Applied General Topology
© Universidad Politécnica de Valencia
Volume 6, No. 2, 2005
pp. 119-133

On some applications of fuzzy points

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Abstract. The notion of preopen sets (see [9] and [14]) play a very important role in General Topology and Fuzzy Topology. Preopen sets are also called nearly open and locally dense (see [4]). The purpose of this paper is to give some applications of fuzzy points in fuzzy topological spaces. Moreover, in section 2 we offer some properties of fuzzy preclosed sets through the contribution of fuzzy points and we introduce new separation axioms in fuzzy topological spaces. Also using the notions of weak and strong fuzzy points, we investigate some properties related to the preclosure of such points, and also their impact on separation axioms. In section 3, using the notion of fuzzy points, we introduce and study the notions of fuzzy pre-upper limit, fuzzy pre-lower limit and fuzzy pre-limit. Finally in section 4, we introduce the fuzzy pre-continuous convergence on the set of fuzzy pre-continuous functions and give a characterization of the fuzzy pre-continuous convergence through the assistance of fuzzy pre-upper limit.

2000 AMS Classification: 54A40

Keywords: Fuzzy Topology, Fuzzy points, Fuzzy convergence, Fuzzy separation axioms, Fuzzy preopen sets.

1. Introduction.

Throughout this paper, the symbol I will denote the unit interval $[0, 1]$. In 1965, Zadeh (see [18]) introduced the fundamental notion of fuzzy set by which fuzzy mathematics emerged. Let X be a nonempty set. A *fuzzy set* in X is a function with domain X and values in I, i.e. an element of I^X.

A member A of I^X is *contained* in a member B of I^X, denoted by $A \leq B$, if $A(x) \leq B(x)$ for every $x \in X$ (see [18]).

Let $A, B \in I^X$. We define the following fuzzy sets (see [18]):

(1) $A \wedge B \in I^X$ by $(A \wedge B)(x) = \min\{A(x), B(x)\}$ for every $x \in X$.
(2) $A \vee B \in I^X$ by $(A \vee B)(x) = \max\{A(x), B(x)\}$ for every $x \in X$.
(3) $A^c \in I^X$ by $A^c(x) = 1 - A(x)$ for every $x \in X$.

(4) Let $f : X \to Y$, $A \in I^X$ and $B \in I^Y$. Then $f(A)$ is a fuzzy set in Y such that $f(A)(y) = \sup\{A(x) : x \in f^{-1}(y)\}$, if $f^{-1}(y) \neq \varnothing$ and $f(A)(y) = 0$, if $f^{-1}(y) = \varnothing$. Also, $f^{-1}(B)$ is a fuzzy set in X, defined by $f^{-1}(B)(x) = B(f(x))$, $x \in X$.

The first definition of a *fuzzy topological space* is due to Chang (see [3]). According to Chang, a fuzzy topological space is a pair (X, τ), where X is a set and τ is a *fuzzy topology* on it, i.e. a family of fuzzy sets ($\tau \subseteq I^X$) satisfying the following three axioms:

(1) $\bar{0}, \bar{1} \in \tau$. By $\bar{0}$ and $\bar{1}$ we denote the characteristic functions $\mathcal{X}_\varnothing$ and $\mathcal{X}_X$, respectively.
(2) If $A, B \in \tau$, then $A \wedge B \in \tau$.
(3) If $\{A_j : j \in J\} \subseteq \tau$, then $\vee\{A_j : j \in J\} \in \tau$.

By using the notion of fuzzy set, Wong (see [15]) was able to introduce and investigate the notions of fuzzy points. In this paper we adopted Pu's definition of a fuzzy point. A fuzzy set in a set X is called a *fuzzy point* if it takes the value 0 for all $y \in X$ except one, say, $x \in X$. If its value at x is λ ($0 < \lambda \leq 1$) we denote the fuzzy point by p_x^λ, where the point x is called its *support*, denoted by $supp(p_x^\lambda)$, that is $supp(p_x^\lambda) = x$. The class of all fuzzy points in X is denoted by $\mathcal{X}$.

