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Uniformly discrete hit-and-miss hypertopology A missing link in hypertopologies

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Abstract. Recently it was shown that the lower Hausdorff metric (uniform) topology is generated by families of uniformly discrete sets as *hit* sets. This result leads to a new hypertopology which is the join of the above topology and the upper Vietoris topology. This *uniformly discrete hit-and-miss hypertopology* is coarser than the locally finite hypertopology and finer than both Hausdorff metric (uniform) topology and Vietoris topology. In this paper this new hypertopology is studied. Here is a Hasse diagram in which each arrow goes from a coarser topology to a finer one and equality follows UC or TB as indicated. The diagram clearly shows that the new (underlined) topology provides the missing link.

	UC	
Proximal locally finite	⟶	**Locally finite**
↑ UC		↑ UC
	UC	
Hausdorff metric	⟶	<u>**Uniformly discrete**</u>
↑ TB		↑ TB
	UC	
Proximal	⟶	**Vietoris**

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†In memory of Professor Enrico Meccariello who made a considerable contribution to this work and who suddendly passed away before his time.

1. Introduction and Preliminaries

During the early part of the last century there were two main studies on the hyperspace $CL(X)$ of all non empty closed subsets of a topological (uniform, metric) space X. Vietoris introduced a topology consisting of two parts (a) the lower *finite hit* part and (b) the upper *miss* part. Hausdorff defined a metric on the hyperspace of a metric space, and the resulting topology depends on the metric rather than on the topology of the base space. Two equivalent metrics on a metrizable space induce equivalent Hausdorff metric topologies if, and only if, they are uniformly equivalent. Recently, it was shown that all known hypertopologies are variations of the prototype: Vietoris hit-and-miss topology, and so they can be described as the joins of lower and upper parts ([11]). In particular, the Hausdorff metric topology coincides with the join $\tau(\mathcal{U})^- \vee \sigma(\delta^+)$, where

(a) $\tau(\mathcal{U})^-$ is the *lower uniformly discrete hit* part, and
(b) $\sigma(\delta^+)$ is the *upper proximal miss* part ([11]).

A new hypertopology arises formally but, in fact, is indeed quite natural. It is **the uniformly discrete hit-and-miss hypertopology** $\tau(\mathcal{U})$, which is the join of the *upper Vietoris* and the *lower uniformly discrete* topologies. This hypertopology, which is well placed in the family of hypertopologies, is finer than both the Vietoris $\tau(V)$ and the Hausdorff metric $\tau(H_d)$ topologies, but is coarser than the locally finite topology. The locally finite hypertopology $\tau(\mathcal{L})$ plays a key role (see [2], [8] and [14]). Given a metric space (X, d), $\tau(\mathcal{L})$ is the sup of all Hausdorff metric topologies $\{\tau(H_\varrho)\}$ induced by all metrics ϱ topologically equivalent to d. The uniformly discrete hit-and-miss hypertopology $\tau(\mathcal{U}(d))$, or simply $\tau(\mathcal{U})$, has a decomposition which sheds more light on the nature of the Hausdorff metric topology and on the relations among these classical hypertopologies.

From this representation and the knowledge that lower and upper parts act separately in any comparison, it is convenient to study the two parts separately. We will show that the two parts, in spite of the definition, play a symmetric role. In fact, they coincide with the corresponding parts of the Hausdorff metric topology if, and only if, the underlying metric space (X, d) is UC (A metric space (X, d) is UC if, and only if, each continuous real valued function defined on X is uniformly continuous). More surprising, the coincidence of either the lower or the upper parts forces the coincidence of the hypertopologies (This phenomenon is due to the symmetric nature of the Hausdorff metric topology). We will focus our attention on metric spaces, but we point out that many results hold in uniform spaces too.

In the second section we introduce the uniformly discrete hit-an-miss hypertopology $\tau(\mathcal{U})$ and state its basic, but important properties.

