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On resolutions of linearly ordered spaces

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Abstract. We define an extended notion of resolution of topological spaces, where the resolving maps are partial instead of total. To show the usefulness of this notion, we give some examples and list several properties of resolutions by partial maps. In particular, we focus our attention on order resolutions of linearly ordered sets. Let X be a set endowed with a Hausdorff topology τ and a (not necessarily related) linear order $\preceq$. A unification of X is a pair $(Y, \imath)$, where Y is a LOTS and $\imath \colon X \hookrightarrow Y$ is an injective, order-preserving and open-in-the-range function. We exhibit a canonical unification $(Y, \imath)$ of $(X, \preceq, \tau)$ such that Y is an order resolution of a GO-space $(X, \preceq, \tau^*)$, whose topology τ^* refines τ. We prove that $(Y, \imath)$ is the unique minimum unification of X. Further, we explicitly describe the canonical unification of an order resolution.

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1. Resolutions by partial maps

Let (X, τ) be a topological space; if the topology is understood, we denote it by X. By neighborhood of a point we mean open neighborhood. For each point $x \in X$, the set of all neighborhoods of x is denoted by $\tau(x)$. Similarly, if $\mathcal{B}$ is a base for X, then $\mathcal{B}(x)$ denotes the set of all basic neighborhoods of x. In this paper topological spaces are assumed to be Hausdorff.

A *chain* is a linearly ordered set. If $(X, \preceq)$ is a chain, then its *reverse chain* is $(X, \preceq^*)$, where $x \preceq^* y$ if and only if $y \preceq x$ for each $x, y \in X$; to simplify notation, we denote this chain by X^*. If A and B are subchains of $(X, \preceq)$, the notation $A \prec B$ stands for $a \prec b$ for each $a \in A$ and $b \in B$; in particular, if $A = \{a\}$, we simplify notation and write $a \prec B$.

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A topological space (X, τ) is *orderable* if there exists a linear order $\preceq$ on X such that the order topology $\tau_{\preceq}$ coincides with τ. A *linearly ordered topological space* (for short, a *LOTS*) is a chain endowed with the order topology; we denote a LOTS by $(X, \preceq, \tau_{\preceq})$.

A topological space (X, τ) is *suborderable* if there exists an orderable topological space (Y, σ) such that (X, τ) embeds homeomorphically into (Y, σ). It is known that a topological space (X, τ) is suborderable if and only if there exists a total order on X (called a *compatible* order) such that (i) the original topology is finer than the order topology, and (ii) each point of X has a local base consisting of (possibly degenerate) intervals. A *generalized ordered space* (for short, a *GO-space*) is a suborderable space endowed with a compatible order. The class of GO-spaces is known to coincide with the class of topological subspaces of LOTS. In the sequel we assume without loss of generality that a GO-space X is a subspace of the LOTS in which it embeds. For a recent survey on LOTS and GO-spaces, see [3] (Section F-7) and references therein.

Now we define the notion of a resolution of a family of topological spaces. This elegant and fundamental idea was introduced by Fedorčuk in 1968 (see [2]) and extensively studied by Watson in 1992 (see [4]).

Definition 1.1. Let (X, τ) be a topological space, $(Y_x, \tau_x)_{x \in X}$ a family of topological spaces and $(f_x \colon X(x) \to Y_x)_{x \in X}$ a family of continuous maps, where $X(x)$ is an open subset of $X \setminus \{x\}$. We endow the set $\bigcup_{x \in X} \{x\} \times Y_x$ with a topology $\tau_{\otimes}$ induced by a base $\mathcal{B}$. For each point $x \in X$, neighborhood $U \subseteq X$ of x and open set $V \subseteq Y_x$, we define basic open sets $U \otimes_x V$ in $\mathcal{B}$ as follows:

$$U \otimes_x V := \{x\} \times V \,\cup \bigcup \left\{ \{x'\} \times Y_{x'} \,:\, x' \in (U \cap f_x^{-1} V) \right\}.$$

The topological space $\left(\bigcup_{x \in X} \{x\} \times Y_x, \tau_{\otimes} \right)$ is called the *resolution of* X *at each* $x \in X$ *into* Y_x *by the map* f_x; we denote it by $\bigotimes_{x \in X} (Y_x, f_x)$. The space X is the *global* space and the Y_x's are the *local* spaces.

Without loss of generality, we assume that the global space X is non-trivial; in fact, if $X = \{x\}$, then the resolution is homeomorphic to Y_x. Our definition slightly extends the classical notion of resolution, in which the resolving functions f_x are defined on the whole set $X \setminus \{x\}$. For sake of clarity, we refer to the classical notion as a *resolution by total functions*. On the other hand, the resolving functions f_x described in Definition 1.1 can be thought of as partial functions $f_x \colon X \setminus \{x\} \to Y_x$; we call the associated topological space a *resolution by partial functions* or simply a *resolution*.

In this section we give some examples and list several properties of resolutions. We start by showing that $\bigotimes_{x \in X} (Y_x, f_x)$ is a well-defined topological space.

Lemma 1.2. *The family $\mathcal{B}$ is a base for a topology on the set $\bigcup_{x \in X} \{x\} \times Y_x$.*

We need a technical result, which we prove first.

Lemma 1.3. *For each $(x', y') \in U \otimes_x V$, there exists $U' \otimes_{x'} V' \in \mathcal{B}$ such that $(x', y') \in U' \otimes_{x'} V' \subseteq U \otimes_x V$.*

Proof. The result is obvious if $x' = x$. Assume that $x' \neq x$. Then $(x', y') \in U \otimes_x V$ implies that $(x', y') \in \bigcup \big\{\{w\} \times Y_w : w \in (U \cap f_x^{-1}V)\big\}$, whence $x' \in U \cap f_x^{-1}V$ and $y' \in Y_{x'}$. Set $U' := U \cap f_x^{-1}V$ and $V' := Y_{x'}$. Note that $(x', y') \in U' \otimes_{x'} V' \in \mathcal{B}$, because f_x is continuous on an open subset of $X \setminus \{x\}$ and so U' is a neighborhood of x'. Furthermore, we have:

$$U' \otimes_{x'} V' \subseteq \bigcup \big\{\{w\} \times Y_w : w \in U \cap f_x^{-1}V\big\} \subseteq U \otimes_x V.$$

Thus $U' \otimes_{x'} V'$ satisfies the claim. $\square$

By Lemma 1.3, we can assume without loss of generality that a basic neighborhood of (x, y) is of the type $U \otimes_x V$, where $U \subseteq X$ is a neighborhood of x and $V \subseteq Y_x$ is an open set.

Proof of Lemma 1.2. It suffices to show that for any two basic open sets $U_1 \otimes_{x_1} V_1$ and $U_2 \otimes_{x_2} V_2$ in $\mathcal{B}$, if $(x', y') \in (U_1 \otimes_{x_1} V_1) \cap (U_2 \otimes_{x_2} V_2)$, then there exists $U' \otimes_{x'} V' \in \mathcal{B}$ such that $(x', y') \in U' \otimes_{x'} V' \subseteq (U_1 \otimes_{x_1} V_1) \cap (U_2 \otimes_{x_2} V_2)$. For each $i \in \{1, 2\}$, if $(x', y') \in U_i \otimes_{x_i} V_i$, then Lemma 1.3 implies that there exists $U'_i \otimes_{x'} V'_i$ such that $(x', y') \in U'_i \otimes_{x'} V'_i \subseteq U_i \otimes_{x_i} V_i$. Set $U' := U'_1 \cap U'_2$ and $V' := V'_1 \cap V'_2$. Then, we obtain:

$$(x', y') \in U' \otimes_{x'} V' = (U'_1 \otimes_{x'} V'_1) \cap (U'_2 \otimes_{x'} V'_2) \subseteq (U_1 \otimes_{x_1} V_1) \cap (U_2 \otimes_{x_2} V_2).$$

This proves the claim. $\square$

A smaller base for the resolution topology $\tau_\otimes$ is the following.

Lemma 1.4. *Let (X, τ) and $(Y_x, \tau_x)_{x \in X}$ be topological spaces. Assume that $\mathcal{B}_X$ is a base for τ and for each $x \in X$, $\mathcal{B}_x$ is a base for τ_x. The family*

$$\mathcal{B}_\otimes := \{U \otimes_x V : x \in X \wedge U \in \mathcal{B}_X(x) \wedge V \in \mathcal{B}_x\}$$

is a base for $\bigotimes_{x \in X}(Y_x, f_x)$.

Proof. Let $U \otimes_x V$ be a basic open set in the resolution topology and z a point of $U \otimes_x V$. By Lemma 1.3, we can assume that $z = (x, y)$ for some $y \in V$. Select $U' \in \mathcal{B}_X$ and $V' \in \mathcal{B}_x$ such that $x \in U' \subseteq U$ and $y \in V' \subseteq V$. Then the set $U' \otimes_x V' \in \mathcal{B}_\otimes$ is such that $(x, y) \in U' \otimes_x V' \subseteq U \otimes_x V$. It follows that $\mathcal{B}_\otimes$ is a base for $\bigotimes_{x \in X}(Y_x, f_x)$. $\square$

Some topological spaces can viewed as a resolution of other spaces only if we use partial resolving functions. In the next example we show that the closed unit interval is a resolution by partial maps of the unit circle.

Example 1.5. Let S^1 be the unit circle (having the origin $(0, 0)$ as its center). Resolve each point $s \in S^1 \setminus \{(1, 0)\}$ into the one-point space $\{y_s\}$ by the constant function f_s. Note that in order to resolve the point $t = (1, 0)$ into a two-point discrete space $\{y_t^-, y_t^+\}$, we cannot use a total function to map continuously $S^1 \setminus \{t\}$ onto $\{y_t^-, y_t^+\}$. Let $t^* = (-1, 0)$ be the antipodal point of t. Define $f_t \colon S^1 \setminus \{t, t^*\} \to \{y_t^-, y_t^+\}$ by $f_t(x, y) := y_t^-$ if $y < 0$ and $f_t(x, y) := y_t^+$ if $y > 0$. The resolution is (homeomorphic to) the unit interval $[0, 1]$.