The fuzzy point p_x^λ is said to be *contained* in a fuzzy set A or to belong to A, denoted by $p_x^\lambda \in A$, if $\lambda \leq A(x)$. Evidently, every fuzzy set A can be expressed as the union of all the fuzzy points which belongs to A (see [10]).

A fuzzy point p_x^λ is said to be *quasi-coincident* with A denoted by $p_x^\lambda qA$ if and only if $\lambda > A^c(x)$ or $\lambda + A(x) > 1$ (see [10]).

A fuzzy set A is said to be *quasi-coincident* with B, denoted AqB, if and only if there exists $x \in X$ such that $A(x) > B^c(x)$ or $A(x) + B(x) > 1$ (see [10]). If A does not quasi-coincident with B, then we write $A \not{q} B$.

Let f be a function from X to Y. Then (see for example [1], [2], [3], [8], [11], [12], [13], [16], and [17]):

(1) $f^{-1}(B^c) = (f^{-1}(B))^c$, for any fuzzy set B in Y.
(2) $f(f^{-1}(B)) \leq B$, for any fuzzy set B in Y.
(3) $A \leq f^{-1}(f(A))$, for any fuzzy set A in X.
(4) Let p be a fuzzy point of X, A be a fuzzy set in X and B be a fuzzy set in Y. Then, we have:
  (i) If $f(p)\ q\ B$, then $p\ q\ f^{-1}(B)$.
  (ii) If $p\ q\ A$, then $f(p)\ q\ f(A)$.
(5) Let A and B be fuzzy sets in X and Y, respectively and p be a fuzzy point in X. Then we have:
  (i) $p \in f^{-1}(B)$ if $f(p) \in B$.
  (ii) $f(p) \in f(A)$ if $p \in A$.

Let Λ be a directed set and X be an ordinary set. The function $S : \Lambda \to \mathcal{X}$ is called a *fuzzy net* in X. For every $\lambda \in \Lambda$, $S(\lambda)$ is often denoted by s_λ and hence a net S is often denoted by $\{s_\lambda, \lambda \in \Lambda\}$ (see [10]).

Let $\{A_n, n \in N\}$ be a net of fuzzy sets in a fuzzy topological space X. Then by $F - \overline{\lim\limits_{N}}(A_n)$, we denote the *fuzzy upper limit* of the net $\{A_n, n \in N\}$ in I^X, that is, the fuzzy set which is the union of all fuzzy points p_x^λ in X such that for every $n_0 \in N$ and for every fuzzy open $Q-$neighborhood U of p_x^λ in X there exists an element $n \in N$ for which $n \geq n_0$ and $A_n q U$. In other cases we set $F - \overline{\lim\limits_{N}}(A_n) = \overline{0}$.

For the notions of fuzzy upper limit and fuzzy lower limit see [6].

Recall that a fuzzy subset A of a fuzzy topological space X is called *fuzzy preopen* (see [5] and [14]) if $A \leq Int(Cl(A))$, where Int and Cl denoted the interior and closure operators. A is called *fuzzy preclosed* if $Cl(Int(A)) \leq A$. We denote the family of all fuzzy preopen (respectively, fuzzy preclosed) sets of X by $FPO(X)$ (respectively, $FPC(X)$). Also the intersection of all fuzzy preclosed sets containing A is called *fuzzy preclosure* of A, denoted by $pCl(A)$, that is

$$pCl(A) = \inf\{K : A \leq K, K \in FPC(X)\}.$$

Similar the *fuzzy preinterior* of A, denoted by $pInt(A)$, is defined as follows:

$$pInt(A) = \sup\{U : U \leq A, U \in FPO(X)\}.$$

Let A be a fuzzy preopen (respectively, preclosed) set of a fuzzy space X. Then, by Theorem 3.7 of [14], $pInt(A) = A$ (respectively, $pCl(A) = A$). Also, by Theorem 3.6 of [14], we have $pCl(A^c) = \overline{1} - pInt(A) = \overline{1} - A = A^c$ (respectively, $pInt(A^c) = \overline{1} - pCl(A) = \overline{1} - A = A^c$). Thus, the fuzzy set A^c is fuzzy preclosed (respectively, preopen).

