

On $b\tau$ -closed sets

MAXIMILIAN GANSTER AND MARKUS STEINER

ABSTRACT. This paper is closely related to the work of Cao, Greenwood and Reilly in [10] as it expands and completes their fundamental diagram by considering b -closed sets. In addition, we correct a wrong assertion in [10] about βgs -spaces.

2000 AMS Classification: 54A05

Keywords: b -closed, qr -closed

1. INTRODUCTION AND PRELIMINARIES

In recent years quite a number of generalizations of closed sets has been considered in the literature. We recall the following definitions:

Definition 1.1. Let (X, τ) be a topological space. A subset $A \subseteq X$ is called

- (1) α -closed [18] if $\text{cl}(\text{int}(\text{cl}(A))) \subseteq A$,
- (2) semi-closed [15] if $\text{int}(\text{cl}(A)) \subseteq A$,
- (3) preclosed [17] if $\text{cl}(\text{int}(A)) \subseteq A$,
- (4) b -closed [3] if $\text{int}(\text{cl}(A)) \cap \text{cl}(\text{int}(A)) \subseteq A$,
- (5) semi-preclosed [2] or β -closed [1] if $\text{int}(\text{cl}(\text{int}(A))) \subseteq A$.

The complement of an α -closed (resp. semi-closed, preclosed, b -closed, β -closed) set is called α -open (resp. semi-open, preopen, b -open, β -open). The smallest α -closed (resp. semi-closed, preclosed, b -closed, β -closed) set containing $A \subseteq X$ is called the α -closure (resp. semi-closure, preclosure, b -closure, β -closure) of A and shall be denoted by $\text{cl}_\alpha(A)$ (resp. $\text{cl}_s(A)$, $\text{cl}_p(A)$, $\text{cl}_b(A)$, $\text{cl}_\beta(A)$).

In 2001, Cao, Greenwood and Reilly [10] introduced the concept of qr -closed sets to deal with various notions of generalized closed sets that had been considered in the literature so far. If $\mathcal{P} = \{\tau, \alpha, s, p, \beta\}$ and $q, r \in \mathcal{P}$ then a subset $A \subseteq X$ is called qr -closed if $\text{cl}_q(A) \subseteq U$ whenever $A \subseteq U$ and U is r -open. (For convenience we denote $\text{cl}(A)$ by $\text{cl}_\tau(A)$ and open (resp. semi-open, preopen) by τ -open (resp. s -open, p -open).)

In the following we shall consider the expanded family $\mathcal{P} \cup \{b\}$. As in Corollary 2.6 of [10], it is easily established that the concept of a $b\tau$ -closed set yields the only new type of sets that can be gained by utilizing the b -closure (resp. the b -interior) in the context of qr -closed sets. Thus we give

Definition 1.2. Let (X, τ) be a topological space. A subset $A \subseteq X$ is called $b\tau$ -closed if $\text{cl}_b(A) \subseteq U$ whenever $A \subseteq U$ and U is open. The complement of a $b\tau$ -closed set is called $b\tau$ -open.

Remark 1.3. The concepts of ss -closed (resp. $s\tau$ -closed, $p\tau$ -closed, $\beta\tau$ -closed) sets have been first introduced in the literature under the name of sg -closed [5] (resp. gs -closed [4], gp -closed [16], gsp -closed [11]) sets.

We also consider the following classes of topological spaces:

Definition 1.4. A topological space (X, τ) is called

- (1) sg -submaximal if every codense subset of (X, τ) is ss -closed,
- (2) T_{gs} if every $s\tau$ -closed subset of (X, τ) is ss -closed,
- (3) extremally disconnected if the closure of each open subset of (X, τ) is open,
- (4) resolvable if (X, τ) is the union of two disjoint dense subsets.

For undefined concepts we refer the reader to [10] and [9] and the references given there.

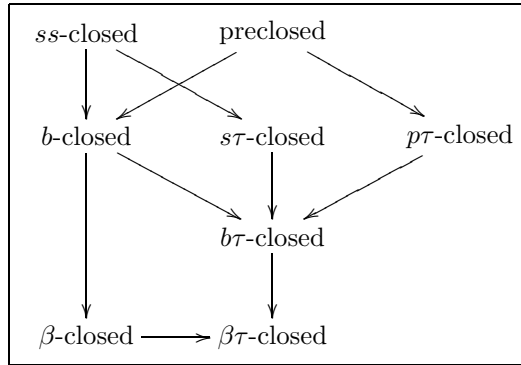
2. $b\tau$ -CLOSED SETS AND THEIR RELATIONSHIPS

In [10] the relationships between various types of generalized closed sets have been summarized in a diagram. We shall expand this diagram by adding b -closed sets and $b\tau$ -closed sets.

Proposition 2.1. Every ss -closed set in a topological space (X, τ) is b -closed.

