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On complete accumulation points of discrete subsets

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Abstract. We introduce a class of spaces in which every discrete subset has a complete accumulation point. Properties of this class are obtained and consistent examples are given to show that this class differs from the class of countably compact and the class of compact spaces. A number of questions are posed.

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1. Introduction

A classical theorem of General Topology states that a Hausdorff space is compact if and only if each infinite subset has a complete accumulation point (for details, we refer the reader to [3], 3.12.1). Additionally, it was shown in [14] that a Hausdorff space is compact if and only if the closure of every discrete subspace is compact. On comparing these results, it is natural to ask whether one can characterize compactness in terms of complete accumulation points of discrete sets:

Question 1.1. *Is it true that if every discrete subspace of a Hausdorff space X has a complete accumulation point in X, then X is compact?*

Shortly we shall see that the answer to this question is consistently, No. We make the following definition.

Definition 1.2. A T_1-space X is said to be *discretely complete* if every infinite discrete subspace has a complete accumulation point in X.

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Clearly every compact topological space is discretely complete and each discretely complete space is countably compact. Furthermore, since an accumulation point of a countable subset of a T_1-space is a complete accumulation point of that set, it follows that:

Remark 1.3. A countably compact T_1-space with countable spread is discretely complete; in particular, each hereditarily separable countably compact T_1-space is discretely complete.

We do not know if there exists a model of ZFC in which every discretely complete Hausdorff space is compact. Since a discretely complete space is countably compact, and it is well-known that a countably compact, linearly Lindelöf space is compact, it is immediate that a completely discrete, linearly Lindelöf space is compact. Furthermore, a space is linearly Lindelöf if and only if every uncountable subset of regular cardinality has a complete accumulation point. So a non-compact T_1-space which is discretely complete must contain some (non-discrete) subset of uncountable regular cardinality which has no complete accumulation point.

In [10], Ostaszewski constructed a non-compact, perfectly normal, hereditarily separable, countably compact Tychonoff space assuming $CH + \clubsuit(\equiv \diamondsuit)$; it follows immediately from Remark 1.3 that this is an example of a discretely complete non-compact space.

In [4], Fedorčuk constructed using $\diamondsuit$, a hereditarily separable compact Hausdorff space in which every infinite closed set has cardinality $2^{\mathfrak{c}}$. In the sequel, this space is used to show that a product of two discretely complete spaces need not be countably compact.

Both of the examples we have just mentioned are S-spaces (regular, hereditarily separable but not hereditarily Lindelöf) and it is well-known that the existence of an S-space is independent of ZFC. We also note that the existence of a discretely complete, non-compact regular space of countable spread is independent of ZFC since it was shown in [2] that under the Proper Forcing Axiom each regular countably compact space of countable spread is compact. Furthermore, Peter Nyikos has informed us that the same result holds in the case of countably compact Hausdorff spaces. However, it seems not to be known whether or not there exists in ZFC a compact Hausdorff space X such that $|X| > 2^{s(X)}$ and as we shall show later, the existence of such a space gives rise to a non-compact discretely complete Hausdorff space.

All spaces in the sequel are assumed to be T_1. All notation and terminology not specifically defined below can be found in [3], [7] and [11].

2. The Results

The examples of Fedorčuk [4] and Ostaszewski [10] cited above in Section 1, both have countable spread, but a simple construction allows us to produce discretely complete, non-compact Tychonoff spaces of arbitrary spread.

Lemma 2.1. *If in a model of ZFC there is a discretely complete, non-compact Tychonoff space of countable spread, then for each cardinal κ there is a discretely complete, non-compact Tychonoff space of spread κ.*

Proof. Let X be a discretely complete non-compact Tychonoff space of countable spread and let $Z = (\kappa \times X) \cup \{\infty\}$ where κ has the discrete topology, $\kappa \times X$ has the product topology and basic neighbourhoods of ∞ are of the form $\{\infty\} \cup [(\kappa \setminus F) \times X]$, where $F \subseteq \kappa$ is finite. We denote the projection from $\kappa \times X \to \kappa$ by π. Clearly $s(Z) = \kappa$ and if $D \subseteq Z$ is an infinite discrete set, there are two possibilities:

1) If $\pi(D)$ is infinite, then ∞ is a complete accumulation point of D, or

2) If $\pi(D) = \{\alpha_0, \dots, \alpha_n\}$ is finite, then D is countable and for some $j \in \{0, \dots, n\}$, $D \cap (\{\alpha_j\} \times X)$ is countably infinite and hence D has a complete accumulation point in $\{\alpha_j\} \times X$. □

The class of discretely complete spaces, like that of compact space is closed under taking continuous images and closed subsets.