2. Fuzzy points, preclosed sets and separations axioms

Definition 2.1. *A fuzzy set A in a fuzzy space X is called a fuzzy pre-neighborhood of a fuzzy point p_x^λ if there exists a $V \in FPO(X)$ such that $p_x^\lambda \in V \leq A$. A fuzzy pre-neighborhood A is said to be preopen if $A \in FPO(X)$.*

Definition 2.2. *A fuzzy set A in a fuzzy space X is called a fuzzy $Q-$pre-neighborhood of p_x^λ if there exists $B \in FPO(X)$ such that $p_x^\lambda q B$ and $B \leq A$.*

Remark 2.3. A fuzzy $Q-$pre-neighborhood of a fuzzy point generally does not contain the point itself. In what follows by $\mathcal{N}_{Q-p-n}(p_x^\lambda)$ we denote the family of all fuzzy preopen $Q-$pre-neighborhoods of the fuzzy point p_x^λ in X. The set $\mathcal{N}_{Q-p-n}(p_x^\lambda)$ with the relation $\leq^*$ (that is, $U_1 \leq^* U_2$ if and only if $U_2 \leq U_1$) form a directed set.

Proposition 2.4. *Let A be a fuzzy set of a fuzzy space X. Then, a fuzzy point $p_x^\lambda \in pCl(A)$ if and only if for every $U \in FPO(X)$ for which $p_x^\lambda q U$ we have UqA.*

Proof. The fuzzy point $p_x^\lambda \in pCl(A)$ if and only if $p_x^\lambda \in F$, for every fuzzy preclosed set F of X for which $A \leq F$. Equivalently $p_x^\lambda \in pCl(A)$ if and only if $\lambda \leq 1 - U(x)$, for every fuzzy preopen set U for which $A \leq \overline{1} - U$. Thus

$p_x^\lambda \in pCl(A)$ if and only if $U(x) \leq 1 - \lambda$, for every fuzzy preopen set U for which $U \leq \bar{1} - A$. So, $p_x^\lambda \in pCl(A)$ if and only if for every fuzzy preopen set U of X such that $U(x) > 1 - \lambda$ we have $U \not\leq \bar{1} - A$. Therefore by Proposition 2.1 of [10], $p_x^\lambda \in pCl(A)$ if and only if for every fuzzy preopen set U of X such that $U(x) + \lambda > 1$ we have UqA. Thus, $p_x^\lambda \in pCl(A)$ if and only if for every fuzzy preopen set U of X such that $p_x^\lambda qU$ we have UqA. □

Definition 2.5. *Let A be a fuzzy set of a fuzzy space X. A fuzzy point p_x^λ is called a pre-boundary point of a fuzzy set A if and only if $p_x^\lambda \in pCl(A) \wedge (\bar{1} - pCl(A))$. By $pBd(A)$ we denote the fuzzy set $pCl(A) \wedge (\bar{1} - pCl(A))$.*

Proposition 2.6. *Let A be a fuzzy set of a fuzzy space X. Then*

$$A \vee pBd(A) \leq pCl(A).$$

Proof. Let $p_x^\lambda \in A \vee pBd(A)$. Then $p_x^\lambda \in A$ or $p_x^\lambda \in pBd(A)$. Clearly, if $p_x^\lambda \in pBd(A)$, then $p_x^\lambda \in pCl(A)$. Let us suppose that $p_x^\lambda \in A$. We have

$$pCl(A) = \wedge\{F : F \in I^X,\ F \text{ is preclosed and } A \leq F\}.$$

So, if $p_x^\lambda \in A$, then $p_x^\lambda \in F$, for every fuzzy preclosed set F of X for which $A \leq F$ and therefore $p_x^\lambda \in pCl(A)$. □

Example 2.7. Let (X, τ) be a fuzzy space such that $X = \{x, y\}$ and $\tau = \{\bar{0}, \bar{1}, p_x^{\frac{1}{2}}\}$.