Another advantage of this extended notion of resolution is that a wide class of subspaces of a resolution by total functions can be seen as a resolution by partial functions.

Lemma 1.6. *Let* $Z := \bigotimes_{x\in X}(Y_x, g_x)$ *be the resolution of* X *at each* $x \in X$ *into* Y_x *by a total map* g_x*. Assume that* W *is a subspace of* Z *with the property that for each* $x \in X$*, there exists an open set* $V_x \subseteq Y_x$ *such that* $W \cap (\{x\} \times Y_x) = \{x\} \times V_x$*. Then* W *is a resolution* $\bigotimes_{x\in X}(Y_x, f_x)$ *by partial maps* f_x*.*

Proof. For each $x \in X$, let V_x be an open subset of Y_x such that $W \cap (\{x\} \times Y_x) = \{x\} \times V_x$. Continuity of the total function $g_x \colon X \setminus \{x\} \to Y_x$ implies that the set $X(x) := g_x^{-1}V_x$ is open in $X \setminus \{x\}$. Denote by f_x the partial maps on $X \setminus \{x\}$, defined as $g_x \restriction X(x)$. Then $W = \bigotimes_{x\in X}(Y_x, f_x)$. □

In the next examples, the symbol $\mathbf{n}$ denotes the discrete LOTS with exactly n elements, *i.e.*, $\mathbf{n} = \{0, 1, \ldots, n-1\}$.

Example 1.7. The *double arrow space* is the lexicographic product $\mathbb{R} \times_{\text{lex}} \mathbf{2}$ endowed with the order topology. This space can be seen as the resolution of $\mathbb{R}$ at each point x into the discrete space $\mathbf{2}$ by the function $f_x \colon \mathbb{R} \setminus \{x\} \to \mathbf{2}$, defined by $f_x(x') = 0$ for $x' < x$ and $f_x(x'') = 1$ for $x'' > x$.

The *Sorgenfrey line* is the subspace $S = \{(x, 1) : x \in \mathbb{R}\}$ of the double arrow space. The Sorgenfrey line can be trivially seen as a resolution by partial maps: the global space is $\mathbb{R}$, the local spaces are all equal to $\mathbf{1}$ and the resolving functions are the constant maps $g_x \colon (x, \to) \to \mathbf{1}$.

Example 1.8. The *Alexandroff duplicate* is the space $\mathbb{R} \times \mathbf{2}$, whose topology is such that the subspace $\mathbb{R} \times \{0\}$ is homeomorphic to $\mathbb{R}$ and the subspace $\mathbb{R} \times \{1\}$ is made of isolated points.

The *Michael line* M is the usual space $\mathbb{R}$ with each irrational isolated. The Michael line is homeomorphic to the subspace $(\mathbb{P} \times \{1\}) \cup (\mathbb{Q} \times \{0\})$ of the Alexandroff duplicate.

The space M can also be viewed as a subspace of the following resolution of $\mathbb{R}$. Resolve each rational into the space $\mathbf{1}$. Further, resolve each irrational x into the discrete space $\mathbf{3}$ by the function $f_x \colon \mathbb{R} \setminus \{x\} \to \mathbf{3}$, defined by $f_x(x') = 0$ for $x' < x$ and $f_x(x'') = 2$ for $x'' > x$. The Michael line is the subspace $\{(x, 1) : x \in \mathbb{R}\}$ of this resolution.

According to Lemma 1.6, we can view the Michael line M as a resolution by partial maps. Resolve the global space $\mathbb{R}$ into $\mathbf{1}$ at each rational x by the constant (total) map $f_x \colon \mathbb{R} \setminus \{x\} \to \mathbf{1}$. Further, resolve $\mathbb{R}$ at each irrational x into $\mathbf{2}$ by the empty map.

Next we list some simple properties of resolutions; their proof is straightforward and is omitted.

Lemma 1.9. (Monotonicity) *Let* U, U_1, U_2 *be neighborhoods of* $x \in X$ *and* V, V_1, V_2 *open sets in* Y_x*. We have:*

(i) *if* $U_1 \subseteq U_2$*, then* $U_1 \otimes_x V \subseteq U_2 \otimes_x V$*;*

(ii) *if* $V_1 \subseteq V_2$*, then* $U \otimes_x V_1 \subseteq U \otimes_x V_2$*;*
(iii) *if* $U_1 \subseteq U_2$ *and* $V_1 \subseteq V_2$*, then* $U_1 \otimes_x V_1 \subseteq U_2 \otimes_x V_2$*.*

Lemma 1.10. (Distributivity) *Let* $(U_i)_{i\in I}$ *be a family of neighborhoods of* $x \in X$ *and* $(V_j)_{j\in J}$ *a family of open sets in* Y_x*. For each* $h \in I$ *and* $k \in J$*, we have:*

(i) $\left(\bigcap_{i\in I} U_i\right) \otimes_x V_k \;=\; \bigcap_{i\in I}\left(U_i \otimes_x V_k\right)$;
(ii) $U_h \otimes_x \left(\bigcap_{j\in J} V_j\right) \;=\; \bigcap_{j\in J}\left(U_h \otimes_x V_j\right)$;
(iii) $\left(\bigcap_{i\in I} U_i\right) \otimes_x \left(\bigcap_{j\in J} V_j\right) \;=\; \bigcap_{i\in I}\bigcap_{j\in J}\left(U_i \otimes_x V_j\right)$;
(i') $\left(\bigcup_{i\in I} U_i\right) \otimes_x V_k \;=\; \bigcup_{i\in I}\left(U_i \otimes_x V_k\right)$;
(ii') $U_h \otimes_x \left(\bigcup_{j\in J} V_j\right) \;=\; \bigcup_{j\in J}\left(U_h \otimes_x V_j\right)$;
(iii') $\left(\bigcup_{i\in I} U_i\right) \otimes_x \left(\bigcup_{j\in J} V_j\right) \;=\; \bigcup_{i\in I}\bigcup_{j\in J}\left(U_i \otimes_x V_j\right)$.

Lemma 1.11. (Decomposability) *Let* U *be a neighborhood of* $x \in X$ *and* V *an open set in* Y_x*. We have:*

$$U \otimes_x V = (U \otimes_x Y_x) \cap (X \otimes_x V)\,.$$

Lemma 1.12. *For each* $x \in X$*, we have:*

$$X \otimes_x Y_x = \bigcup \left\{\{x'\} \times Y_{x'} : x' \in \mathrm{dom} f_x \cup \{x\}\right\}.$$

In particular, if $\mathrm{dom} f_x = X \setminus \{x\}$*, then* $X \otimes_x Y_x = \bigcup_{x\in X}\{x\} \times Y_x$*.*

Now we define the projections of the resolution on the global and the local spaces.

Definition 1.13. The *global projection* is the function $\pi\colon \bigcup_{x\in X}\{x\}\times Y_x \to X$ defined by $\pi(x,y) := x$ for each $(x,y) \in \mathrm{dom}\pi$. Further, for each $x \in X$, the *local projection at* x is the function $\pi_x\colon \bigcup_{x'\in X(x)\cup\{x\}}\{x'\} \times Y_{x'} \to Y_x$ defined as follows for each $(x',y') \in \mathrm{dom}\pi_x$:

$$\pi_x(x',y') := \begin{cases} y' & \text{if} \quad x' = x \\ f_x(x') & \text{if} \quad x' \in X(x). \end{cases}$$

Note that for each $x \in X$ such that $\mathrm{dom} f_x = X \setminus \{x\}$, the domain of π_x is $\bigcup_{x\in X}\{x\} \times Y_x$.

If the resolution functions f_x are total maps, then the global and local projections are continuous (see [4], Theorem 6). The same holds also in the case that the resolution functions are partial maps and their domain is endowed with the subspace topology of $\tau_\otimes$. The next lemma summarizes some related facts; its proof is easy and is omitted.

Lemma 1.14. *For each neighborhood* $U \subseteq X$ *of* x *and open set* $V \subseteq Y_x$*, we have:*

(i) $\pi^{-1}U \;=\; (U \otimes_x Y_x) \cup \bigcup_{x'\in U\setminus \mathrm{dom} f_x}\{x'\} \times Y_{x'}$;
(ii) *if* $U \setminus \{x\} \subseteq \mathrm{dom} f_x$*, then* $\pi^{-1}U = U \otimes_x Y_x$*;*
(iii) $\pi_x^{-1}V = X \otimes_x V$*.*

Corollary 1.15. *The global and the local projections are continuous functions.*

Let $U \subseteq X$ be a neighborhood of x and $V \subseteq Y_x$ an open set. The *global section* and the *local section* of $U \otimes_x V$ are defined, respectively, by $\mathrm{glob}_x(U,V) := \pi^{-1}U$ and $\mathrm{loc}_x(U,V) := \pi_x^{-1}V$. Each basic open set is the intersection of its global and local section.

Corollary 1.16. *For each neighborhood $U \subseteq X$ of x and open set $V \subseteq Y_x$, we have*

$$U \otimes_x V = \pi^{-1}U \cap \pi_x^{-1}V = \mathrm{glob}_x(U,V) \cap \mathrm{loc}_x(U,V).$$

Proof. Since $\bigcup\{\{x'\} \times Y_{x'} : x' \in U \setminus \mathrm{dom}f_x\} \cap (X \otimes_x V) = \{x\} \times V$, the claim follows from Lemmas 1.11 and 1.14. □

2. Order resolutions

In this section we focus our attention on particular types of resolutions, in which both the global space $(X, \preceq, \tau_\preceq)$ and the local spaces $(Y_x, \preceq_x, \tau_{\preceq_x})_{x\in X}$ are LOTS. If the spaces Y_x are compact, there is a standard way to define the resolving functions f_x (called *order maps* in this setting); the resulting topological space is called an *order resolution.* In our definition we allow the resolving functions f_x to be partially defined, in order to deal also with the cases in which some of the local spaces Y_x have no maximum and/or no minimum.

