual order topology. Other terms and symbols follow [2].

2. σ-starcompact spaces and related spaces

In this section, we give two examples which show the converses in the above diagram in the section 1 do not hold.

Example 2.1. There exists a Tychonoff σ-starcompact space which is not star-Lindelöf.

Proof. Let D be a discrete space of the cardinality $\mathfrak{c}$. Define

$$X = (\beta(D) \times (\omega + 1)) \setminus ((\beta(D) \setminus D) \times \{\omega\}).$$

Then, X is σ-starcompact, since $\beta(D) \times \omega$ is a σ-compact dense subset of X.

Next, we show that X is not star-Lindelöf. Since $|D| = \mathfrak{c}$, then we can enumerate D as $\{d_\alpha : \alpha < \mathfrak{c}\}$. For each $\alpha < \mathfrak{c}$, let $U_\alpha = \{d_\alpha\} \times [0, \omega]$. Then $U_\alpha \cap U_{\alpha'} = \varnothing$ for $\alpha \neq \alpha'$. Let us consider the open cover

$$\mathcal{U} = \{U_\alpha : \alpha < \mathfrak{c}\} \cup \{\beta(D) \times \omega\}.$$

of X. Let F be a countable subset of X. Then, there exists a $\alpha_0 < \mathfrak{c}$ such that $F \cap U_{\alpha_0} = \varnothing$. Since U_{α_0} is the only element of $\mathcal{U}$ containing the point $\langle d_{\alpha_0}, \omega\rangle$ and $U_{\alpha_0} \cap F = \varnothing$, then $\langle d_{\alpha_0}, \omega\rangle \notin St(F, \mathcal{U})$, which shows that X is not star-Lindelöf. $\square$

Example 2.2. There exists a Tychonoff $\mathcal{L}$-starcompact space which is not σ-star-compact.

Proof. Let $D = \{d_\alpha : \alpha < \mathfrak{c}\}$ be a discrete space of the cardinality $\mathfrak{c}$ and let

$$Y = D \cup \{\infty\}, \text{ where } \infty \notin D$$

be the one-point Lindelöfication of D. Then, every compact subset of Y is finite by the construction of the topology of Y. Hence, Y is not σ-compact. Define

$$X = (Y \times (\omega + 1)) \setminus (\langle \infty, \omega\rangle).$$

Then, X is $\mathcal{L}$-starcompact, since $Y \times \omega$ is a Lindelöf dense subset of X.

Now, we show that X is not σ-starcompact. For each $\alpha < \mathfrak{c}$, let $U_\alpha = \{d_\alpha\} \times [0,\omega]$. Then $U_\alpha \cap U_{\alpha'} = \varnothing$ for $\alpha \neq \alpha'$. Let us consider the open cover

$$\mathcal{U} = \{U_\alpha : \alpha < \mathfrak{c}\} \cup \{Y \times \{n\} : n \in \omega\}.$$

of X. Let C be σ-compact subset of X. Then, $C \cap (D \times \{\omega\})$ is countable, since $D \times \{\omega\}$ is discrete closed in X. On the other hand, for each $n \in \omega$, $C \cap (Y \times \{n\})$ is countable in $Y \times \{n\}$, since $Y \times \{n\}$ is open and close in X. Thus, C is a countable subset of X. Since C is countable, then $\{\alpha : C \cap U_\alpha \neq \varnothing\}$ is countable, Hence, there exists a $\alpha_\omega \in \mathfrak{c}$ such that

$$C \cap U_\alpha = \varnothing \text{ for each } \alpha > \alpha_\omega.$$

If we pick $\alpha' > \alpha_\omega$. Then, $\langle d_{\alpha'}, \omega\rangle \notin St(C,\mathcal{U})$, since $U_{\alpha'}$ is the only element of $\mathcal{U}$ containing $\langle d_{\alpha'}, \omega\rangle$ and $U_{\alpha'} \cap C = \varnothing$, which shows that X is not σ-starcompact. □

Remark 2.3. The author does not know if there exists a normal $\mathcal{L}$-starcompact which is not σ-starcompact and a normal σ-starcompact space which is not star-Lindelöf.

3. Properties of σ-starcompact spaces

In Example 2.1, the closed subset $D \times \{\omega\}$ of X is not σ-starcompact, which shows that a closed subset of a σ-starcompact space need not be σ-starcompact. In the following, we construct an example which shows that a regular-closed G_δ-subspace of a σ-starcompact space need not be σ-starcompact.

Example 3.1. There exists a star-Lindelöf (hence, σ-starcompact) Tychonoff space having a regular-closed G_δ-subspace which is not σ-starcompact.