The third section is devoted to comparisons and equalities: the role of the uniformly discrete hit-and-miss hypertopology $\tau(\mathcal{U})$ in the lattice of all hypertopologies on $CL(X)$ is carefully investigated.

2. The uniformly discrete hit-and-miss hypertopology

Let (X, d), or simply X, be a metric space and $CL(X)$ the family of all non empty closed subsets of X. For $A \in CL(X)$, $\varepsilon > 0$, the set

$$\{S_\varepsilon(x)^- : x \in Q \subset A, \text{ where } Q \text{ is } \varepsilon\text{-discrete}\}$$

is the (*discrete*) *ε-screen* of A *centered* at Q and is denoted by $S_\varepsilon(Q, A)^-$. $L_\varepsilon(A)^-$ is the family of *all discrete screens* of A, namely

$$\{S_\varepsilon(Q, A)^- : Q \subset A, \text{ where } Q \text{ is } \varepsilon\text{-discrete}, \varepsilon > 0\}.$$

Lemma 2.1. *For $A \in CL(X)$, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, let Q_1 be ε_1-discrete and Q_2 be ε_2-discrete subsets of A. Then $A \in S_\varepsilon(Q, A)^- \subset S_{\varepsilon_1}(Q_1, A)^- \cap S_{\varepsilon_2}(Q_2, A)^-$, for $\varepsilon = \frac{1}{3} min\{\varepsilon_1, \varepsilon_2\}$.*

Definition 2.2. Given a metric space (X, d), the *lower uniformly discrete topology* $\tau(\mathcal{U}(d))^-$ on $CL(X)$, briefly *the lower discrete topology,* has the family $\{L_\varepsilon(A)^- : \varepsilon > 0\}$ as a local base at $A \in CL(X)$.

Proposition 2.3. [11] *Let (X, d) be a metric space. The lower discrete topology $\tau(\mathcal{U}(d))^-$ on $CL(X)$ equals the lower Hausdorff metric topology $\tau(H_d^-)$.*

Definition 2.4. Given a metric space (X, d), the *uniformly discrete hit-and-miss topology* $\tau(\mathcal{U}(d))$, or briefly $\tau(\mathcal{U})$, on $CL(X)$ is *the join of the lower (uniformly) discrete topology and the upper Vietoris topology,* i.e. $\tau(\mathcal{U}(d)) = \tau(\mathcal{U}(d))^- \vee \tau(V^+)$. We call $\tau(\mathcal{U}(d))$ the **UD-topology**.

We first study the countability properties of $\tau(\mathcal{U}(d))^-$ and $\tau(\mathcal{U}(d)) = \tau(\mathcal{U}(d))^- \vee \tau(V^+)$. The first countability of the lower discrete topology $\tau(\mathcal{U}(d))^-$ on $CL(X)$ is obvious since it is induced by a pseudometric (Proposition 2.3), but we would like to offer a different proof.

Proposition 2.5. *The lower discrete topology $\tau(\mathcal{U}(d))^-$ on $CL(X)$ is first countable.*

Proof. Given $A \in CL(X)$, let Q_n be a $\frac{1}{n}$-maximal discrete subset of A. Then the family $\mathcal{L}(A)^- = \{S_{\frac{1}{n}}(x)^- : x \in Q_n : n \in \mathbb{N}\}$, is a countable local base at A. □

Proposition 2.6. *Let (X, d) be a metric space. The following are equivalent.*

(1) *$(CL(X), \tau(\mathcal{U}(d)))$ is first countable;*
(2) *$(CL(X), \tau(V^+))$ is first countable;*
(3) *(X, d) is topologically UC.*

Proof. (1) $\Rightarrow$ (2) use the same argument as in Theorem 1.1 in [5].

(2) $\Leftrightarrow$ (3) is Corollary 1.9 in [5] (a metric space (X, d) is topologically UC if, and only if, admits a compatible metric ϱ such that (X, ϱ) is UC).