We use the following notation for intervals in X (rays are considered as particular intervals):

- $\mathcal{I}$ is the family of all intervals in X (including X); further, $\mathcal{I}(x) := \{I \in \mathcal{I} : x \in I\}$;
- $\overrightarrow{\mathcal{I}} := \{(x', \to) : x' \in X\} \cup \{X\}$ and $\overleftarrow{\mathcal{I}} := \{(\leftarrow, x'') : x'' \in X\} \cup \{X\}$;
- $\overrightarrow{\mathcal{I}(x)} := \{I \in \overrightarrow{\mathcal{I}} : x \in I\}$ and $\overleftarrow{\mathcal{I}(x)} := \{I \in \overleftarrow{\mathcal{I}} : x \in I\}$.

Similarly, intervals in Y_x are denoted as follows:

- $\mathcal{I}_x$: the family of all intervals in Y_x (including Y_x);
- $\overrightarrow{\mathcal{I}_x} := \{(y', \to) : y' \in Y_x\}$ and $\overleftarrow{\mathcal{I}_x} := \{(\leftarrow, y'') : y'' \in Y_x\}$.

Definition 2.1. For each $x \in X$, let $X(x)$ be the following subset of $X \setminus \{x\}$:

$$X(x) := \begin{cases} X \setminus \{x\} & \text{if } \exists\,\mathrm{min}Y_x \;\wedge\; \exists\,\mathrm{max}Y_x \\ (\leftarrow, x) & \text{if } \exists\,\mathrm{min}Y_x \;\wedge\; \nexists\,\mathrm{max}Y_x \\ (x, \to) & \text{if } \nexists\,\mathrm{min}Y_x \;\wedge\; \exists\,\mathrm{max}Y_x \\ \varnothing & \text{if } \nexists\,\mathrm{min}Y_x \;\wedge\; \nexists\,\mathrm{max}Y_x. \end{cases}$$

The *order map* $f_x \colon X(x) \to Y_x$ is defined as follows for each $x' \in X(x)$ (if any):

$$f_x(x') := \begin{cases} \min Y_x & \text{if } x' \prec x \\ \max Y_x & \text{if } x' \succ x. \end{cases}$$

The space $\left(\bigcup_{x\in X}\{x\} \times Y_x, \tau_\otimes\right)$ is called the *resolution of X at each $x \in X$ into Y_x by the order map f_x* and is denoted by $\bigotimes_{x\in X}^{\mathrm{Ord}} Y_x$ or simply by $\bigotimes_{x\in X} Y_x$. Also, we denote by $\mathcal{B}_\otimes$ the following base (cf. Lemma 1.4) for the resolution topology $\tau_\otimes$:

$$\mathcal{B}_\otimes := \{U \otimes_x V : \; x \in X \;\wedge\; U \in \mathcal{I}(x) \;\wedge\; V \in \mathcal{I}_x\}.$$

Example 2.2. Resolve each ordinal α in ω_1 into the half-open interval $[0,1)$ by the order map. Denote this resolution by $\bigotimes_{\alpha\in\omega_1}[0,1)$. For each successor ordinal α, a neighborhood base is given by sets of the type $\{\alpha\}\times(a,b)$ and $\{\alpha\}\times[0,b)$. Further, if α is a limit ordinal, then a neighborhood base is composed of all sets of the type $\{\alpha\}\times(a,b)$ and $\{\alpha\}\times[0,b)\cup\left(\bigcup_{\beta\in(\gamma,\alpha)}\{\beta\}\times[0,1)\right)$, where $\gamma<\alpha$. Note that the resolution space described above has the same underlying set of the lexicographic product $\omega_1\times_{\mathrm{lex}}[0,1)$, but its topology $\tau_\otimes$ is finer than the order topology $\tau_{\preceq_{\mathrm{lex}}}$. In fact, neighborhood bases at limit ordinals are the same for the two topologies, but at successor ordinals neighborhood bases for the resolution topology are strictly finer than for the order topology (cf. Example 2.10).

Next we compute $U\otimes_x V$ in the case that both U and V are rays. The notation $\bigcup_{(x,\rightarrow)}\{w\}\times Y_w$ stands for $\bigcup_{w\in(x,\rightarrow)}\{w\}\times Y_w$; a similar meaning have the other symbols.

Lemma 2.3. *Let $(x',\rightarrow)$ and $(\leftarrow,x'')$ be open rays in X containing x, and $(y',\rightarrow)$ and $(\leftarrow,y'')$ open rays in Y_x. The following equalities hold:*

$$(x',\rightarrow)\otimes_x(y',\rightarrow) = \begin{cases}\{x\}\times(y',\rightarrow) & \textit{if } \nexists \max Y_x\\ \{x\}\times(y',\rightarrow)\ \cup\ \bigcup\limits_{(x,\rightarrow)}\{w\}\times Y_w & \textit{if } \exists \max Y_x\end{cases}$$

$$(\leftarrow,x'')\otimes_x(y',\rightarrow) = \begin{cases}\{x\}\times(y',\rightarrow) & \textit{if } \nexists \max Y_x\\ \{x\}\times(y',\rightarrow)\ \cup\ \bigcup\limits_{(x,x'')}\{w\}\times Y_w & \textit{if } \exists \max Y_x\end{cases}$$

$$(x',\rightarrow)\otimes_x(\leftarrow,y'') = \begin{cases}\{x\}\times(\leftarrow,y'') & \textit{if } \nexists \min Y_x\\ \{x\}\times(\leftarrow,y'')\ \cup\ \bigcup\limits_{(x',x)}\{w\}\times Y_w & \textit{if } \exists \min Y_x\end{cases}$$

$$(\leftarrow,x'')\otimes_x(\leftarrow,y'') = \begin{cases}\{x\}\times(\leftarrow,y'') & \textit{if } \nexists \min Y_x\\ \{x\}\times(\leftarrow,y'')\ \cup\ \bigcup\limits_{(\leftarrow,x)}\{w\}\times Y_w & \textit{if } \exists \min Y_x\end{cases}$$

$$X\otimes_x(y',\rightarrow) = (x'\rightarrow)\otimes_x(y',\rightarrow)$$

$$X\otimes_x(\leftarrow,y'') = (\leftarrow,x'')\otimes_x(\leftarrow,y'')$$

$$(x', \rightarrow) \otimes_x Y_x \quad = \quad \begin{cases} \{x\} \times Y_x & \textit{if} \quad \nexists \mathrm{min} Y_x \wedge \nexists \mathrm{max} Y_x \\ \bigcup_{(x',x]} \{w\} \times Y_w & \textit{if} \quad \exists \mathrm{min} Y_x \wedge \nexists \mathrm{max} Y_x \\ \bigcup_{[x,\rightarrow)} \{w\} \times Y_w & \textit{if} \quad \nexists \mathrm{min} Y_x \wedge \exists \mathrm{max} Y_x \\ \bigcup_{(x',\rightarrow)} \{w\} \times Y_w & \textit{if} \quad \exists \mathrm{min} Y_x \wedge \exists \mathrm{max} Y_x \end{cases}$$

$$(\leftarrow, x'') \otimes_x Y_x \quad = \quad \begin{cases} \{x\} \times Y_x & \textit{if} \quad \nexists \mathrm{min} Y_x \wedge \nexists \mathrm{max} Y_x \\ \bigcup_{(\leftarrow,x]} \{w\} \times Y_w & \textit{if} \quad \exists \mathrm{min} Y_x \wedge \nexists \mathrm{max} Y_x \\ \bigcup_{[x,x'')} \{w\} \times Y_w & \textit{if} \quad \nexists \mathrm{min} Y_x \wedge \exists \mathrm{max} Y_x \\ \bigcup_{(\leftarrow,x'')} \{w\} \times Y_w & \textit{if} \quad \exists \mathrm{min} Y_x \wedge \exists \mathrm{max} Y_x \end{cases}$$

and

$$X \otimes_x Y_x \quad = \quad \begin{cases} \{x\} \times Y_x & \textit{if} \quad \nexists \mathrm{min} Y_x \wedge \nexists \mathrm{max} Y_x \\ \bigcup_{(\leftarrow,x]} \{w\} \times Y_w & \textit{if} \quad \exists \mathrm{min} Y_x \wedge \nexists \mathrm{max} Y_x \\ \bigcup_{[x,\rightarrow)} \{w\} \times Y_w & \textit{if} \quad \nexists \mathrm{min} Y_x \wedge \exists \mathrm{max} Y_x \\ \bigcup_{X} \{w\} \times Y_w & \textit{if} \quad \exists \mathrm{min} Y_x \wedge \exists \mathrm{max} Y_x . \end{cases}$$

Proof. Straightforward from definition. □

We use rays to define a subfamily $\mathcal{S}_\otimes \subseteq \mathcal{B}_\otimes$, which is a subbase for $\tau_\otimes$. Set

- $\overrightarrow{\mathcal{S}} := \{U \otimes_x V : x \in X \ \wedge \ U \in \overrightarrow{\mathcal{I}(x)} \ \wedge \ V \in \overrightarrow{\mathcal{I}_x}\}$,
- $\overleftarrow{\mathcal{S}} := \{U \otimes_x V : x \in X \ \wedge \ U \in \overleftarrow{\mathcal{I}(x)} \ \wedge \ V \in \overleftarrow{\mathcal{I}_x}\}$,
- $\mathcal{S}_\otimes := \overrightarrow{\mathcal{S}} \cup \overleftarrow{\mathcal{S}}$.