The family of all fuzzy preclosed sets of X contains the following fuzzy sets A of X:

i) $A \in I^X$ such that $A(x) \in [0, \frac{1}{2})$ and $A(y) \in [0, 1]$.
Indeed,

$$Cl(Int(A)) = Cl(\bar{0}) = \bar{0} \leq A.$$

ii) $A \in I^X$ such that $A(x) \in [\frac{1}{2}, 1]$ and $A(y) = 1$.
Indeed,

$$Cl(Int(A)) = Cl(p_x^{\frac{1}{2}}) \leq (p_x^{\frac{1}{2}})^c \leq A.$$

Also, the family of all fuzzy preopen sets of X are the following fuzzy sets U of X:

i) $U \in I^X$ such that $U(x) \in [0, \frac{1}{2}]$ and $U(y) = 0$.
Indeed,

$$Int(Cl(U)) = Int((p_x^{\frac{1}{2}})^c) = p_x^{\frac{1}{2}} \geq U.$$

ii) $U \in I^X$ such that $U(x) \in (\frac{1}{2}, 1]$ and $U(y) \in [0, 1]$.
Indeed,

$$Int(Cl(U)) = Int(\bar{1}) = \bar{1} \geq U.$$

We consider the fuzzy set $B \in I^X$ such that $B = p_x^{\frac{2}{3}}$. By the above we have:

$$pCl(B) = (p_x^{\frac{1}{3}})^c,$$

where $(p_x^{\frac{1}{3}})^c(z) = \frac{2}{3}$, if $z = x$ and $(p_x^{\frac{1}{3}})^c(z) = 1$, if $z = y$.

Also, we have

$$\bar{1} - pCl(B) = p_x^{\frac{1}{3}}$$

and

$$pBd(B) = pCl(B) \wedge (\bar{1} - pCl(B)) = p_x^{\frac{1}{3}}.$$

Thus

$$B \vee pBd(B) = B \neq pCl(B).$$

Definition 2.8. *A fuzzy space X is called pre-T_0 if for every two fuzzy points p_x^λ and p_y^μ such that $p_x^\lambda \neq p_y^\mu$, either $p_x^\lambda \notin pCl(p_y^\mu)$ or $p_y^\mu \notin pCl(p_x^\lambda)$.*

Definition 2.9. *A fuzzy space X is called pre-T_1 if every fuzzy point is fuzzy preclosed.*

Remark 2.10. Clearly, every pre-T_1 fuzzy space is pre-T_0.

Proposition 2.11. *A fuzzy space X is pre-T_1 if and only if for each $x \in X$ and each $\lambda \in [0,1]$ there exists a fuzzy preopen set A such that $A(x) = 1 - \lambda$ and $A(y) = 1$ for $y \neq x$.*

Proof. $\Rightarrow$) Let $\lambda = 0$. We set $A = \bar{1}$. Then A is fuzzy preopen set such that $A(x) = 1 - 0$ and $A(y) = 1$ for $y \neq x$. Now, let $\lambda \in (0,1]$ and $x \in X$. We set $A = (p_x^\lambda)^c$. The set A is fuzzy preopen such that $A(x) = 1 - \lambda$ and $A(y) = 1$ for $y \neq x$.

$\Leftarrow$) Let p_x^λ be an arbitrary fuzzy point of X. We prove that the fuzzy point p_x^λ is fuzzy preclosed. By assumption there exists a fuzzy preopen set A such that $A(x) = 1 - \lambda$ and $A(y) = 1$ for $y \neq x$. Clearly, $A^c = p_x^\lambda$. Thus the fuzzy point p_x^λ is fuzzy preclosed and therefore the fuzzy space X is pre-T_1. □

Definition 2.12. *A fuzzy space X is called a pre-Hausdorff space if for any fuzzy points p_x^λ and p_y^μ for which $supp(p_x^\lambda) = x \neq supp(p_y^\mu) = y$, there exist two fuzzy preopen Q-pre-neighbourhoods U and V of p_x^λ and p_y^μ, respectively, such that $U \wedge V = \bar{0}$.*

Example 2.13. Let (X, τ) be a fuzzy space such that $X = \{x, y\}$ and $\tau = \{\bar{0}, \bar{1}, p_x^{\frac{1}{2}}\}$.