Lemma 2.2. *The continuous image of a discretely complete space is discretely complete.*

Proof. Suppose X is discretely complete and $f : X \to Y$ is continuous and surjective. Let $D = \{d_\alpha : \alpha \in \kappa\}$ be a discrete subset of Y; then if for each $\alpha \in \kappa$ we pick $x_\alpha \in f^{-1}[\{d_\alpha\}]$ it is clear that $\{x_\alpha : \alpha \in \kappa\}$ is a discrete subset of X and hence must have a complete accumulation point, p say. Then if $q = f(p)$ and V is an open neighbourhood of q, it follows that $|\{\alpha : x_\alpha \in f^{-1}[V]\}| = \kappa$ and hence $|\{\alpha : d_\alpha \in V\}| = \kappa$, showing that q is a complete accumulation point of D. □

The proof of the following trivial result is left to the reader.

Lemma 2.3. *A closed subspace of a discretely complete space is discretely complete*

However, unlike the class of compact spaces, the class of discretely complete spaces is not closed under the taking of products.

Example 2.4. It is consistent with ZFC that there exist two discretely complete Tychonoff spaces whose product is not countably compact.

Proof. Let X denote the (above mentioned) compact space constructed in [4] under $\diamondsuit$. Let S be a fixed countably infinite subset of X; note that since X has no non-trivial convergent sequences, S is not compact and hence not countably compact. We enumerate the infinite discrete subsets of S as $\mathcal{D} = \{D_\alpha : \alpha < \mathfrak{c}\}$. Since $|\mathrm{cl}(D_0)| = 2^{\mathfrak{c}}$, we may choose distinct points $a_0^0, b_0^0 \in \mathrm{cl}(D_0) \setminus S$. If for some $\alpha < \mathfrak{c}$, we have chosen sets $A_{0,\alpha} = \{a_\beta^0 : \beta < \alpha\}$ and $B_{0,\alpha} = \{b_\beta^0 : \beta < \alpha\}$, such that $A_{0,\alpha} \cap B_{0,\alpha} = \varnothing$ and $a_\beta^0, b_\beta^0 \in \mathrm{cl}(D_\beta) \setminus S$ for each $\beta < \alpha$, then again, since $|\mathrm{cl}(D_\alpha)| = 2^{\mathfrak{c}}$ and $|S \cup A_{0,\alpha} \cup B_{0,\alpha}| \leq \mathfrak{c}$, we can choose distinct points $a_\alpha^0, b_\alpha^0 \in \mathrm{cl}(D_\alpha) \setminus (S \cup A_{0,\alpha} \cup B_{0,\alpha})$. Let $K_1 = S \cup \{a_\alpha^0 : \alpha < \mathfrak{c}\}$ and $L_1 = S \cup \{b_\alpha^0 : \alpha < \mathfrak{c}\}$.

Clearly K_1 and L_1 have cardinality $\mathfrak{c}$ and hence $|[K_1]^\omega| = |[L_1]^\omega| = \mathfrak{c}$. Enumerate the countably infinite discrete subsets of K_1 and L_1 as $\{D_{1,\alpha} : \alpha < \mathfrak{c}\}$ and $\{E_{1,\alpha} : \alpha < \mathfrak{c}\}$ respectively. Since, for each $\alpha < \mathfrak{c}$, the sets $\mathrm{cl}(D_{1,\alpha})$ and $\mathrm{cl}(E_{1,\alpha})$ have cardinality $2^{\mathfrak{c}}$ we can repeat the process described in the previous paragraph so as to obtain, for each $\alpha < \mathfrak{c}$, sets $A_{1,\alpha} = \{a^1_\beta : \beta < \mathfrak{c}\}$ and $B_{1,\alpha} = \{b^1_\beta : \beta < \mathfrak{c}\}$, such that $A_{1,\alpha} \cap B_{1,\alpha} = \varnothing$ and $a^1_\beta \in \mathrm{cl}(D_{1,\beta}) \setminus (K_1 \cup L_1)$ and $b^1_\beta \in \mathrm{cl}(E_{1,\beta}) \setminus (K_1 \cup L_1)$; finally. let $K_2 = K_1 \cup A_{1,\alpha}$ and $L_2 = L_1 \cup B_{1,\alpha}$.