Lemma 2.4. *For each $U_1 \in \overrightarrow{\mathcal{I}(x)}$, $U_2 \in \overleftarrow{\mathcal{I}(x)}$, $V_1 \in \overrightarrow{\mathcal{I}_x} \setminus \{Y_x\}$ and $V_2 \in \overleftarrow{\mathcal{I}_x} \setminus \{Y_x\}$, we have:*

(i) $U_2 \otimes_x V_1 \ = \ (X \otimes_x V_1) \cap (U_2 \otimes_x Y_x) \ \subseteq \ X \otimes_x V_1 \ = \ U_1 \otimes_x V_1$;
(ii) $U_1 \otimes_x V_2 \ = \ (U_1 \otimes_x Y_x) \cap (X \otimes_x V_2) \ \subseteq \ X \otimes_x V_2 \ = \ U_2 \otimes_x V_2$;

(iii) $(U_1 \cap U_2) \otimes_x (V_1 \cap V_2) = (U_1 \otimes_x V_2) \cap (U_2 \otimes_x V_1) = (U_1 \otimes_x V_1) \cap (U_2 \otimes_x V_2) = \{x\} \times (V_1 \cap V_2)$;

(iv) $(U_1 \cap U_2) \otimes_x Y_x = (U_1 \otimes_x Y_x) \cap (U_2 \otimes_x Y_x)$.

Proof. Parts (i) and (ii), as well as the second and third equality in (iii) follow from Lemma 2.3. The first equality in (iii) follows from part (i), part (ii) and Lemma 1.10 (iii). Finally, part (iv) is an immediate consequence of Lemma 1.10 (i). □

Corollary 2.5. *For each $x \in X$ and $U \otimes_x V \in \mathcal{B}_\otimes$, there exist $U_1 \otimes_x V_1 \in \overrightarrow{\mathcal{S}}$ and $U_2 \otimes_x V_2 \in \overleftarrow{\mathcal{S}}$ such that*

$$U \otimes_x V = (U_1 \otimes_x V_1) \cap (U_2 \otimes_x V_2).$$

In particular, $\mathcal{S}_\otimes$ is a subbase for $\tau_\otimes$.

Proof. Let $x \in X$. The open interval $U \subseteq X$ containing x can have the following forms: (i) $U = X$; (ii) $U = (x', \to)$, with $x' \prec x$; (iii) $U = (\leftarrow, x'')$, with $x \prec x''$; (iv) $U = (x', x'')$, with $x' \prec x \prec x''$. Similarly, we have four possibilities for the open interval $V \subseteq Y_x$, namely: (i) $V = Y_x$; (ii) $V = (y', \to)$; (iii) $V = (\leftarrow, y'')$; (iv) $V = (y', y'')$. In all sixteen cases, the claim follows from Lemma 2.4. □

Since both the global space X and the local spaces Y_x are LOTS, we can obtain another topological space as follows. Consider the reverse of all chains and endow them with the relative order topology. By applying the resolution operator to the LOTS X^* and $(Y_x^*)_{x\in X}$, we obtain a new topological space, the *reverse order resolution* $\bigotimes^{\mathrm{Ord}}_{x\in X^*} Y_x^*$. To simplify notation, we denote this space by $\bigotimes^*_{x\in X} Y_x$. We shall show that $\bigotimes^*_{x\in X} Y_x = \bigotimes_{x\in X} Y_x$ (see Corollary 2.8).

Note that if $I = (y, z)$ is an open interval in Y_x, then $I^* \subseteq Y_x^*$ is the open interval $(y, z)^* = (z, y)$. Fix $x \in X$. Assume that Y_x has a minimum element y_x but no maximum element. The order map f_x and the reverse order map f_x^* are defined, respectively, as follows:

$$\begin{array}{lll} f_x \colon (\leftarrow, x) \to Y_x & , & w \longmapsto y_x \\ f_x^* \colon (x, \to) \to Y_x^* & , & w \longmapsto y_x. \end{array}$$

Thus, if I is the open interval $[y_x, y'') \subseteq Y_x$, then

$$(f_x)^{-1} I = (\leftarrow, x) \subseteq X \quad \text{and} \quad (f_x^*)^{-1} I^* = (x, \to) \subseteq X^*$$

whence $(f_x^*)^{-1} I^* = \left({f_x}^{-1} I\right)^*$. Similar considerations can be done in the case that Y_x has a maximum but not a minimum, or has both a minimum and a maximum. The following lemma summarizes the above results.

Lemma 2.6. *Let $x \in X$ and assume that Y_x has a minimum (respectively, maximum) element y_x. If $I \subseteq Y_x$ is the open interval $[y_x, y)$ (respectively, $(y, y_x]$), then $(f_x^*)^{-1} I^* = \left({f_x}^{-1} I\right)^*$.*

Order resolution and reverse order resolution have the same basic open sets.

Theorem 2.7. *Each basic open set $U \otimes_x V$ in $\bigotimes_{x\in X} Y_x$ is equal to the basic open set $U^* \otimes_x^* V^*$ in $\bigotimes_{x\in X}^* Y_x$.*

Proof. Several cases have to be considered: (i) $x = \min X$; (ii) $x = \max X$; (iii) $x \neq \min X$ and $x \neq \max X$. We examine only case (iii), since the others are similar.

Let $U \otimes_x V$ be a basic open set in $\bigotimes_{x\in X} Y_x$; without loss of generality, let $U = (x', x'')$, where $x' \prec x \prec x''$. For $V \subseteq Y_x$, we can assume that one of the following cases occurs: (a) $V = [y, y'')$, with $y = \min Y_x$; (b) $V = (y', y]$, with $y = \max Y_x$; (c) $V = (y', y'')$.

In case (a), Lemma 2.6 yields

$$U \otimes_x V \;=\; (x', x'') \otimes_x [y, y'') \;=\; \{x\} \times [y, y'') \;\cup\; \bigcup \{\{w\} \times Y_w : w \in (x', x)\}$$

and

$$U^* \otimes_x^* V^* \;=\; (x', x'')^* \otimes_x^* [y, y'')^* \;=\; \{x\} \times (y'', y] \cup \bigcup \{\{w\} \times Y_w^* : w \in (x, x')\}\,.$$

Therefore, $U \otimes_x V = U^* \otimes_x^* V^*$, as claimed. Case (b) is similar to (a). For case (c), we have:

$$U \otimes_x V \;=\; (x', x'') \otimes_x (y', y'') \;=\; \{x\} \times (y', y'')$$

and

$$U^* \otimes_x^* V^* \;=\; (x', x'')^* \otimes_x^* (y', y'')^* \;=\; \{x\} \times (y'', y')$$

whence $U \otimes_x V = U^* \otimes_x^* V^*$ also in this case. □

Corollary 2.8. $\bigotimes_{x\in X} Y_x = \bigotimes_{x\in X}^* Y_x$.

Proof. Since the underlying set is the same for both spaces, it suffices to show that their topologies coincide. Theorem 2.7 yields $U \otimes_x V = U^* \otimes_x^* V^*$ for any basic open set $U \otimes_x V$ in $\bigotimes_{x\in X} Y_x$. By duality, we obtain $U^* \otimes_x^* V^* = U^{**} \otimes_x^{**} V^{**} = U \otimes_x V$ for any basic open set $U^* \otimes_x^* V^*$ in $\bigotimes_{x\in X}^* Y_x$. The result follows. □

In the last part of this section we study the relationship between two natural topologies defined on the set $\bigcup_{x\in X}\{x\} \times Y_x$. Namely, we compare the order resolution $\bigotimes_{x\in X} Y_x$ and the chain $\sum_{x\in X} Y_x$ endowed with the order topology τ_Σ.

Lemma 2.9. *The resolution topology $\tau_\otimes$ on $\bigotimes_{x\in X} Y_x$ is finer than the order topology τ_Σ on $\sum_{x\in X} Y_x$.*

Proof. Let I be an open ray in $\sum_{x\in X} Y_x$ and (x, y) a point in I. To prove the claim we exhibit a neighborhood $W \in \tau_\otimes(x, y)$ such that $W \subseteq I$. By duality (see Theorem 2.7), it suffices to examine the case $I = (\leftarrow, (x'', y''))$, where $x \preceq x''$.

First assume that $x \prec x''$. Select $U \in \tau(x)$ such that $x'' \notin U$ and set $W := \pi^{-1}U$. Continuity of the global projection π (see Corollary 1.15) implies that W is a neighborhood of (x, y) in $\bigotimes_{x\in X} Y_x$, which does not contain the point (x'', y''). Thus $(x, y) \in W \subseteq I$, as claimed.

Next, let $x = x''$ and $y \prec_x y''$. Set $U := X$ and $V := (\leftarrow, y'')$. Note that ${f_x}^{-1}V$ is equal either to the empty set (if $x = \min X$ or Y_x has no minimum element) or to the open ray $(\leftarrow, x)$ (if $x \neq \min X$ and Y_x has a minimum element). Thus we obtain:

$$(x, y) \in U \otimes_x V = \{x\} \times (\leftarrow, y'') \cup \bigcup \left\{ \{x'\} \times Y_{x'} : x' \in \left({f_x}^{-1}(\leftarrow, y'') \cap X\right) \right\} \subseteq I.$$

The open set $W := U \otimes_x V$ satisfies the claim. □

The converse of Lemma 2.9 does not hold in general. In particular, the order resolution of a LOTS into LOTS is a GO-space that is not necessarily a LOTS.

Example 2.10. Let X be the discrete LOTS $\mathbf{2} = \{0, 1\}$ and $Y_0 = Y_1$ the half-open interval $[a, b) \subseteq \mathbb{R}$. We show that the resolution topology $\tau_\otimes$ on $\bigotimes_{x \in X} Y_x = \bigotimes_{i \in 2} [a, b)$ is strictly finer than the order topology τ_Σ on the chain $\sum_{x \in X} Y_x = \mathbf{2} \times_{\text{lex}} [a, b)$.

Consider the point $(1, a) \in \mathbf{2} \times [a, b)$, the open neighborhood $\{1\}$ of 1 and the open set $[a, b) = Y_1$. Observe that f_0 is undefined and $f_1 : X(1) \to [a, b)$ is defined by $f_1(0) = a$. Thus, the basic open set $\{1\} \otimes_1 [a, b)$ is the half-open interval $[(1, a), (1, b))$. On the other hand, any interval $I \in \tau_\Sigma$ satisfying $(1, a) \in I \subseteq \{1\} \otimes_1 [a, b)$ must contain a subinterval of the type $I' = ((0, y_1), (1, y_2))$, where $y_1, y_2 \in [a, b)$. Thus, $\{1\} \otimes_1 [a, b)$ is not open in $\mathbf{2} \times_{\text{lex}} [a, b)$.