The fuzzy point $p_x^{\frac{1}{2}}$ is not fuzzy preclosed. Indeed, we have:

$$Cl(Int(p_x^{\frac{1}{2}})) = Cl(p_x^{\frac{1}{2}}) = (p_x^{\frac{1}{2}})^c \not\leq p_x^{\frac{1}{2}}.$$

Thus the fuzzy space X is not pre-T_1. Also, it is clear that the fuzzy space X is pre-T_0.

Example 2.14. Let (X, τ) be a fuzzy space such that $X = \{x, y\}$ and $\tau = \{\bar{0}, \bar{1}\}$.

We observe that every fuzzy point p_x^λ is fuzzy preclosed. Indeed, we have

$$Cl(Int(p_x^\lambda)) = \bar{0} \leq p_x^\lambda.$$

Thus the fuzzy space X is pre-T_1 and therefore is pre-T_0. Also, it is clear that the fuzzy space X is pre-Hausdorff.

It is not difficult to see that the fuzzy space X is not T_0, T_1 and Hausdorff. For the definitions of T_0, T_1 and Hausdorff fuzzy spaces see [10].

Definition 2.15. *A fuzzy space X is called a pre-regular space if for any fuzzy point p_x^λ and a fuzzy preclosed set F not containing p_x^λ, there exist $U, V \in FPO(X)$ such that $p_x^\lambda \in U$, $F \leq V$ and $U \wedge V = \bar{0}$.*

Example 2.16. Let (X, τ) be a fuzzy space such that $X = \{x, y\}$ and $\tau = \{\bar{0}, \bar{1}\}$.

The fuzzy space X is pre-Hausdorff but it is not pre-regular. We prove only that the fuzzy space X is not pre-regular. We consider the fuzzy point $p_x^{\frac{1}{3}}$ and the fuzzy set A of X such that $A(x) = \frac{1}{4}$ and $A(y) = 1$.

For the fuzzy set A we have

$$Cl(Int(A)) = \bar{0} \leq A.$$

Thus the fuzzy set A is fuzzy preclosed. Also, we have $p_x^{\frac{1}{3}} \notin A$.

If U and V are two arbitrary fuzzy preopen sets such that $p_x^{\frac{1}{3}} \in U$ and $A \leq V$, then $(U \wedge V)(x) \geq \frac{1}{4}$ and therefore $U \wedge V \neq \bar{0}$. Thus the fuzzy space X is not pre-regular.

Definition 2.17. *A fuzzy space X is called a quasi pre-T_1 if for any fuzzy points p_x^λ and p_y^μ for which $supp(p_x^\lambda) = x \neq supp(p_y^\mu) = y$, there exists a fuzzy preopen set U such that $p_x^\lambda \in U$ and $p_y^\mu \notin U$ and another V such that $p_x^\lambda \notin V$ and $p_y^\mu \in V$.*

Example 2.18. Let (X, τ) be a fuzzy space such that $X = \{x, y\}$ and $\tau = \{\bar{0}, \bar{1}, p_x^{\frac{1}{2}}\}$.

The fuzzy space X is quasi pre-T_1 but it is not pre-T_1.

Definition 2.19. *(see [7]) A fuzzy point p_x^λ is called weak (respectively, strong) if $\lambda \leq \frac{1}{2}$ (respectively, $\lambda > \frac{1}{2}$).*

Definition 2.20. *A fuzzy set A of a fuzzy space X is called pre-generalized closed (briefly fpg-closed) if $pCl(A) \leq U$ whenever $A \leq U$ and U fuzzy preopen set of X.*

Proposition 2.21. *Let X be a fuzzy space X. Suppose that p_x^λ and p_y^μ are weak and strong fuzzy points, respectively. If p_x^λ is pre-generalized closed, then*

$$p_y^\mu \in pCl(p_x^\lambda) \Rightarrow p_x^\lambda \in pCl(p_y^\mu).$$

Proof. Suppose that $p_y^\mu \in pCl(p_x^\lambda)$ and $p_x^\lambda \notin pCl(p_y^\mu)$. Then $pCl(p_y^\mu)(x) < \lambda$. Also $\lambda \leq \frac{1}{2}$. Thus $pCl(p_y^\mu)(x) \leq 1 - \lambda$ and therefore $\lambda \leq 1 - pCl(p_y^\mu)(x)$. So $p_x^\lambda \in (pCl(p_y^\mu))^c$.