Having defined sets K_β and L_β for each $\beta < \alpha < \omega_1 = \mathfrak{c}$, if α is a limit ordinal, then define $K_\alpha = \bigcup\{K_\beta : \beta < \alpha\}$, and $L_\alpha = \bigcup\{L_\beta : \beta < \alpha\}$; if α is a successor ordinal then repeat the process of the previous paragraph. Thus we define K_α and L_α of X for each $\alpha < \omega_1 = \mathfrak{c}$. Clearly $S \subseteq K_\alpha \subseteq K_\gamma$ and $S \subseteq L_\alpha \subseteq L_\gamma$ whenever $\alpha < \gamma < \mathfrak{c}$ and $K_\alpha \cap L_\alpha = S$. Let $K = \bigcup\{K_\alpha : \alpha < \mathfrak{c}\}$ and $L = \bigcup\{L_\alpha : \alpha < \mathfrak{c}\}$. If T is a countable subset of K (respectively, L), then there is some $\alpha < \mathfrak{c}$ such that $T \subseteq K_\alpha$ (respectively, $T \subseteq L_\alpha$) and hence T has an accumulation point in $K_{\alpha+1}$ (respectively, $L_{\alpha+1}$). Thus both K and L are countably compact. Since X is hereditarily separable, it follows that both K and L have countable spread and are not compact since they are proper dense subsets of $\mathrm{cl}(S)$. It follows from Remark 1.3 that both K and L are discretely complete. However, $\{(s, s) : s \in S\} \cong S$ is a closed subspace of $K \times L$ which is not countably compact. Hence $K \times L$ is not countably compact. □

As our next result shows, even the product of a compact Hausdorff space and a discretely complete space need not be discretely complete (although it will certainly be countably compact).

Theorem 2.5. *If X is a discretely complete, but non-compact T_1-space, then there is a compact Hausdorff space Y such that $X \times Y$ is not discretely complete.*

Proof. Since X is not compact, there is a subset $A = \{a_\alpha : \alpha \in \kappa\} \subseteq X$ which has no complete accumulation point. Since X is discretely complete, it is countably compact, and hence κ must be uncountable. Let Y be the Alexandroff compactification of the discrete space $D(\kappa)$ of cardinality κ and let $\{d_\alpha : \alpha \in \kappa\}$ be an enumeration of $D(\kappa)$. We consider the set $C = \{(a_\alpha, d_\alpha) : \alpha \in \kappa\} \subseteq X \times Y$. Since $\{d_\alpha\}$ is open for each $\alpha \in \kappa$, it follows that C is discrete and if $X \times Y$ were discretely complete, then C would have a complete accumulation point, say $p = (x_0, y_0)$. Thus if U is a neighbourhood of x_0 and V is a neighbourhood of y_0, then $|\{\alpha : (a_\alpha, d_\alpha) \in U \times V\}| = \kappa$. Thus for each neighbourhood U of x_0, $|\{\alpha : a_\alpha \in U\}| = \kappa$, showing that x_0 is a complete accumulation point of A, a contradiction. □

Corollary 2.6. *The property of being discretely complete is not an inverse invariant of perfect mappings.*

In contrast to Theorem 2.5, we have the following results. Recall that a space is *initially κ-compact* if every open cover of size at most κ has a finite subcover. The following two lemmas are immediate consequences of Theorems 2.2 and 5.2 of [12].

Lemma 2.7. *A space is initially κ-compact if and only if every infinite subset of cardinality at most κ has a complete accumulation point.*

Lemma 2.8. *If X is compact and Y is initially κ-compact, then $X \times Y$ is initially κ-compact.*

Theorem 2.9. *If X is a compact space of weight κ and Y is a discretely complete space which is initially κ-compact, then $X \times Y$ is discretely complete.*

Proof. Note that Lemma 2.8 implies that $X \times Y$ is initially κ-compact. Suppose that $D = \{(x_\alpha, y_\alpha) : \alpha \in \lambda\}$ is an infinite discrete subset of $X \times Y$. There are three cases to be considered.