The global projection $\pi \colon \bigcup_{x \in X} \{x\} \times Y_x \to X$ is continuous whenever its domain is endowed with the resolution topology. On the other hand, continuity of π is not ensured if its domain is endowed with the order topology. For example, $\pi \colon (\mathbf{2} \times_{\text{lex}} [a, b), \tau_\Sigma) \to \mathbf{2}$ fails to be continuous, because $(\mathbf{2} \times_{\text{lex}} [a, b), \tau_\Sigma)$ is homeomorphic to the connected interval $[a, b)$.

The local projections $\pi_x \colon \bigcup_{x' \in X(x) \cup \{x\}} \{x'\} \times Y_{x'} \to Y_x$, with $x \in X$, are continuous if their domain is endowed with the subspace topology of the resolution (see Corollary 1.15). Next we show that they are continuous also if we endow their domain with the subspace topology of the LOTS $\left(\sum_{x \in X} Y_x, \tau_\Sigma\right)$. Before proving this fact, we mention a technical lemma.

Lemma 2.11. *Let $V \subseteq Y_x$ be an open set. The set $\{x\} \times V$ is open in $\left(\sum_{x \in X} Y_x, \tau_\Sigma\right)$ if both of the following conditions are verified:*

(a) *if V contains $\max Y_x$, then either $x = \max X$ or x has an immediate successor x'' and $Y_{x''}$ has a minimum;*

(b) *if V contains $\min Y_x$, then either $x = \min X$ or x has an immediate predecessor x' and $Y_{x'}$ has a maximum.*

In particular, $\{x\} \times V$ is open in $\left(\sum_{x \in X} Y_x, \tau_\Sigma\right)$ whenever V contains neither $\max Y_x$ nor $\min Y_x$.

Proof. Straightforward from definition. □

Lemma 2.12. *For each $x \in X$, the function π_x is continuous with respect to the subspace topology of τ_Σ.*

Proof. Let $V \subseteq Y_x$ be an open set. Then $\pi_x^{-1}V = X \otimes_x V$, using Lemma 1.14. To prove the result, it suffices to show that $X \otimes_x V$ is open (in the subspace topology of τ_Σ) for $V = (\leftarrow, y)$. If Y_x has no minimum element, then $X \otimes_x (\leftarrow, y) = \{x\} \times (\leftarrow, y)$ is open by Lemma 2.11. On the other hand, if Y_x has a minimum element $y_{\min}$, then $\pi_x^{-1}(\leftarrow, y) = X \otimes_x [y_{\min}, y) = (\leftarrow, (x, y))$. □

Next we introduce the notion of pseudojump.

Definition 2.13. A *pseudojump* in the chain $\sum_{x\in X} Y_x$ is a jump (x', x'') in X (*i.e.*, a pair of consecutive points of X) with the property that either (a) $\exists\, \max Y_{x'}$ and $\nexists\, \min Y_{x''}$, or (b) $\nexists\, \max Y_{x'}$ and $\exists\, \min Y_{x''}$.

The notion of pseudojump of a chain $L = \sum_{x\in X} Y_x$ obviously depends on the chosen representation of L as a sum of other chains. Consider, e.g., the isomorphic chains $(0, 1)$ (which lacks pseudojumps), $(0, 1/2) \oplus [1/2, 1)$ (which has exactly one pseudojumps) and $\sum_{n\in\omega} \left(\frac{1}{n+3}, \frac{1}{n+2}\right] \oplus (1/2, 1)$ (which has countably many pseudojumps). On the other hand, if we endow the chains with additional structure, then the notion becomes significant. The next theorem characterizes the order resolution as a LOTS.

Theorem 2.14. *Let $(X, \preceq, \tau_\preceq)$ and $\{(Y_x, \preceq_x, \tau_{\preceq_x})\}_{x\in X}$ be LOTS. The following statements are equivalent:*

(i) *the order topology τ_Σ on $\sum_{x\in X} Y_x$ is equal to the resolution topology $\tau_\otimes$ on $\bigotimes_{x\in X} Y_x$, i.e., $\bigotimes_{x\in X} Y_x$ is a LOTS;*
(ii) *the global projection $\pi\colon \left(\sum_{x\in X} Y_x, \tau_\Sigma\right) \to (X, \tau_\preceq)$ is continuous;*
(iii) *the chain $\sum_{x\in X} Y_x$ has no pseudojumps.*

Proof. (i) ⇒ (ii). This implication follows from Corollary 1.15.

(ii) ⇒ (i). By Lemma 2.9, it suffices to show that $\tau_\otimes \subseteq \tau_\Sigma$. Let $U \otimes_x V$ be a basic open set for $\tau_\otimes$. Since $U \otimes_x V = \left(\pi^{-1}U\right) \cap \left(\pi_x^{-1}V\right)$ by Corollary 1.16, the claim follows from hypothesis and Lemma 2.12.

(ii) ⇒ (iii). We prove the contrapositive. Without loss of generality, assume that there exists a jump (x', x'') in X such that $Y_{x'}$ has no maximum and $Y_{x''}$ has a minimum $y''_{\min}$. Consider the nonempty open ray $(x', \rightarrow) \subseteq X$. Since the set $\pi^{-1}(x', \rightarrow) = [(x'', y''_{\min}), \rightarrow)$ is not open in $\left(\sum_{x\in X} Y_x, \tau_\Sigma\right)$ (cf. Example 2.10), it follows that π is not continuous with respect to τ_Σ.

(iii) ⇒ (ii). Assume that $\sum_{x\in X} Y_x$ has no pseudojumps. It suffices to prove that $\pi^{-1}(x', \rightarrow)$ is open in τ_Σ for each $x' \in X$. Let $(x, y) \in \pi^{-1}(x', \rightarrow)$. We find an interval $I \subseteq \sum_{x\in X} Y_x$ such that $(x, y) \in I \subseteq \pi^{-1}(x', \rightarrow)$. If $y \neq \min Y_x$, then there exists $t \in Y_x$ such that $t \prec_x y$. Set $I := ((x, t), \rightarrow)$.

Next assume that $y = \min Y_x$. If (x', x) is a jump in X, then by hypothesis there exists $y'_{\max} := \max Y_{x'}$. Thus $I := ((x', y'_{\max}), \rightarrow)$ satisfies the claim. On the other hand, if (x', x) is not a jump in X, then we can select $w \in (x', x)$ and $t \in Y_w$, and set $I := ((w, t), \rightarrow)$. □

3. Unifications and resolutions

By *TO-space* we mean a triple $(X, \preceq, \tau)$ such that X is a nonempty set, $\preceq$ is a linear order on X and τ is a Hausdorff topology on X (not necessarily related to the order $\preceq$). We describe how a TO-space can be canonically embedded into a LOTS that is an order resolution.

Definition 3.1. Let $(X, \preceq, \tau)$ and $(Y, \leq, \sigma)$ be TO-spaces. A function $f\colon X \to Y$ is a *TO-homomorphism* if $f\colon (X, \tau) \to (Y, \sigma)$ is open in the range (i.e., open sets of X are mapped into open sets of $f(X)$) and $f\colon (X, \preceq) \to (Y, \leq)$ is order-preserving. In particular, a *TO-embedding* (respectively, a *TO-isomorphism*) is an injective (respectively, bijective) TO-homomorphism.

Next we list some simple but useful properties of order-preserving maps between chains. Their proof is easy and is omitted.

Lemma 3.2. *Let $f\colon X \to Y$ be an order-preserving map between chains. We have:*

- (i) *the f-preimage of a convex subset of Y is a convex subset of X;*
- (ii) *if f is surjective, then the f-preimage of an open [closed, half-open] interval is an open [closed, half-open] interval;*
- (iii) *if X is a GO-space, Y is a LOTS and f is surjective, then f is continuous;*
- (iv) *if X is a LOTS, Y is a GO-space and f is injective, then f is a TO-embedding;*
- (v) *if X and Y are LOTS and f is bijective, then f is a homeomorphism.*

As the next example shows, the hypothesis that Y is a LOTS is necessary in (iii) and (v). In particular, a TO-isomorphism may fail to be a homeomorphism.

Example 3.3. Let X be the unit interval $[0, 1]$ (with the order topology) and Y the topological sum $[0, 1) \oplus \{1\}$ (with the natural order). The identity map is a TO-isomorphism but is not continuous. Note that X is a LOTS, whereas Y is a GO-space but not a LOTS.

Let X be a TO-space. Observe that if $f\colon X \to Y$ is a TO-embedding of X into a LOTS, then its (partial) inverse $f^{-1}\colon f(X) \to X$ is a continuous and order-preserving map of a GO-space onto X. Vice versa, assume that $g\colon Z \to X$ is a continuous and order-preserving map of a GO-space onto X. Let Y be a LOTS such that $Z \subseteq Y$. Choose $z_x \in Z$ is such that $g(z_x) = x$. The function $h\colon X \to Y$, defined by $h(x) := z_x$ for each $x \in X$, is a TO-embedding of X into a LOTS.

Definition 3.4. Let $(X, \preceq, \tau)$ be a TO-space and $\imath\colon X \hookrightarrow Y$ is a TO-embedding of X into a LOTS $(Y, \leq, \tau_{\leq})$. The pair $((Y, \leq, \tau_{\leq}), \imath)$ is called a *unification of* (the topology and the order of) X; if there is no risk of confusion, we simplify the notation and write $(Y, \imath)$.

Without loss of generality, we assume that a unification of a TO-space X is a pair $(Y, \imath)$ such that X is a subchain of Y and $\imath$ is the canonical inclusion. (Note that X and $\imath(X)$ have different topologies, in general.)