(2) $\Rightarrow$ (1) $\tau(\mathcal{U}(d)) = \tau(\mathcal{U}(d))^- \vee \tau(V^+)$ is first countable since it is the join of two first countable topologies. □

Proposition 2.7. *Let (X, d) be a metric space. The following are equivalent:*

(1) *The lower discrete topology $\tau(\mathcal{U}(d))^-$ on $CL(X)$ is second countable;*
(2) *X is totally bounded (TB).*

Proof. (2) $\Rightarrow$ (1) If X is TB, then $\tau(\mathcal{U}(d))^- = \tau(V^-)$ and the claim since X is also second countable (see [3] and [5]).

(1) $\Rightarrow$ (2) Suppose that X fails to be TB. So it admits for some positive ε an infinite ε-discrete subset $D = \{x_n : n \in \mathbb{N}\}$. Let $\mathcal{D}$ be the uncountable set of all nonempty subsets of D. Let D_1, $D_2 \in \mathcal{D}$ with $D_1 \neq D_2$ and $d^\star \in D_1 \setminus D_2$. Then $D_2 \in \{S_\varepsilon(d)^- : d \in D_2\}$, but $D_2 \notin \{S_\varepsilon(d)^- : d \in D_1\}$ and the claim. □

Proposition 2.8. *Let (X, d) be a metric space. The following are equivalent:*

(1) *The UD-topology $\tau(\mathcal{U}(d))$ on $CL(X)$ is second countable;*
(2) *$\tau(V^+)$ is second countable;*
(3) *X is compact;*
(4) *The UD-topology $\tau(\mathcal{U}(d))$ on $CL(X)$ is compact and metrizable.*

Proof. See [3] and [5], use Theorem 4.9.7 in [9] and observe that if X is compact all the involved hypertopologies coincide with the Vietoris topology $\tau(V)$ (see section 3). □

We omit the proof of the following technical Lemma.

Lemma 2.9. *Let (X, d) be a metric space and $A \in CL(X)$. Then $(CL(A), \tau(\mathcal{U}(d \mid A))) = (CL(A), \tau(\mathcal{U}(d)) \mid CL(A))$.*

Proposition 2.10. *Let (X, d) be a metric space. The following are equivalent:*

(1) *$\tau(\mathcal{U}(d))$ is normal;*
(2) *X is compact;*
(3) *$\tau(\mathcal{U}(d))$ is compact and metrizable.*

Proof. We show (1) $\Rightarrow$ (2). Suppose that X fails to be compact. Then there exists a countable closed discrete set A. Now use Lemma 2.9 and Ivanova argument (see [6] or [7]). □

We state the following proposition without a proof.

Proposition 2.11. *Let X be metrizable and d and e compatible metrics. Then $\tau(\mathcal{U}(d)) = \tau(\mathcal{U}(e))$ if, and only if, d and e are uniformly equivalent.*

3. Comparisons

The following proposition can be easily deduced from the definitions. Nevertheless, it is important since it is useful to locate in a natural way the Vietoris and Hausdorff metric topologies in the lattice of all hypertopologies on $CL(X)$. We recall that the lower locally finite topology $\tau(\mathcal{L}^-)$ has as a basis: the family of all sets of the form $\{E^- : E \in \mathcal{L}, \mathcal{L}$ is locally finite$\}$.

Proposition 3.1. *Let (X, d) be a metric space. The following inclusions hold on $CL(X)$:*

$\tau(\mathcal{U}(d))^- \subset \tau(\mathcal{L}^-)$,
$\tau(V^-) \subset \tau(\mathcal{U}(d))^-$.

Hence, the following Hasse diagram can be drawn: it is formed by two rectangles (upper/lower) with a common side (recall that the proximal locally finite topology $\sigma(\mathcal{L})$ has been studied in [2]).

Each arrow goes from a coarser topology to a finer one and equality follows UC (see below) or TB (totally bounded) as indicated. The diagram clearly shows that the new topology provides the missing link.