Proof. Let

$$S_1 = (Y \times (\omega + 1)) \setminus (\langle \infty, \omega\rangle).$$

be the same space as the space X in the proof of Example 2.2. As we prove above, S_1 is not σ-starcompact. Let $S_2 = \omega \cup \mathcal{R}$ be the Isbell-Mrówka space [7], where $\mathcal{R}$ is a maximal almost disjoint family of infinite subsets of ω with $|\mathcal{R}| = \mathfrak{c}$. Then, S_2 is star-Lindelöf, since ω is a countable dense subset of S_2. Hence, it is σ-starcompact.

We assume $S_1 \cap S_2 = \varnothing$. Let $\pi : D \times \{\omega\} \to \mathcal{R}$ be a bijection and let X be the quotient image of the disjoint sum $S_1 \oplus S_2$ by identifying $\langle d_\alpha, \omega\rangle$ of S_1 with $\pi(\langle d_\alpha, \omega\rangle)$ of S_2 for each $\langle d_\alpha, \omega\rangle$ of $D \times \{\omega\}$. Let $\varphi : S_1 \oplus S_2 \to X$ be the quotient map. Then, $\varphi(S_1)$ is a regular-closed G_δ-subspace of X which is not σ-starcompact.

We shall show that X is star-Lindelöf. To this end, let $\mathcal{U}$ be an open cover of X. Since $\varphi(\omega)$ is a countable dense subset of $\pi(S_2)$, then

$$\varphi(S_2) \subseteq St(\varphi(\omega), \mathcal{U}).$$

On the other hand, since $\varphi(Y \times \omega)$ is Lindelöf there exists a countable subset F_1 of $\varphi(Y \times \omega)$ such that $\varphi(Y \times \omega) \subseteq St(F_1, \mathcal{U})$. Let $F = \varphi(\omega) \cup F_1$. Then, $X = St(F, \mathcal{U})$. Hence, X is star-Lindelöf, which completes the proof. □

We give a positive result:

Theorem 3.2. *An open F_δ-subset of a σ-starcompact space is σ-starcompact.*

Proof. Let X be an σ-starcompact space and let $Y = \cup\{H_n : n \in \omega\}$ be an open F_δ-subset of X, where the set H_n is closed in X for each $n \in \omega$. To show that Y is σ-starcompact, let $\mathcal{U}$ be an open cover of Y. we have to find a σ-compact subset C of Y such that $St(C,\mathcal{U}) = Y$. For each $n \in \omega$, consider the open cover

$$\mathcal{U}_n = \mathcal{U} \cup \{X \setminus H_n\}$$

of X. Since X is σ-starcompact, there exists a σ-compact subset C_n of X such that $St(C_n,\mathcal{U}_n) = X$. Let $D_n = C_n \cap Y$. Since Y is a F_δ-set, D_n is σ-compact, and clearly $H_n \subseteq St(D_n,\mathcal{U})$. Thus, if we put $C = \cup\{D_n : n \in \omega\}$, then C is a σ-compact subset of Y and $St(C,\mathcal{U}) = Y$. Hence, Y is σ-starcompact. □

A *cozero-set* in a space X is a set of the form $f^{-1}(R \setminus \{0\})$ for some real-valued continuous function f on X. Since a cozero-set is an open F_σ-set, we have the following corollary:

Corollary 3.3. *A cozero-set of a σ-starcompact space is σ-starcompact.*

Since a continuous image of a σ-compact space is σ-compact, then it is not difficult to show the following result.

Theorem 3.4. *A continuous image of a σ-starcompact space is σ-starcompact.*

Next, we turn to consider preimages. To show that the preimage of a σ-starcompact space under a closed 2-to-1 continuous map need not be σ-starcompact we use the Alexandorff duplicate $A(X)$ of a space X. The underlying set of $A(X)$ is $X \times \{0,1\}$; each point of $X \times \{1\}$ is isolated and a basic neighborhood of a point $\langle x, 0\rangle \in X \times \{0\}$ is of the from $(U \times \{0\}) \cup ((U \times \{1\}) \setminus \{\langle x, 1\rangle\})$, where U is a neighborhood of x in X.

Example 3.5. There exists a closed 2-to-1 continuous map $f : X \to Y$ such that Y is a σ-starcompact space, but X is not σ-starcompact.