A unification $(Y, \imath)$ of X is *continuous* if the TO-embedding $\imath$ is a continuous map. Further, $(Y, \imath)$ is a *minimum* unification if for any other unification (Z, φ) of X, there exists a TO-embedding $\psi\colon Y \hookrightarrow Z$ such that $\varphi = \psi \circ \imath$.

Example 3.5. Let $X = \mathbf{2} \times_{\mathrm{lex}} [0,1) = [0,1) \oplus [0,1)$ be a chain endowed with the resolution topology. The following LOTS (together with the canonical inclusions $\imath_Y$, $\imath_W$ and $\imath_Z$, respectively) are examples of unifications of X: $Y = [0,1] \oplus [0,1)$, $W = [0,2] \oplus [0,1)$ and $Z = [0,1) \oplus (\frac{1}{n+1})_{n\in\omega} \oplus [0,1)$. Observe that $(Y, \imath_Y)$ is a continuous minimum unification of X, whereas $(W, \imath_W)$ and $(Z, \imath_Z)$ are continuous unifications of X, which fail to be minimum. Note also that despite Y and Z are homeomorphic LOTS, the unifications $(Y, \imath_Y)$ and $(Z, \imath_Z)$ are different.

Example 3.6. Define on the set $X = \omega \cup \{\omega+1\}$, endowed with the natural order $\leq$, a topology σ as follows. Let $\mathcal{U}$ be an ultrafilter on ω containing the cofinite filter. We define σ as the topology on X such that all natural numbers are isolated points and a system of σ-neighborhoods for the point $\omega+1$ is given by $\{U \cup \{\omega+1\} : U \in \mathcal{U}\}$. Then $(X, \leq, \sigma)$ is a TO-space that fails to be a GO-space, because its character $\chi(X,\sigma)$ is uncountable but its pseudo-character $\psi(X,\sigma)$ is countable (see [1], Problem 3.12.4). A (minimum) unification for X is given by $(Y, \imath)$, where Y is the LOTS $(\omega+2, \leq, \tau_{\leq})$ and $\imath$ is the canonical embedding.

Minimum unifications of a TO-space are essentially unique. To prove uniqueness, we first define the so-called *canonical* unification of a TO-space and then show that this is (up to a relabeling) its unique minimum unification.

The canonical unification of a TO-space $(X, \preceq, \tau)$ is defined in two steps: (i) we obtain a *minimum* (in the sense of Definition 3.7) refinement τ^* of the topology τ such that $(X, \preceq, \tau^*)$ is a GO-space; (ii) we embed the GO-space $(X, \preceq, \tau^*)$ into a *minimum* (in the sense of Definition 3.9) LOTS that is an order resolution $\bigotimes_{x\in X} Y_x$.

Definition 3.7. Let $(X, \preceq, \tau)$ be a TO-space. The *GO-cone above* X is the (nonempty) family of all GO-spaces $(X, \preceq, \sigma)$ such that σ refines τ. The *GO-extension of* X, denoted by $(X, \preceq, \tau^*)$, is the minimum of the GO-cone above X, in the sense that if $(X, \preceq, \sigma)$ is another GO-space such that σ refines τ, then σ refines also τ^*.

The GO-extension of X is well-defined.

Lemma 3.8. *For each TO-space $(X, \preceq, \tau)$, the GO-cone above X has a minimum $(X, \preceq, \tau^*)$.*

Proof. We define τ^*. Let $x \in X$. For each τ-neighborhood U of x, denote by $C(x,U)$ the union of all $\preceq$-convex subsets of U containing x. Let $\mathcal{B}_x := \{C(x,U) : U \in \tau(x)\}$. Then $\mathcal{B}^* := \bigcup_{x\in X} \mathcal{B}_x$ is a base for a topology τ^* on X. The topology τ^* refines both the original topology τ and the order topology $\tau_{\preceq}$ (because (X,τ) is Hausdorff). Further, $(X, \preceq, \tau^*)$ is a GO-space.

Next we prove that $(X, \preceq, \tau^*)$ is the minimum of the GO-cone above X. Let $(X, \preceq, \sigma)$ be a GO-space such that σ refines τ. Let $\mathcal{B}$ be a base for σ composed of convex sets and $C(x, U)$ a basic open set in τ^*. For each $x' \in C(x, U)$, there exists $B \in \mathcal{B}$ such that $x' \in B \subseteq U$. Thus $x' \in B \subseteq C(x, U)$, hence $C(x, U)$ is open in σ. This shows that σ refines τ^*. □

Definition 3.9. Let $(X, \preceq, \tau)$ be a GO-space. A *completion* of X is a pair $(Y, \imath)$, where Y is a LOTS and $\imath\colon X \hookrightarrow Y$ is an order-preserving homeomorphic embedding. A completion $(Y, \imath)$ is *minimum* if for any other completion (Z, φ), there exists an order-preserving homeomorphic embedding $\Psi(\imath, \varphi)\colon Y \hookrightarrow Z$ such that $\varphi = \Psi(\imath, \varphi) \circ \imath$.

Note that if X is a GO-space, then both notions of unification of X (as a TO-space) and completion of X make sense. We show that the unique minimum unification of X is indeed its unique minimum completion.

Theorem 3.10. *Let $(X, \preceq, \sigma)$ be a GO-space.*

(i) *There exists a minimum completion $(\bigotimes_{x\in X} Y_x, \imath)$ of X such that the space $\bigotimes_{x\in X} Y_x$ is the order resolution of X at each point x into a chain Y_x with either one, two or three points.*
(ii) *The image $\imath(X)$ is topologically dense in $\bigotimes_{x\in X} Y_x$.*
(iii) *If (W, η) is another minimum completion of X, then the maps $\Psi(\imath, \eta)$ and $\Psi(\eta, \imath)$ are order-preserving homeomorphisms between $\bigotimes_{x\in X} Y_x$ and W such that $\Psi(\imath, \eta) = \Psi(\eta, \imath)^{-1}$.*

Proof. For each $x \in X$, we define a chain Y_x with either one, two or three points as follows. If x has a σ-neighborhood base consisting of open intervals, then set $Y_x := \{y_x\}$. On the other hand, if x has no neighborhood base consisting of open intervals, then we define Y_x according to cases (I)-(VII) described below.

Let x be a non-endpoint of X. If (I) x has an immediate predecessor and no immediate successor, set $Y_x := \{y_x, y_x^+\}$. If (II) x has an immediate successor and no immediate predecessor, set $Y_x := \{y_x^-, y_x\}$. Further, if x has neither immediate predecessor nor immediate successor, it follows that a σ-neighborhood base at x has one of the following forms: (III) the family of all half-open intervals $(a, x]$ containing a point different from x; (IV) the family of all half-open intervals $[x, b)$ containing a point different from x; (V) the singleton $\{x\}$. Let Y_x be the chain $\{y_x, y_x^+\}$ in case (III), $\{y_x^-, y_x\}$ in case (IV), and $\{y_x^-, y_x, y_x^+\}$ in case (V). Finally, if (VI) x is the minimum point of X and has no immediate successor, or (VII) x is the maximum point of X and has no immediate predecessor, set $Y_x := \{y_x, y_x^+\}$ in case (VI), and $Y_x := \{y_x^-, y_x\}$ in case (VII).

Let $\bigotimes_{x\in X} Y_x$ be the order resolution of the LOTS $(X, \preceq, \tau_\preceq)$ at each point x into the LOTS Y_x defined as above. By Theorem 2.14, the space $\bigotimes_{x\in X} Y_x$ is a LOTS. Further, the correspondence $x \mapsto (x, y_x)$ gives a order-preserving homeomorphic embedding $\imath\colon (X, \preceq, \sigma) \to \bigotimes_{x\in X} Y_x$. Thus $(\bigotimes_{x\in X} Y_x, \imath)$ is a completion of X. Next we prove that it is minimum.

Assume that $\varphi\colon X \hookrightarrow Z$ is an order-preserving homeomorphic embedding of X into a LOTS. We define an order-preserving homeomorphic embedding

$\Psi = \Psi(\imath, \varphi)\colon \bigotimes_{x \in X} Y_x \hookrightarrow Z$ such that $\varphi = \Psi \circ \imath$. For each $x \in X$, denote $z_x := \varphi(x)$ and let $\Psi(x, y_x) := z_x$. To define Ψ for the points of $\bigotimes_{x \in X} Y_x$ that are not in the range of $\imath$, we carry a case by case analysis.

Since φ is a TO-embedding into a LOTS, in cases (I) and (VI) the point z_x is isolated in the range of φ, whereas in case (III) the interval $(\leftarrow, z_x]$ is open in the range of φ. It follows that there exists $z_x^+ \in Z$ that is the immediate successor of z_x. Set $\Psi(x, y_x^+) := z_x^+$. Further, cases (II), (IV) and (VII) are dual to, respectively, (I), (III) and (VI). Thus, we set $\Psi(x, y_x^-) := z_x^-$, where z_x^- is the immediate predecessor of z_x in Z. Finally, in case (V), a combination of the arguments given above yields that there exist $z_x^-, z_x^+ \in Z$, which are, respectively, the immediate predecessor and successor of z_x. Set $\Psi(x, y_x^-) := z_x^-$ and $\Psi(x, y_x^+) := z_x^+$. By construction, Ψ is an injective order-preserving map such that $\varphi = \Psi \circ \imath$. Thus Ψ is a TO-embedding by Lemma 3.2.

Next we prove continuity of Ψ using continuity of φ. It suffices to show that for any open ray $(\leftarrow, z) \subseteq Z$, the preimage $\Psi^{-1}(\leftarrow, z)$ is open in $\bigotimes_{x \in X} Y_x$. If z belongs to the image of Ψ, then $\Psi^{-1}(\leftarrow, z) = (\leftarrow, \Psi^{-1}(z))$ is open in $\bigotimes_{x \in X} Y_x$.