	UC	
Proximal locally finite	$\longrightarrow$	**Locally finite**
$\uparrow$ UC		$\uparrow$ UC
	UC	
Hausdorff metric	$\longrightarrow$	**Uniformly discrete**
$\uparrow$ TB		$\uparrow$ TB
	UC	
Proximal	$\longrightarrow$	**Vietoris**

Observe that all the topologies in the first (resp. second) column have the same upper part: the proximal upper topology $\sigma(V^+)$ (resp. the upper Vietoris topology $\tau(V^+)$); similarly all topologies in the first (resp. second, third) row have the same lower part: lower locally finite $\tau(\mathcal{L}^-)$ (resp. lower uniformly discrete $\tau(\mathcal{U}(d))^-$, lower Vietoris $\tau(V^-)$).

To state comparisons the notion of UC metric is essential. Again, we recall that a metric space (X, d) is UC if each real valued continuous function on (X, d) is uniformly continuous. This class of spaces has been investigated since 1950 and intensely in the last decades (see [4] where further references can be found). We use the following characterization of a UC space due to Atsuji ([1]). A metric space (X, d) is UC if each sequence $(x_n : n \in \mathbb{N})$ without accumulation points is finally uniformly isolated, i.e. there exists a positive ε and an index n_0 such that for all $n \geq n_0$, $S(x_n, \varepsilon) \cap X = \{x_n\}$.

The following is an unexpected result.

Proposition 3.2. *Let (X, d) be a metric space. The following are equivalent:*
(1) $\tau(\mathcal{L}^-) = \tau(\mathcal{U}(d))^-$;
(2) (X, d) *is UC.*

Proof. (1) $\Rightarrow$ (2) By contradiction, suppose (X, d) fails to be UC. Then, passing to a subsequence if necessary, there is a sequence $(x_n : n \in \mathbb{N})$ without accumulation points such that for each n there is $y_n \in X$ such that $d(x_n, y_n) \leq \frac{1}{n}$. Since $(x_n : n \in \mathbb{N})$ is without accumulation points, the same occurs to $(y_n : n \in \mathbb{N})$. Set $r_n = d(x_n, y_n)$ and consider $\{S(x_n, r_n) : n \in \mathbb{N}\}$. Since X is metric (paracompact), we may suppose that $\{S(x_n, r_n) : n \in \mathbb{N}\}$ is a locally finite family. Let $A = (x_n : n \in \mathbb{N})$ and $A_n = \{z_k : k \in \mathbb{N},$ with $z_k = x_k$ for $k \leq n$ and $z_k = y_k$ for $k > n\}$. The sequence A_n clearly converges to A w.r.t. $\tau(\mathcal{U}(d))^-$, but

A_n fails to converge to A w.r.t. $\tau(\mathcal{L}^-)$. In fact, $A \in \{S(x_n, r_n) : n \in \mathbb{N}\}^-$, but for no n, $A_n \in \{S(x_n, r_n) : n \in \mathbb{N}\}^-$.

(2) $\Rightarrow$ (1) Suppose that (X, d) is UC. Let $\mathcal{V} = \{S(x_i, r_i) : i \in I\}$ be a discrete family. For each $i \in I$ there exists a continuous function $g_i \colon X \to [0, 1]$ such that $g_i(x_i) = 1$, $g_i(S(x_i, r_i)) \subseteq [0, 1]$, and $g_i(x) = 0$ for $x \notin S(x_i, r_i)$. Set $\varrho(x, y) = d(x, y) + \sum \mid g_i(x) - g_i(y) \mid$.

By a routine argument ϱ is a compatible metric. Since the identity map $id \colon (X, d) \to (X, \varrho)$ is continuous, it must be uniformly continuous. So, there exists ε such that $d(x, y) < \varepsilon$ implies $\varrho(x, y) < 1$. Thus, the set $\{x_i : i \in I\}$ is ε-discrete and the claim $\tau(\mathcal{L}^-) = \tau(\mathcal{U}(d))^-$. □

We recall the following visual characterization of UC spaces due to Nagata ([10]): a metric space (X, d) is UC if, and only if, disjoint closed sets are a positive distance apart. Thus, the corresponding results for the upper parts are easily stated.