1) If $\lambda \leq \kappa$, then since $X \times Y$ is initially κ-compact, it follows from Lemma 2.7 that D has a complete accumulation point in $X \times Y$.

2) If $cof(\lambda) > \kappa$, then fix a base $\mathcal{B}$ of X of size κ and for each $B \in \mathcal{B}$, define

$$I_B = \{\alpha \in \lambda : \text{there is an open neighbourhood } W_\alpha \text{ of } y_\alpha \text{ such that } (B \times W_\alpha) \cap D = \{(x_\alpha, y_\alpha)\}\}.$$

Since $cof(\lambda) > \kappa$, there is some $B \in \mathcal{B}$ such that $|I_B| = \lambda$. The set $Y_B = \{y_\alpha : \alpha \in I_B\}$ is discrete in Y and hence has a complete accumulation point $q \in Y$. But then, if for each $x \in X$, (x, q) is not a complete accumulation point of D, then for each $x \in X$ we can find open neighbourhoods U_x of x and V_x of q such that $|(U_x \times V_x) \cap D| < |D|$. The open cover $\{U_x : x \in X\}$ of X has a finite subcover $\{U_{x_1}, \ldots, U_{x_n}\}$ and if we let $V = \bigcap\{V_{x_k} : 1 \leq k \leq n\}$, it follows that $|(X \times V) \cap D| < |D|$ which contradicts the fact that q is a complete accumulation point of Y_B.

3) If $\lambda > \kappa \geq cof(\lambda)$, then we can find regular cardinals $\{\lambda_\alpha : \alpha \in cof(\lambda)\}$ such that $\kappa < \lambda_\alpha < \lambda$ and $\sup\{\lambda_\alpha : \alpha \in cof(\lambda)\} = \lambda$. Now write $D = \bigcup\{D_\alpha : \alpha \in cof(\lambda)\}$ where $|D_\alpha| = \lambda_\alpha$. By 2), each of the discrete sets D_α has a complete accumulation point $(p_\alpha, q_\alpha) \in X \times Y$, and since this latter space is initially κ-compact, it again follows from Lemma 2.7 that the set $\{(p_\alpha, q_\alpha) : \alpha \in cof(\lambda)\}$ has a complete accumulation point $(p, q) \in X \times Y$. Now any neighbourhood V of (p, q) contains $cof(\lambda)$-many points (p_α, q_α) and hence λ_α-many points of D_α for $cof(\lambda)$-many α. It follows that (p, q) is a complete accumulation point of D. □

Corollary 2.10. *If X is a compact metrizable space and Y is discretely complete, then $X \times Y$ is discretely complete.*

Proof. Since X is metrizable, $w(X) = \omega$. The space Y is discretely complete, hence countably compact, that is to say, initially ω-compact; the result now follows from the theorem. □

We now show that a construction very similar to that used in Example 2.4 can in fact be carried out on any compact Hausdorff space in which $|X| > 2^{s(X)}$ in order to construct a non-compact discretely complete space. The construction is reminiscent of the classical construction of a countably compact, non-compact dense subspace of $\beta\omega$ of size $\mathfrak{c}$ (see [5], 9.15).

Theorem 2.11. *In any model of ZFC in which there exists a compact Hausdorff space X with $|X| > 2^{s(X)}$, there exists a non-compact discretely complete Tychonoff space.*

Proof. Suppose $\kappa = s(X)$; by Theorem 2.17 of [7], there is a dense subspace $E_0 \subseteq X$ of cardinality at most 2^κ. We enumerate the discrete subsets of E_0 as $\{D^0_\alpha : \alpha \in \lambda_0\}$, where $\lambda_0 \leq |E_0|^{s(X)} \leq (2^\kappa)^\kappa = 2^\kappa < |X|$. Since X is compact, each subset D^0_α has a complete accumulation point $x^0_\alpha \in X$; let $E_1 = E_0 \cup \{x^0_\alpha : \alpha \in \lambda_0\}$. Clearly $|E_1| \leq \max\{\lambda_0, 2^\kappa\} < |X|$. Having constructed subsets E_α for each $\alpha < \beta < \kappa^+$, with the property that $|E_\alpha| \leq 2^\kappa < |X|$ and such that $E_\mu \subseteq E_\nu$ whenever $\mu < \nu$, we construct E_β as follows:

If β is a limit ordinal then $E_\beta = \bigcup\{E_\alpha : \alpha \in \beta\}$.