Now let $(\leftarrow, z)$ be such that z is not in the image of Ψ. Without loss of generality, assume that $\varphi^{-1}(\leftarrow, z) \neq \varnothing$. Let $(x, y) \in \Psi^{-1}(\leftarrow, z)$. We claim that either (x, y) has an immediate successor (x'', y''), or there exists $(x'', y'') \in \bigotimes_{x \in X} Y_x$ such that $(x, y) \prec (x'', y'')$ and $\Psi(x'', y'') \preceq z$. Continuity of Ψ follows from the claim, since $(x, y) \in (\leftarrow, (x'', y'')) \subseteq \Psi^{-1}(\leftarrow, z)$.

To prove the claim, assume by contradiction that (x, y) has no immediate successor and for each $(x'', y'') \in \bigotimes_{x \in X} Y_x$, if $(x, y) \prec (x'', y'')$ then $z \prec \Psi(x'', y'')$. Then x has no immediate successor and $\imath(x) = (x, y_x)$ is such that y_x is the maximum of Y_x. It follows that $\varphi^{-1}(\leftarrow, z) = (\leftarrow, x]$ is not open in X, which contradicts the continuity of φ. This finishes the proof of (i).

To prove (ii), observe that any two points in $\bigotimes_{x \in X} Y_x \setminus \imath(X)$ cannot be consecutive. Furthermore, it is easy to show that all points in $\bigotimes_{x \in X} Y_x \setminus \imath(X)$ are not isolated in $\tau_\otimes$. It follows that $\overline{\imath(X)} = \bigotimes_{x \in X} Y_x$.

Finally, assume that (W, η) is another minimum completion of X. We show that the compositions $\Psi(\eta, \imath) \circ \Psi(\imath, \eta)$ and $\Psi(\imath, \eta) \circ \Psi(\eta, \imath)$ are the identity maps on $\bigotimes_{x \in X} Y_x$ and W, respectively. This will prove (iii).

By hypothesis, there exist order-preserving homeomorphic embedding $\Psi(\imath, \eta)$ and $\Psi(\eta, \imath)$ such that $\eta = \Psi(\imath, \eta) \circ \imath$ and $\imath = \Psi(\eta, \imath) \circ \eta$. Thus the composition $\Psi(\eta, \imath) \circ \Psi(\imath, \eta)\colon \bigotimes_{x \in X} Y_x \hookrightarrow \bigotimes_{x \in X} Y_x$ is an order-preserving homeomorphic embedding, whose restriction to $\imath(X)$ is the canonical inclusion of $\imath(X)$ into $\bigotimes_{x \in X} Y_x$. By (ii), it follows that $\Psi(\eta, \imath) \circ \Psi(\imath, \eta)$ is the identity on $\bigotimes_{x \in X} Y_x$.

To prove that $\Psi(\imath, \eta) \circ \Psi(\eta, \imath)$ is the identity on W, we first show that $\eta(X)$ is topologically dense in W. By way of contradiction, assume that there exists $w \in W$ and an open interval $(a, b) \subseteq W$ such that $w \in (a, b)$ and $(a, b) \cap \eta(X) = \varnothing$. It follows that $(\Psi(a), \Psi(b))$ is an open interval containing $\Psi(w)$, which does not intersect $\imath(X)$. This contradicts property (ii). Now the equality $\Psi(\imath, \eta) \circ \Psi(\eta, \imath) = \mathrm{id}_W$ follows by an argument similar to that given above. $\square$

Note that the topology of $\bigotimes_{x\in X} Y_x$ is the same as its order topology (see Theorem 2.14). An immediate consequence of Theorem 3.10 is

Corollary 3.11. *For each GO-space $(X, \preceq, \sigma)$, the pair $(\bigotimes_{x\in X} Y_x, \imath)$ is the unique (up to a relabeling) minimum completion.*

Remark 3.12. For any GO-space $(X, \preceq, \sigma)$, a completion $(Y, \imath)$ of X is also a unification of X. The proof of Theorem 3.10 yields that the unique (up to a relabeling) minimum completion $(\bigotimes_{x\in X} Y_x, \imath)$ of X has the property that for each TO-embedding $\varphi\colon X \hookrightarrow Z$ of X into a LOTS, there exists a TO-embedding $\Psi(\imath, \varphi)\colon \bigotimes_{x\in X} Y_x \hookrightarrow Z$ such that $\varphi = \Psi(\imath, \varphi) \circ \imath$. Thus *the* minimum completion of X is also *a* minimum unification of X (indeed, *the* minimum unification of X, cf. Corollary 3.15).

Definition 3.13. Let $(X, \preceq, \tau)$ be a TO-space. Using Lemma 3.8 and Theorem 3.10, we define the *canonical unification* $(\widehat{\bigotimes}_{x\in X} Y_x, \hat{\imath})$ of X as follows. The LOTS $\widehat{\bigotimes}_{x\in X} Y_x$ is an order resolution obtained as in the proof of Theorem 3.10: the global space is the GO-extension $(X, \preceq, \tau^*)$ of $(X, \preceq, \tau)$, whereas the local spaces Y_x are the discrete LOTS with either one, two or three points. The TO-embedding $\hat{\imath}$ is the composition of the identity $\mathrm{id}_\tau^{\tau^*}\colon (X, \preceq, \tau) \to (X, \preceq, \tau^*)$ and the homeomorphic embedding $\imath\colon (X, \preceq, \tau^*) \to \widehat{\bigotimes}_{x\in X} Y_x$ defined in the proof of Theorem 3.10.

Theorem 3.14. *Let $(X, \preceq, \tau)$ be a TO-space. The canonical unification of X is a minimum unification. Further, for any other minimum unification $(W, \hat{\eta})$ of X, there exist order-preserving homeomorphisms $\psi\colon \widehat{\bigotimes}_{x\in X} Y_x \to W$ and $\chi\colon W \to \widehat{\bigotimes}_{x\in X} Y_x$ such that $\psi \circ \hat{\imath} = \chi$, $\chi \circ \hat{\imath} = \psi$ and $\psi^{-1} = \chi$.*

Proof. First we show that $(\widehat{\bigotimes}_{x\in X} Y_x, \hat{\imath})$ is a minimum unification. Let $(Z, \hat{\varphi})$ be a unification of X. We define a TO-embedding $\psi\colon \widehat{\bigotimes}_{x\in X} Y_x \to Z$ such that $\hat{\varphi} = \psi \circ \hat{\imath}$. Note that $\hat{\varphi}(X) \subseteq Z$ is a GO-space. Let σ be the topology on X such that $\hat{\varphi}$ gives a homeomorphism between (X, σ) and $\hat{\varphi}(X)$. Thus (X, σ) is an element of the GO-cone above (X, τ). Further, we have $\hat{\varphi} = \varphi \circ \mathrm{id}_\tau^\sigma$, where $\mathrm{id}_\tau^\sigma\colon (X, \tau) \to (X, \sigma)$ is the identity map and $\varphi\colon (X, \sigma) \to Z$ is the homeomorphic embedding defined as $\hat{\varphi}$. By definition of GO-extension of $(X, \preceq, \tau)$, the identities $\mathrm{id}_\tau^{\tau^*}\colon (X, \tau) \to (X, \tau^*)$, $\mathrm{id}_{\tau^*}^\sigma\colon (X, \tau^*) \to (X, \sigma)$ and $\mathrm{id}_\tau^\sigma\colon (X, \tau) \to (X, \sigma)$ are order-preserving open maps.

Set $\varphi' := \varphi \circ \mathrm{id}_{\tau^*}^\sigma$. Then $\varphi'\colon (X, \preceq, \tau^*) \hookrightarrow Z$ is a TO-embedding. By Theorem 3.10, there exists a TO-embedding $\Psi(\imath, \varphi')\colon \widehat{\bigotimes}_{x\in X} Y_x \to Z$ such that $\varphi' = \Psi(\imath, \varphi') \circ \imath$. It follows that $\hat{\varphi} = \varphi' \circ \mathrm{id}_\tau^{\tau^*} = \Psi(\imath, \varphi') \circ \hat{\imath}$. Thus $\psi := \Psi(\imath, \varphi')$ satisfies the claim.

Now let $(W, \hat{\eta})$ be another minimum unification of X. Using the same notation as above (and as in Theorem 3.10), one can show that the maps

$$\psi := \Psi(\imath, \eta')\colon \widehat{\bigotimes_{x\in X}} Y_x \to W \qquad \text{and} \qquad \chi := \Psi(\eta, \imath')\colon W \to \widehat{\bigotimes_{x\in X}} Y_x$$

are TO-embeddings such that $\psi \circ \hat{\imath} = \chi$ and $\chi \circ \hat{\imath} = \psi$. Theorem 3.10 (iii) yields that ψ and χ are order-preserving homeomorphisms such that $\psi^{-1} = \chi$. □

Remark 3.12 can now be restated as follows.

Corollary 3.15. *The canonical unification of a GO-space is its unique (up to a relabeling) minimum unification and minimum completion.*

We conclude the paper by describing explicitly the minimum unification (and completion) of an order resolution. Let $(X, \preceq)$ and $(Y_x, \preceq_x)_{x\in X}$ be chains. Endow the set $\bigcup_{x\in X}\{x\} \times Y_x$ with the lexicographic order $\preceq_{\text{lex}}$ and the order resolution topology $\tau_\otimes$. We denote this GO-space by $\left(\bigotimes_{x\in X} Y_x, \preceq_{\text{lex}}\right)$.

Definition 3.16. For each $x \in X$, define a chain $(Y_x^+, \preceq_x^+)$ such that $Y_x \subseteq Y_x^+$ and $\preceq_x^+$ extends $\preceq_x$ in the following way:

(i) if x has an immediate predecessor x' and an immediate successor x'' such that $\exists\,\max Y_{x'}$, $\nexists\,\min Y_x$, $\nexists\,\max Y_x$ and $\exists\,\min Y_{x''}$, then set $Y_x^+ := Y_x \cup \{y_0, y_1\}$ and $y_0 \prec_x^+ y \prec_x^+ y_1$ for all $y \in Y_x$;

(ii) if (i) does not hold and x has an immediate predecessor x' such that $\exists\,\max Y_{x'}$ but $\nexists\,\min Y_x$, then set $Y_x^+ := Y_x \cup \{y_0\}$ and $y_0 \prec_x^+ y$ for all $y \in Y_x$;

(iii) if (i) does not hold and x has an immediate successor x'' such that $\exists\,\min Y_{x''}$ but $\nexists\,\max Y_x$, then set $Y_x^+ := Y_x \cup \{y_1\}$ and $y \prec_x^+ y_1$ for all $y \in Y_x$;

(iv) in all other cases, set $Y_x^+ := Y_x$.