But p_x^λ is pre-generalized closed and $(pCl(p_y^\mu))^c$ is fuzzy preopen. Thus

$$pCl(p_x^\lambda) \leq (pCl(p_y^\mu))^c.$$

By assumption we have $p_y^\mu \in pCl(p_x^\lambda)$. Thus

$$p_y^\mu \in (pCl(p_y^\mu))^c.$$

We prove that this is a contradiction.

Indeed, we have

$$\mu \leq 1 - pCl(p_y^\mu)(y)$$

or

$$pCl(p_y^\mu)(y) \leq 1 - \mu.$$

Also $p_y^\mu \in pCl(p_y^\mu)$. Thus

$$\mu \leq 1 - \mu.$$

But p_y^μ is a strong fuzzy point, that is $\mu > \frac{1}{2}$. So the above relation $\mu \leq 1-\mu$ is a contradiction. Thus $p_x^\lambda \in pCl(p_y^\mu)$. □

Proposition 2.22. *If X is a quasi pre-T_1 fuzzy space and p_x^λ a weak fuzzy point in X, then $(p_x^\lambda)^c$ is a fuzzy pre-neighborhood of each fuzzy point p_y^μ with $y \neq x$.*

Proof. Let $y \neq x$ and p_y^μ be a fuzzy point of X. Since the space X is a quasi pre-T_1 there exists a fuzzy preopen U of X such that $p_y^\mu \in U$ and $p_x^\lambda \notin U$. This implies that $\lambda > U(x)$. Also, $\lambda \leq \frac{1}{2}$. Thus $U(x) \leq 1 - \lambda$. Therefore $U(y) \leq 1 = (p_x^\lambda)^c(y)$, for every $y \in X \setminus \{x\}$. So $U \leq (p_x^\lambda)^c$. Therefore the fuzzy point p_x^λ is a pre-neighborhood of p_y^μ. □

Proposition 2.23. *If X is a pre-regular fuzzy space, then for any strong fuzzy point p_x^λ and any fuzzy preopen set U containing p_x^λ, there exists a fuzzy preopen set W containing p_x^λ such that $pCl(W) \leq U$.*

Proof. Suppose that p_x^λ is any strong fuzzy point contained in $U \in FPO(X)$. Then $\frac{1}{2} < \lambda \leq U(x)$. Thus the complement of U, that is the fuzzy set U^c, is a fuzzy preclosed set to which does not belong the fuzzy point p_x^λ. Thus, there exist $W, V \in FPO(X)$ such that $p_x^\lambda \in W$ and $U^c \leq V$ with $W \wedge V = \bar{0}$. Hence, we have $W \leq V^c$ and by Theorem 3.8 of [14] $pCl(W) \leq pCl(V^c) = V^c$. Now $U^c \leq V$ implies $V^c \leq U$. This means that $pCl(W) \leq U$ which completes the proof. □

Proposition 2.24. *If X is a fuzzy pre-regular space, then the strong fuzzy points in X are fpg-closed.*

Proof. Let p_x^λ be any strong fuzzy point in X and U be a fuzzy open set such that $p_x^\lambda \in U$. By Proposition 2.23 there exists a $W \in FPO(X)$ such that $p_x^\lambda \in W$ and $pCl(W) \leq U$. By Theorem 3.8 of [14], we have

$$pCl(p_x^\lambda) \leq pCl(W) \leq U.$$

Thus the fuzzy point p_x^λ is fpg-closed. □

Definition 2.25. *A fuzzy space X is called a weakly pre-regular space if for any weak fuzzy point p_x^λ and a fuzzy preclosed set F not containing p_x^λ, there exist $U, V \in FPO(X)$ such that $p_x^\lambda \in U$, $F \leq V$ and $U \wedge V = \bar{0}$.*

Observe that every pre-regular fuzzy space is weakly pre-regular.