Proof. Let Y be the space X in the proof of Example 2.1. Then Y is σ-starcompact and has the infinite discrete closed subset $F = D \times \{\omega\}$. Let X be the Alexandroff duplicate $A(Y)$ of Y. Then, X is not σ-starcompact, since $F \times \{1\}$ is an infinite discrete, open and closed set in X. Let $f : X \to Y$ be the natural map. Then, f is a closed 2-to-1 continuous map, which completes the proof. □

Now, we give a positive result:

Theorem 3.6. *Let f be an open perfect map from a space X to a σ-starcompact space Y. Then, X is σ-starcompact*

Proof. Since $f(X)$ is open and closed in Y, we may assume that $f(X) = Y$. Let $\mathcal{U}$ be an open cover of X and let $y \in Y$. Since $f^{-1}(y)$ is compact, there exists a finite subcollection $\mathcal{U}_y$ of $\mathcal{U}$ such that $f^{-1}(y) \subseteq \cup\mathcal{U}_y$ and $U \cap f^{-1}(y) \neq \varnothing$ for each $U \in \mathcal{U}_y$. Pick an open neighbourhood V_y of y in Y such that $f^{-1}(V_y) \subseteq \cup\{U : U \in \mathcal{U}_y\}$, and we can assume that

$$V_y \subseteq \cap\{f(U) : U \in \mathcal{U}_y\} \tag{1}$$

because f is open. Taking such open set V_y for each $y \in Y$, we have an open cover $\mathcal{V} = \{V_y : y \in Y\}$ of Y. Hence, there exists a σ-compact subset C of Y such that $St(C, \mathcal{V}) = Y$, since Y is σ-compact. Since f is perfect, the set $f^{-1}(C)$ is a σ-compact subset of X. To show that $St(f^{-1}(C), \mathcal{V}) = X$, let $x \in X$. Then, there exists $y \in Y$ such that $f(x) \in V_y$ and $V_y \cap C \neq \varnothing$. Since

$$x \in f^{-1}(V_y) \subseteq \cup\{U : U \in \mathcal{U}_y\},$$

we can choose $U \in \mathcal{U}_y$ with $x \in U$. Then $V_y \subseteq f(U)$ by (1), and hence $U \cap f^{-1}(C) \neq \varnothing$. Therefore, $x \in St(f^{-1}(C), \mathcal{U})$. Consequently , we have that $St(f^{-1}(C), \mathcal{U}) = X$. □

By Theorem 3.6, we have the following Corollary 3.7.

Corollary 3.7. *Let X be a σ-starcompact space and Y a compact space. Then, $X \times Y$ is $\mathcal{C}$-starcompact.*

The following theorem is a generalization of Corollary 3.7.

Theorem 3.8. *Let X be a σ-starcompact space and Y a locally compact, Lindelöf space. Then, $X \times Y$ is σ-starcompact.*

Proof. Let $\mathcal{U}$ be an open cover of $X \times Y$. For each $y \in Y$, there exists an open neighbourhood V_y of y in Y such that $cl_Y V_y$ is compact. By the Corollary 3.7, the subspace $X \times cl_Y V_y$ is σ-starcompact. Thus, there exists a σ-compact subset $C_y \subseteq X \times cl_Y V_y$ such that

$$X \times cl_Y V_y \subseteq St(C_y, \mathcal{U}).$$

Since Y is Lindelöf, there exists a countable cover $\{V_{y_i} : i \in \omega\}$ of Y. Let $C = \cup\{C_{y_i} : i \in \omega\}$. Then, C is a σ-compact subset of $X \times Y$ such that $St(C, \mathcal{U}) = X \times Y$. Hence, $X \times Y$ is σ-starcompact. □

In the following, we give an example showing that the condition of the locally compact space in Theorem 3.8 is necessary.

Example 3.9. There exist a countably compact space X and a Lindelöf space Y such that $X \times Y$ is not σ-starcompact.

Proof. Let $X = \omega_1$ with the usual order topology. $Y = \omega_1 + 1$ with the following topology. Each point α with $\alpha < \omega_1$ is isolated and a set U containing ω_1 is open if and only if $Y \setminus U$ is countable. Then, X is countably compact and Y is Lindelöf. Now, we show that $X \times Y$ is not σ-starcompact. For each $\alpha < \omega_1$, let $U_\alpha = [0, \alpha] \times [\alpha, \omega_1]$, and $V_\alpha = [\alpha, \omega_1) \times \{\alpha\}$. Consider the open cover

$$\mathcal{U} = \{U_\alpha : \alpha < \omega_1\} \cup \{V_\alpha : \alpha < \omega_1\}$$

of $X \times Y$ and let C be a σ-compact subset of $X \times Y$. Then, $\pi_X(C)$ is a σ-compact subset of X, where $\pi_X : X \times Y \to X$ is the projection. Thus, there exists $\beta < \omega_1$ such that

$$\pi_X(C) \cap (\beta, \omega_1) = \varnothing$$

by the definition of the topology of X. Pick α_0 with $\alpha_0 > \beta$. Then, $V_{\alpha_0} \cap C = \varnothing$. If we pick $\alpha' > \alpha_0$, then $\langle \alpha', \alpha_0 \rangle \notin St(C, \mathcal{U})$ since V_{α_0} is the only element of $\mathcal{U}$ containing $\langle \alpha', \alpha_0 \rangle$. Hence, $X \times Y$ is not σ-starcompact, which completes the proof. □