Endow the chain $\sum_{x\in X} Y_x^+$ with the order topology τ_Σ^+. We call the LOTS $\left(\sum_{x\in X} Y_x^+, \tau_\Sigma^+\right)$ the *lexicographic completion* of the family $(Y_x, \preceq_x)_{x\in X}$. Open intervals in this LOTS are denoted by $((x', y'), (x'', y''))^+$; a similar notation is used for the other types of intervals and for rays. (Note that the lexicographic completion of a family of chains is obtained by inserting a jump per pseudojump, thus eliminating all pseudojumps in the representation of the sum.)

The set $\bigcup_{x\in X}\{x\} \times Y_x$ can be endowed with two topologies (apart from the resolution topology): the order topology τ_Σ and the subspace topology of τ_Σ^+ (inherited from the lexicographic completion of the family $(Y_x)_{x\in X}$). Note that $\sum_{x\in X} Y_x$ is an open dense subspace of $\left(\sum_{x\in X} Y_x^+, \tau_\Sigma^+\right)$.

In the next result we list some sets that are always open in the subspace topology (but possibly fail to be open in the order topology).

Lemma 3.17. *For each $x \in X$, the following sets are open in the space $\left(\bigcup_{x\in X}\{x\} \times Y_x, \tau_\Sigma^+\right)$:*

(i) *$A = \bigcup_{w\in(x,\to)}\{w\} \times Y_w$;*

(ii) *$B = \bigcup_{w\in[x,\to)}\{w\} \times Y_w$, if either $\nexists\min Y_x$, or $\exists\min Y_x$ and x has an immediate predecessor;*

(iii) *$C = \bigcup_{w\in[x,x'')}\{w\} \times Y_w$, if either $\nexists\min Y_x$, or $\exists\min Y_x$ and x has an immediate predecessor. (If $x = \max X$, then $[x, x'') = \{x\}$.)*

Proof. For (i), note that A is always open in τ_Σ (and hence in τ_Σ^+), except for the following case: Y_x has no maximum, x has an immediate successor x'' and $Y_{x''}$ has a minimum $y''_{\min}$. In this case, there exists $y_{\max} := \max Y_x^+ \in Y_x^+ \setminus Y_x$. It follows that

$$A \;=\; [(x'', y''_{\min}), \rightarrow) \;=\; ((x, y_{\max}), \rightarrow)^+ \cap \big(\bigcup_{x\in X} \{x\} \times Y_x\big)$$

is open in τ_Σ^+.

To prove (ii), first assume that Y_x has no minimum and let (s,t) be a point of B. If $s = x$, then we can select $y \in Y_x$ such that $y \prec_x t$ and so $(s,t) \in ((s,y), \rightarrow) \subseteq B$. If $s \neq x$, then $(s,t) \in ((x,y), \rightarrow) \subseteq B$, where y is an arbitrary point of Y_x. Thus B is open in this case.

Next assume that Y_x has a minimum $y_{\min}$ and x has an immediate predecessor x'. If $Y_{x'}$ has no maximum, then there exists $y'_{\max} := \max Y_{x'}^+ \in Y_{x'}^+ \setminus Y_{x'}$. Since

$$B \;=\; [(x, y_{\min}), \rightarrow) \;=\; ((x', y'_{\max}), \rightarrow)^+ \cap \big(\bigcup_{x\in X} \{x\} \times Y_x\big)$$

it follows that B is open. On the other hand, if $Y_{x'}$ has a maximum, then B is open in the order topology τ_Σ and thus the same holds in the subspace topology of τ_Σ^+.

To prove (iii), observe that $C = A \cap B$, where $A := \bigcup_{w\in(\leftarrow,x'')}\{w\} \times Y_w$ and $B := \bigcup_{w\in[x,\rightarrow)}\{w\} \times Y_w$. Therefore, the claim follows from (the dual of) (i) and (ii). □

Lemma 3.18. *The inclusion* $\imath_\otimes \colon \big(\bigotimes_{x\in X} Y_x, \preceq_{\mathrm{lex}}\big) \hookrightarrow \big(\sum_{x\in X} Y_x^+, \tau_\Sigma^+\big)$ *is an order-preserving homeomorphic embedding.*

Proof. It suffices to prove that $\imath_\otimes$ is a homeomorphic embedding. First we show that $\imath_\otimes$ is continuous. Let $S = (\leftarrow, (x,y))^+ \cap \big(\bigcup_{x\in X}\{x\} \times Y_x\big)$ be a subbasic open set in $\imath_\otimes\big(\bigotimes_{x\in X} Y_x\big)$; we show that $\imath_\otimes^{-1}(S) = S$ is open in $\bigotimes_{x\in X} Y_x$.

If (x,y) is such that $y \in Y_x$, then Lemma 2.9 implies that $S = (\leftarrow, (x,y))$ is open in $\tau_\otimes$. Next, assume that $y \in Y_x^+ \setminus Y_x$; thus, either $y = \min Y_x^+$ or $y = \max Y_x^+$. If $y = \min Y_x^+$, then x has an immediate predecessor x' and $Y_{x'}$ has a maximum $y'_{\max}$. By Lemma **??**, $S = (\leftarrow, (x', y'_{\max})] = \pi^{-1}(\leftarrow, x)$ is open in $\tau_\otimes$. On the other hand, if $y = \min Y_x^+$, then S is the open ray $(\leftarrow, (x'', y''_{\min}))$, where x'' is the immediate successor of x and $y''_{\min}$ is the minimum of $Y_{x''}$.

Next we show that $\imath_\otimes^{-1} \colon \imath_\otimes\big(\bigotimes_{x\in X} Y_x\big) \to \bigotimes_{x\in X} Y_x$ is continuous. Let $U \otimes_x V$ be a subbasic open set in $\mathcal{S}_\otimes$; we prove that $U \otimes_x V$ is open in $\imath_\otimes\big(\bigotimes_{x\in X} Y_x\big) \subseteq \sum_{x\in X} Y_x^+$. By Lemma 2.3 and duality, it suffices to examine the following two cases: (a) $U \otimes_x V = X \otimes_x (y, \rightarrow)$ for some $y \in Y_x$; (b) $U \otimes_x V = (x', \rightarrow) \otimes_x Y_x$ for some $x' \prec x$.

For (a), Lemma 2.3 yields that $X \otimes_x (y, \rightarrow)$ is equal to either $\{x\} \times (y, \rightarrow)$ (if Y_x has no maximum) or $((x,y), \rightarrow)$ (if Y_x has a maximum); in both cases the claim holds. For (b), the claim follows from Lemma 2.3, Lemma 2.11 and Lemma 3.17. □

Theorem 3.19. $\left(\left(\sum_{x\in X} Y_x^+, \tau_\Sigma^+\right), \imath_\otimes\right)$ *is the minimum unification of the GO-space* $\left(\bigotimes_{x\in X} Y_x, \preceq_{\text{lex}}\right)$.

Proof. Since Lemma 3.18 implies that $\left(\left(\sum_{x\in X} Y_x^+, \tau_\Sigma^+\right), \imath_\otimes\right)$ is a unification of $\left(\bigotimes_{x\in X} Y_x, \preceq_{\text{lex}}\right)$, it suffices to show that it is minimum. Let (Z,φ) be a unification of $\bigotimes_{x\in X} Y_x$. We define a TO-embedding $\psi\colon \sum_{x\in X} Y_x^+ \to Z$ such that $\varphi = \psi \circ \imath_\otimes$. Let $(x,y) \in \sum_{x\in X} Y_x^+$. If $(x,y) \in \bigcup_{x\in X}\{x\} \times Y_x$, then let $\psi(x,y) := \varphi(\imath_\otimes^{-1}(x,y))$.

If $(x,y) \in \sum_{x\in X} Y_x^+ \setminus \left(\bigcup_{x\in X}\{x\} \times Y_x\right)$, then either (i) x has an immediate predecessor x', $Y_{x'}$ has a maximum and Y_x has no minimum, or (ii) x has an immediate successor x'', $Y_{x''}$ has a minimum and Y_x has no maximum. We define ψ in case (ii) only, since (i) is dual to (ii). Observe that $y \in Y_x^+ \setminus Y_x$ is the maximum of Y_x^+. If we denote $y''_{\min} := \min Y_{x''}$, then $((x,y),\to)^+ = [(x'', y''_{\min}),\to)^+$ is an open ray in $\sum_{x\in X} Y_x^+$. Thus the set

$$[(x'', y''_{\min}),\to) \;=\; \imath_\otimes^{-1}\left([(x'', y''_{\min}),\to)^+ \,\cap\, \imath_\otimes\left(\otimes_{x\in X} Y_x\right)\right)$$

is open in $\bigotimes_{x\in X} Y_x$. Since φ is a TO-embedding, the sets $A := \varphi(\leftarrow, (x'', y''_{\min}))$ and $B := \varphi[(x'', y''_{\min}),\to)$ are open in $\varphi\left(\bigotimes_{x\in X} Y_x\right)$; further, $A \prec \min B = \varphi(x'', y''_{\min})$. Note that there exists $z_0 \in Z$ such that $A \prec z_0 \prec \min B$, since otherwise B would fail to be open in $\varphi\left(\bigotimes_{x\in X} Y_x\right) \subseteq Z$. Select such a $z_0 \in Z$ and define $\psi(x,y) := z_0$. This completes the definition of ψ in case (ii).

It is immediate to check that ψ is injective and order-preserving. Thus ψ is a TO-embedding by Lemma 3.2. $\square$

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