If $\beta = \gamma + 1$, then since $|E_\gamma| \leq 2^\kappa$, we can enumerate the discrete subsets of E_γ as $\{D^\gamma_\alpha : \alpha \in \lambda_\gamma\}$ where $\lambda_\gamma \leq 2^\kappa < |X|$. Again, since X is compact, each of the sets D^γ_α has a complete accumulation point in X and for each $\alpha \in \lambda_\gamma$ we choose one such, x^γ_α. Let $E_\beta = E_\gamma \cup \{x^\gamma_\alpha : \alpha \in \lambda_\gamma\}$. It is immediate that $|E_\beta| \leq 2^\kappa < |X|$.

Now let $E = \bigcup\{E_\alpha : \alpha < \kappa^+\}$; clearly $|E| \leq \kappa^+.2^\kappa = 2^\kappa$ and so $E \subsetneq X$. Furthermore, if $S \subseteq E$ is discrete, then $|S| \leq \kappa$ and hence there is some $\alpha \in \kappa^+$ such that $S \subseteq E_\alpha$ and so S has a complete accumulation point in $E_{\alpha+1}$ and hence in E as well. $\square$

We note in passing that by applying the technique of Gryzlov [6], the space E in the previous theorem can even be made to be normal. Before stating a generalization of this result we need some definitions.

Recall that a space X is a *KC-space* if every compact subspace of X is closed, X is an *SC-space* (see for example, [1]) if every convergent sequence together with its limit forms a closed subset of X and X is a *US-space* (see, [8]) if every convergent sequence in X has a unique limit. It is easy to see that $KC \Rightarrow SC \Rightarrow US \Rightarrow T_1$.

A similar technique to that used in Theorem 2.11 can be used to prove the following result. We leave the details to the reader.

Theorem 2.12. *In any model of ZFC in which there exists a compact KC-space X with $|X| > d(X)^{s(X)}$, there exists a non-compact discretely complete KC-space.*

As mentioned in the Introduction, we do not know if there is a ZFC example of a discretely complete, non-compact Hausdorff space; however, discretely complete, non-compact US-spaces exist in ZFC.

Example 2.13. There is in ZFC a non-compact US-space which is discretely complete.

Proof. Our aim is to define a topology σ on the set ω_1 such that (ω_1, σ) is a non-compact, discretely complete US-space; we begin by defining a topology τ (used in [13]) generated by the following sub-base:

$$\{\{\beta : \beta < \alpha\} : \alpha \in \omega_1\} \cup \{C : \omega_1 \setminus C \text{ is finite}\}.$$

Clearly τ is a T_1-topology which is weaker than the order topology on ω_1 and hence (ω_1, τ) is countably compact but not Lindelöf, since the open cover $\{\{\beta : \beta < \alpha\} : \alpha \in \omega_1\}$ has no countable subcover. Furthermore, if $A \subset \omega_1$ has order type (induced by the order on ω_1) greater than or equal to $\omega + 1$, then A is not discrete, and hence every discrete subset of (ω_1, τ) is countable. That this space is discretely complete but not Lindelöf and hence not compact is now a consequence of the remarks following Definition 1.2.

However, in (ω_1, τ), every injective sequence converges to an uncountable number of points. To obtain a US-space we use a construction used for compact spaces by Künzi and van der Zypen, [8].