Proof. Let $A \subseteq X$ be ss -closed and let $x \in \text{cl}_b(A)$. Since singletons are either preopen or nowhere dense (see [14]), we distinguish two cases. If $\{x\}$ is preopen, it is also b -open and hence $\{x\} \cap A \neq \emptyset$, i.e. $x \in A$. If $\{x\}$ is nowhere dense, then $X \setminus \{x\}$ is semi-open. Suppose that $x \notin A$. Then $A \subseteq X \setminus \{x\}$ and, since A is ss -closed, we have $\text{cl}_b(A) \subseteq \text{cl}_s(A) \subseteq X \setminus \{x\}$. Hence $x \notin \text{cl}_b(A)$, a contradiction. Therefore $\text{cl}_b(A) \subseteq A$, and so A is b -closed. \square

The remaining relationships in the following diagram can easily be established.



We now address the question of when the above implications can be reversed.

Proposition 2.2. *Let (X, τ) be a topological space. Then:*

- (1) *Each b -closed set is ss -closed iff (X, τ) is sg -submaximal.*
- (2) *Each b -closed set is $s\tau$ -closed iff (X, τ) is sg -submaximal.*
- (3) *Each $s\tau$ -closed set is b -closed iff (X, τ) is T_{gs} .*
- (4) *Each $b\tau$ -closed set is b -closed iff (X, τ) is T_{gs} .*
- (5) *Each $b\tau$ -closed set is $s\tau$ -closed iff (X, τ) is sg -submaximal.*
- (6) *Each $b\tau$ -closed set is $p\tau$ -closed iff (X, τ) is extremally disconnected.*
- (7) *Each $b\tau$ -closed set is β -closed iff (X, τ) is T_{gs} .*
- (8) *Each β -closed set is b -closed iff $\text{cl}(W)$ is open for every open resolvable subspace W of (X, τ) .*

Proof. We will only show (1). The other assertions can be proved in a similar manner using the standard methods that can be found in [10], and (8) has been shown in [13].

First recall that a space is sg -submaximal iff every preclosed set is ss -closed (see [7]). If every b -closed set is ss -closed then every preclosed set ss -closed, i.e. (X, τ) is sg -submaximal.

Conversely, suppose that (X, τ) is sg -submaximal and let A be b -closed. Then A is the intersection of a semi-closed and a preclosed set (see [3]). Since every semi-closed set is ss -closed, by hypothesis, A is the intersection of two ss -closed sets. Since the arbitrary intersection of ss -closed sets is always ss -closed (see [12]), we conclude that A is ss -closed. \square

Proposition 2.3. *Let (X, τ) be a topological space. Then the following statements are equivalent:*

- (1) *Each $\beta\tau$ -closed set is $b\tau$ -closed.*
- (2) *Each β -closed set is $b\tau$ -closed.*

Proof. The necessity is clear, so we only have to show the sufficiency. Let A be a $\beta\tau$ -closed set and U be an open subset of X such that $A \subseteq U$. Since

A is $\beta\tau$ -closed we have $\text{cl}_\beta(A) \subseteq U$. Now, $\text{cl}_\beta(A)$ is β -closed and hence $b\tau$ -closed by hypothesis. Therefore $\text{cl}_b(A) \subseteq \text{cl}_b(\text{cl}_\beta(A)) \subseteq U$ and thus our claim is proved. \square

Remark 2.4. If $A \subseteq X$, then the largest b -open subset of A is called the b -interior of A and is denoted by $\text{bint}(A)$. It is well known that $\text{bint}(A) = (\text{cl}(\text{int}(A)) \cup \text{int}(\text{cl}(A))) \cap A$ (see [3]). Consequently, a subset A is $b\tau$ -open iff for every closed subset F satisfying $F \subseteq A$ we have $F \subseteq \text{cl}(\text{int}(A)) \cup \text{int}(\text{cl}(A))$.

We shall now present one of our major results.

Theorem 2.5. *Let (X, τ) be a topological space. Then the following are equivalent:*

- (1) *Each β -closed set is b -closed.*
- (2) *Each β -closed set is $b\tau$ -closed.*
- (3) *$\text{cl}(W)$ is open for every open resolvable subspace W of (X, τ) .*

Proof. It is obvious that (1) \Rightarrow (2). Furthermore, it has been shown in [13] that (3) \Leftrightarrow (1), so we only have to prove that (2) \Rightarrow (3).