Proposition 3.3. *Let (X, d) be a metric space. The following are equivalent:*

(1) $\tau(\mathcal{L}^+) = \sigma(\mathcal{L}^+)$;
(2) $\tau(\mathcal{L}) = \sigma(\mathcal{L})$;
(3) $\tau(\mathcal{U}(d))^+ = \tau(H_d^+)$;
(4) $\tau(\mathcal{U}(d)) = \tau(H_d)$;
(5) $\tau(V^+) = \sigma(V^+)$;
(6) $\tau(V) = \sigma(V)$;
(7) (X, d) *is UC.*

Proof. Note that the hypertopologies $\tau(\mathcal{L}^+)$ and $\tau(\mathcal{U}(d))^+$ have as upper part the upper Vietoris topoloy $\tau(V^+)$, whereas the upper part of $\sigma(\mathcal{L}^+)$ and $\tau(H_d^+)$ is the upper proximal topology $\sigma(V^+)$. Clearly, according to the above mentioned characterization of UC spaces, $\tau(V^+)$ and $\sigma(V^+)$ coincide if, and only if, the metric space (X, d) is UC. □

Proposition 3.4. *Let (X, d) be a metric space. The following are equivalent:*

(1) $\tau(V^-) = \tau(\mathcal{U}(d))^-$;
(2) $\tau(V) = \tau(\mathcal{U}(d))$;
(3) $\tau(H_d^-) = \sigma(V^-)$;
(4) $\tau(H_d) = \sigma(V)$;
(5) (X, d) *is totally bounded (TB).*

Proof. See [11]. □

Observe that in the upper rectangle horizontal and vertical equalities are described by the same single property, i.e. UC.

Theorem 3.5. *Let (X, d) be a metric space. The following are equivalent:*

(1) $\tau(\mathcal{L}) = \sigma(\mathcal{L}) = \tau(H_d) = \tau(\mathcal{U}(d))$;
(2) (X, d) *is UC.*

Proof. Use Propositions 3.2 and 3.3 or the following conceptual argument which uses the full power of the whole hypertopologies. The locally finite hypertopology $\tau(\mathcal{L})$ equals the Hausdorff-Boubaki hypertopology $\tau(\mathcal{W})$ induced by the fine uniformity $\mathcal{W}$ on (X, d) (Theorem 2.2 in [12]). Recall that (X, d) is UC if, and only if, the metric uniformity $\mathcal{V}(d)$ is the fine uniformity $\mathcal{W}$ (see [10], [12] and [13]). Observe that the Hausdorff-Boubaki hypertopology $\tau(\mathcal{V}(d))$ induced by the metric uniformity $\mathcal{V}(d)$ is the Hausdorff metric hypertopology $\tau(H_d)$. Thus, (X, d) is UC if, and only if, $\tau(\mathcal{L}) = \tau(\mathcal{W}) = \tau(H_d)$ (Corollary 2.2 in [12]). It follows easily that (X, d) is UC if, and only if, $\tau(\mathcal{L}) = \sigma(\mathcal{L}) = \tau(H_d) = \tau(\mathcal{U}(d))$. □

Note that in the lower rectangle the horizontal (resp. vertical) equalities occur when the space is UC (resp. TB). To squeeze the lower rectangle to a point we need two properties, namely TB and UC. Consequently, due to the UC property, also the upper rectangle reduces to a point. But, TB + UC is equivalent to compactness. So, we have that compactness is equivalent to the equality of all the six studied hypertopologies.

Corollary 3.6. *Let (X, d) be a metric space. The following are equivalent:*

(1) $\sigma(V) = \tau(V) = \tau(\mathcal{U}(d)) = \tau(H_d) = \tau(\mathcal{L}) = \sigma(\mathcal{L})$;
(2) (X, d) *is compact.*

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