Let $\mathcal{A} = \{A_\alpha : \alpha \in I\}$ be a maximal almost disjoint (MAD) family of injective sequences in ω_1, where $A_\alpha = \{x_\alpha^n : n \in \omega\}$ and for each $\alpha \in I$, choose a limit $\ell_\alpha \notin A_\alpha$. Denote the set $\{x_\alpha^n : n \geq m\} \cup \{\ell_\alpha\}$ by A_α^m and let σ be the topology generated by the subbase $\tau \cup \{X \setminus A_\alpha^m : m \in \omega,\ \alpha \in I\}$. We claim that (ω_1, σ) is a US-space which is discretely complete and since it is not Lindelöf, it is not compact. In order to show that the space is discretely complete, we first show that its spread is countable. To this end, suppose that B is an uncountable subset of ω_1 and we write $B = \bigcup\{B_\alpha : \alpha \in \omega_1\}$ where the sets B_α are mutually disjoint and countably infinite. Consider the countably infinite set $B_0 \equiv B_{\beta_0}$. Since $\mathcal{A}$ is a MAD family, there must exist $A_{\alpha_0} \in \mathcal{A}$ such that $B_0 \cap A_{\alpha_0}$ is infinite. Let $b_0 = \sup(A_{\alpha_0} \cup B_0 \cup \{\ell_{\alpha_0}\}) < \omega_1$, clearly $B \cap (b_0, \omega_1)$ is uncountable. Let $\beta_1 = \min\{\beta \in \omega_1 : |B_\beta \cap (b_0, \omega_1)| = \omega\}$. Again, since $\mathcal{A}$ is a MAD family, there is some $A_{\alpha_1} \in \mathcal{A}$ such that $A_{\alpha_1} \cap B_{\beta_1} \cap (b_0, \omega_1)$ is infinite; let $b_1 = \sup(A_{\alpha_1} \cup B_{\beta_1} \cup \{\ell_{\alpha_1}\})$. Continuing this process, we obtain a family $\{B_{\beta_n} : n \in \omega\}$ of countably infinite, mutually disjoint subsets of B, each of which intersects an element of $\mathcal{A}$ in an infinite set. Let $S = \bigcup\{A_{\alpha_n} \cap B_{\beta_n} \cap (b_{n-1}, \omega_1) : n \in \omega\}$ and let $b \in B \cap (\sup(S), \omega_1)$. We claim that $b \in \mathrm{cl}_\sigma(S)$, thus showing that B is not discrete. Suppose to the contrary that $b \notin \mathrm{cl}_\sigma(S)$; then there is some basic closed set containing S but not b. Since all τ-closed sets contain a cofinal interval of ω_1, it follows that there must be some finite subset of $\mathcal{A}$, say $\{A_{\gamma_1}, A_{\gamma_1}, \ldots, A_{\gamma_n}\}$ such that $b \notin \bigcup\{A_{\gamma_m} : 1 \leq m \leq n\} \cup \{\ell_{\gamma_m} : 1 \leq m \leq n\} \supseteq S$. Clearly then there is some $j \in \{1, \ldots, n\}$ and some $k \in \omega$ such that $A_{\gamma_j} \cap A_{\alpha_k} \cap B_{\beta_k}$ is infinite, which since there are only finitely many possible such j but infinitely many such k, contradicts the fact that $\mathcal{A}$ is an almost disjoint family. Thus $b \in \mathrm{cl}(S)$ and to show that (ω_1, σ) is discretely complete, it now suffices to show that it is countably compact. However, if $T = \{t_n : n \in \omega\}$ is a countably infinite subset of ω_1, then there is some $A_\lambda \in \mathcal{A}$ such that $A_\lambda \cap T$ is infinite. It is immediate that $\ell_\lambda \in \mathrm{cl}(T)$.

Finally, the proof that (ω_1, σ) is a US-space follows exactly as in [8]. $\square$

Recall that a space is *weakly Lindelöf* if every open cover has a countable dense subsystem.

The spaces $\beta\omega$ and $\omega^* = \beta\omega \setminus \omega$ are the source of many examples and counterexamples, thus the following is of interest.