The Theorem 3.9 also shows the product of two σ-starcompact spaces need not be σ-starcompact. Next, we give a well-known example showing that the product of two countably compact spaces need not be σ-starcompact. We give the proof roughly for the sake of completeness.

Example 3.10. There exist two countably compact spaces X and Y such that $X \times Y$ is not σ-starcompact.

Proof. Let D be a discrete space of the cardinality $\mathfrak{c}$. We can define $X = \cup_{\alpha<\omega_1} E_\alpha$, $Y = \cup_{\alpha<\omega_1} F_\alpha$, where E_α and F_α are the subsets of $\beta(D)$ which are defined inductively so as to satisfy the following conditions (1), (2) and (3):

(1) $E_\alpha \cap F_\beta = D$ if $\alpha \neq \beta$;
(2) $|E_\alpha| \leq \mathfrak{c}$ and $|F_\alpha| \leq \mathfrak{c}$;
(3) every infinite subset of E_α (resp. F_α) has an accumulation point in $E_{\alpha+1}$ (resp. $F_{\alpha+1}$).

Those sets E_α and F_α are well-defined since every infinite closed set in $\beta(D)$ has the cardinality $2^{\mathfrak{c}}$ (see [5]). Then, $X \times Y$ is not σ-starcompact, because the diagonal $\{\langle d, d \rangle : d \in D\}$ is a discrete open and closed subset of $X \times Y$ with the cardinality $\mathfrak{c}$ and σ-starcompactness is preserved by open and closed subsets. □

Example 3.11. There exist a separable space X and a Lindelöf space Y such that $X \times Y$ is not σ-starcompact.

Proof. Let $X = Y$ be the same space Y in the proof of Example 2.2. Then, Y is Lindelöf, however is not σ-starcompact. Let $Y = \omega \cup \mathcal{R}$ be the Isbell-Mrówka space [7], where $\mathcal{R}$ is a maximal almost disjoint family of infinite subsets of ω with $|\mathcal{R}| = \mathfrak{c}$. Then, Y is separable. Since $|\mathcal{R}| = \mathfrak{c}$, then we can enumerate $\mathcal{R}$ as $\{r_\alpha : \alpha < \mathfrak{c}\}$. To show that $X \times Y$ is not σ-starcompact. For each $\alpha < \mathfrak{c}$, let $U_\alpha = \{d_\alpha\} \times Y$ and $V_\alpha = (X \setminus \{d_\alpha\}) \times (\{r_\alpha\} \cup r_\alpha)$. For $n \in \omega$, let $W_n = X \times \{n\}$. We consider the open cover

$$\mathcal{U} = \{U_\alpha : \alpha < \mathfrak{c}\} \cup \{V_\alpha : \alpha < \mathfrak{c}\} \cup \{W_n : n \in \omega\}$$

of $X \times Y$. Let C be a σ-compact subset of $X \times Y$. Then, $\pi_X(C)$ is a σ-compact subset of X, where $\pi_X : X \times Y \to X$ is the projection. Thus, there exists $\alpha < \mathfrak{c}$ such that $C \cap U_\alpha = \varnothing$. Hence, $\langle d_\alpha, r_\alpha \rangle \notin St(C, \mathcal{U})$ since U_α is the only element of $\mathcal{U}$ containing $\langle d_\alpha, r_\alpha \rangle$. Hence, $X \times Y$ is not σ-starcompact. which completes the proof. □

Theorem 3.12. *Every Tychonoff space can be embedded in a σ-starcompact Tychonoff space as a closed G_δ-subspace.*

Proof. Let X be a Tychonoff space. If we put

$$Z = (\beta(X) \times (\omega + 1)) \setminus ((\beta(X) \setminus X) \times \{\omega\}),$$

then $X \times \{\omega\}$ is a closed subset of Z, which is homeomorphic to X. Since $\beta(D) \times \omega$ is a σ-compact dense subset of Z, then Z is σ-starcompact, which completes the proof. $\square$

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