If $W = \emptyset$, we are done, so let W be a nonempty open resolvable subspace and let E_1 and E_2 be disjoint dense subsets of $(W, \tau|_W)$. Suppose that there exists a point $x \in \text{cl}(W) \setminus \text{int}(\text{cl}(W))$. Let $S = E_1 \cup \text{cl}(\{x\})$. It is easily checked that $\text{int}(S) = \emptyset$, $\text{cl}(S) = \text{cl}(W)$ and that S is β -open. By hypothesis, S is $b\tau$ -open and so, since $\text{cl}(\{x\}) \subseteq S$, we conclude that $\{x\} \subseteq \text{cl}(\{x\}) \subseteq \text{int}(\text{cl}(S)) = \text{int}(\text{cl}(W))$. This is, however, a contradiction to our assumption and so $\text{cl}(W)$ has to be open. \square

3. A REMARK ON βgs -SPACES

In [10] a space has been called βgs if every $\beta\tau$ -closed subset of (X, τ) is $s\tau$ -closed. We observe that this is equivalent to the property that every β -closed subset is $s\tau$ -closed, see [6]. Using some of our previous results, we are now able to give the following characterization.

Theorem 3.1. *Let (X, τ) be a topological space. Then the following are equivalent:*

- (1) *(X, τ) is a βgs -space.*
- (2) *Every β -closed set is b -closed and (X, τ) is sg -submaximal.*
- (3) *Every β -closed set is ss -closed.*

Note that the last property in the above theorem has been fully characterized in [8]. It was shown there that every β -closed set is ss -closed iff (X, τ^α) is g -submaximal, where τ^α denotes the α -topology of (X, τ) (see [18]), and a space is called g -submaximal if each codense set is $\tau\tau$ -closed. So the claim in [10] that there exists a βgs -space whose α -topology is not g -submaximal turns out to be wrong now. In fact, one can easily check that Example 3.5 of [10] is false. So βgs is not a new topological property as we have just seen.

REFERENCES

- [1] M. E. Abd El-Monsef, S.N. El-Deeb, and R.A. Mahmoud, *β -open sets and β -continuous mappings*, Bull. Fac. Sci. Assiut Univ. **12** (1983), 77–90.
- [2] D. Andrijević, *Semi-preopen sets*, Mat. Vesnik **38** (1986), 24–32.
- [3] D. Andrijević, *On b -open sets*, Mat. Vesnik **48** (1996), 69–64.
- [4] S. Arya and T. Nour, *Characterizations of s -normal spaces*, Indian J. Pure Appl. Math. **21** (1990), 717–719.
- [5] P. Bhattacharya and B. Lahiri, *Semi-generalized closed sets in topology*, Indian J. Math. **29** (1987), 375–382.
- [6] J. Cao, M. Ganster, Ch. Konstadilaki, and I. Reilly, *On preclosed sets and their generalizations*, Houston J. Math. **28** (2002), 771–780.
- [7] J. Cao, M. Ganster, and I. Reilly, *Submaximality, extremal disconnectedness and generalized closed sets*, Houston J. Math. **24** (1998), 681–688.
- [8] J. Cao, M. Ganster, and I. Reilly, *On sg -closed sets and $g\alpha$ -closed sets*, Mem. Fac. Sci. Kochi Univ. Ser A, Math. **20** (1999), 1–5.
- [9] J. Cao, M. Ganster, and I. Reilly, *On generalized closed sets*, Topology Appl. **123** (2002), 37–46.
- [10] J. Cao, S. Greenwood, and I. Reilly, *Generalized closed sets: a unified approach*, Applied General Topology **2** (2001), 179–189.
- [11] J. Dontchev, *On generalizing semi-preopen sets*, Mem. Fac. Sci. Kochi Univ. Ser. A, Math. **16** (1995), 35–48.
- [12] J. Dontchev and H. Maki, *On sg -closed sets and semi- λ -closed sets*, Questions Answers Gen. Topology **15** (1997), 259–266.
- [13] M. Ganster and M. Steiner, *On some questions about b -open sets*, Questions Answers Gen. Topology, to appear.
- [14] D. Janković and I. Reilly, *On semi-separation properties*, Indian J. Pure Appl. Math. **16** (1985), 957–964.
- [15] N. Levine, *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly **70** (1963), 36–41.
- [16] H. Maki, J. Umehara, and T. Noiri, *Every topological space is $pre-T_{1/2}$* , Mem. Fac. Sci. Kochi Univ. Ser. A, Math. **17** (1996), 33–42.
- [17] A.S. Mashhour, M.E. Abd El-Monsef, and S.N. El-Deeb, *On pre-continuous and weak pre-continuous mappings*, Proc. Math. Phys. Soc. Egypt **53** (1982), 47–53.
- [18] O. Njåstad, *On some classes of nearly open sets*, Pacific J. Math. **15** (1965), 961–970.

RECEIVED MARCH 2006

ACCEPTED JULY 2006

M. GANSTER (ganster@weyl.math.tu-graz.ac.at)
Department of Mathematics, Graz University of Technology, Steyrergasse 30,
A-8010 Graz, AUSTRIA

M. STEINER (msteiner@sbox.tugraz.at)
Department of Mathematics, Graz University of Technology, Steyrergasse 30,
A-8010 Graz, AUSTRIA