Theorem 2.14. *[CH] For each $p \in \omega^*$, $\omega^* \setminus \{p\}$ is not discretely complete.*

Proof. In the proof of Corollary 1.5.4 of [9] it is shown (in ZFC) that $\omega^* \setminus \{p\}$ is not weakly Lindelöf. Hence there is an open cover $\mathcal{U}$ of $\omega^* \setminus \{p\}$ with no countable dense subsystem. Now, using CH, we may enumerate $\mathcal{U}$ as

$$\{U_\alpha : \alpha < \omega_1\}.$$

Choose $x_0 \in U_0$ and let $\alpha_0 = 0$. Suppose now that for some $\lambda < \omega_1$, we have chosen points x_γ, indices $\alpha_\gamma \in \omega_1$ and elements $U_{\alpha_\gamma} \in \mathcal{U}$ for each $\gamma < \lambda$. Note that since α_γ is a countable ordinal for all $\gamma < \lambda$, it follows that $\mathrm{cl}(\bigcup\{U_\xi : \xi < \alpha_\gamma, \gamma < \lambda\}) \neq \omega^* \setminus \{p\}$ and hence we can define

$$\alpha_\lambda = \min\{\beta < \omega_1 : U_\beta \setminus \mathrm{cl}(\bigcup\{U_\xi : \xi < \alpha_\gamma, \gamma < \lambda\}) \neq \varnothing\}$$

and choose $x_\lambda \in U_{\alpha_\lambda} \setminus \mathrm{cl}(\bigcup\{U_\xi : \xi < \alpha_\gamma, \gamma < \lambda\})$.

Note that this construction ensures that $x_\lambda \notin \mathrm{cl}\{x_\gamma : \gamma < \lambda\}$ and $x_\gamma \notin \mathrm{cl}(U_{\alpha_\lambda})$ for each $\gamma > \lambda$ and so $\{x_\alpha : \alpha \in \omega_1\}$ is discrete. Furthermore, the discrete subset $\{x_\alpha : \alpha \in \omega_1\}$ has no complete accumulation point in $\omega^* \setminus \{p\}$, since each of the open sets U_α contains only countably many points of the set $\{x_\alpha : \alpha \in \omega_1\}$. □

3. Open Questions

Below, we repeat the principal open questions regarding discretely complete spaces.

Question 3.1. *Is it consistent with ZFC that every Tychonoff discretely complete space is compact?*

Question 3.2. *Is there in ZFC, an SC (or even a KC or Hausdorff) example of a discretely complete space which is not compact?*

References

[1] O. T. Alas and R. G. Wilson, *Minimal properties between T_1 and T_2*, Houston J. Math. **32**, no. 2 (2006), 493–504.

[2] Z. Balogh, A. Dow, D. H. Fremlin and P. J. Nyikos, *Countable tightness and proper forcing*, Bulletin A.M.S. (N. S.) **19**, no. 1 (1988), 295–298.

[3] R. Engelking, *General Topology*, Heldermann Verlag, Berlin 1989.

[4] V. Fedorčuk, *On the cardinality of hereditarily separable compact Hausdorff spaces*, Soviet Math. Doklady **16** (1975), 651–655.

[5] L. Gillman and M. Jerison, *Rings of Continuous Functions*, Van Nostrand, Princeton, 1960.

[6] A. Gryzlov, *Cardinal invariants and compactifications*, Comm. Math. Univ. Carolinae **35**, no. 1 (1994), 403–408.
[7] I. Juhasz, Cardinal Funcions in Topology - Ten years later, Mathematical Centre Tracts 123, Mathematisch Centrum, Amsterdam, 1980.
[8] H.-P. Künzi and D. van der Zypen, *Maximal (sequentially) compact topologies*, preprint, http : //arxiv.org/PS_cache/math/pdf/0306/0306082.pdf
[9] J. van Mill, *An introduction to $\beta\omega$*, in *Handbook of Set-theoretic Topology*, North Holland, Amsterdam, 1984.
[10] A. J. Ostaszewski, *On countably compact perfectly normal spaces*, J. London Math. Soc. **14** (1976), 501–516.
[11] M. E. Rudin, *Lectures on Set Theoretic Topology*, CBMS, Regional Conference Series in Mathematics, No. 23, A. M. S., Providence, 1975.
[12] R. M. Stephenson, *Initially κ-compact and related spaces*, in *Handbook of Set-theoretic Topology*, North Holland, Amsterdam, 1984.
[13] A. Tamariz and R. G. Wilson, *Example of a T_1 topological space without a Noetherian base*, Proceedings A.M.S. **104**, no. 1 (1988), 310–312.
[14] V. V. Tkachuk, *Spaces that are projective with respect to classes of mappings,* Trans. Moscow Math. Soc. **50** (1988), 139–156.

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