Definition 2.26. *Let X be a fuzzy space. A fuzzy set U in X is said to be fuzzy pre-nearly crisp if $pCl(U) \wedge (pCl(U))^c = \bar{0}$.*

Proposition 2.27. *Let X be a fuzzy space. If for any weak fuzzy point p_x^λ and any $U \in FPO(X)$ containing p_x^λ, there exists a fuzzy preopen and pre-nearly crisp fuzzy set W containing p_x^λ such that $pCl(W) \leq U$, then X is fuzzy weakly pre-regular.*

Proof. Assume that F is a fuzzy preclosed set not containing the weak fuzzy point p_x^λ. Then F^c is a fuzzy preopen set containing p_x^λ. By hypothesis, there exists a fuzzy preopen and pre-nearly crisp fuzzy set W such that $p_x^\lambda \in W$ and $pCl(W) \leq F^c$. We set $N = pInt(pCl(W))$ and $M = 1 - pCl(W)$. Then N is fuzzy preopen, $p_x^\lambda \in N$ and $F \leq M$. We are going to prove that $M \wedge N = \bar{0}$. Now assume that there exists $y \in X$ such that $(N \wedge M)(y) = \mu \neq \bar{0}$. Then $p_y^\mu \in N \wedge M$. Hence, $p_y^\mu \in pCl(W)$ and $p_y^\mu \in \bar{(}pCl(W))^c$. This is a contradiction since W is pre-nearly crisp. Thus the fuzzy space X is weakly pre-regular. □

Definition 2.28. *Let X be a fuzzy space. A fuzzy point p_x^λ in X is said to be well-preclosed if there exists $p_y^\mu \in pCl(p_x^\lambda)$ such that $supp(p_x^\lambda) \neq supp(p_y^\mu)$.*

Proposition 2.29. *If X is a fuzzy space and p_x^λ is a fpg-closed, well-preclosed fuzzy point, then X is not quasi pre-T_1 space.*

Proof. Let X be a fuzzy quasi pre-T_1 space. By the fact p_x^λ is well-preclosed, there exists a fuzzy point p_y^μ with $supp(p_x^\lambda) \neq supp(p_y^\mu)$ such that $p_y^\mu \in pCl(p_x^\lambda)$. Then there exists $U \in FPO(X)$ such that $p_x^\lambda \in U$ and $p_y^\mu \notin U$. Therefore $pCl(p_x^\lambda) \leq U$ and $p_y^\mu \in U$. But this is a contradiction and hence X can not be quasi pre-T_1 space. □

Definition 2.30. *Let X be a fuzzy space. A fuzzy point p_x^λ is said to be just-preclosed if the fuzzy set $pCl(p_x^\lambda)$ is again fuzzy point.*

Clearly, in a fuzzy pre-T_1 space every fuzzy point is just-preclosed.

Proposition 2.31. *Let X be a fuzzy space. If p_x^λ and p_x^μ are two fuzzy points such that $\lambda < \mu$ and p_x^μ is fuzzy preopen, then p_x^λ is just-preclosed if it is fpg-closed.*

Proof. We prove that the fuzzy set $pCl(p_x^\lambda)$ is again a fuzzy point. We have $p_x^\lambda \in p_x^\mu$ and the fuzzy set p_x^μ is fuzzy preopen. Since p_x^λ is fpg-closed we have $pCl(p_x^\lambda) \leq p_x^\mu$. Thus $pCl(p_x^\lambda)(x) \leq \mu$ and $pCl(p_x^\lambda)(z) \leq 0$, for every $z \in X \setminus \{x\}$. So the fuzzy set $pCl(p_x^\lambda)$ is a fuzzy point. □

3. Fuzzy pre-convergence and fuzzy points

Definition 3.1. *Let $\{A_n, n \in N\}$ be a net of fuzzy sets in a fuzzy space X. Then by $F - pre - \overline{\lim_N}(A_n)$, we denote the fuzzy pre-upper limit of the net $\{A_n, n \in N\}$ in I^X, that is, the fuzzy set which is the union of all fuzzy* *points* p_x^λ *in* X *such that for every* $n_0 \in N$ *and for every fuzzy preopen* $Q-$*pre-neighborhood* U *of* p_x^λ *in* X *there exists an element* $n \in N$ *for which* $n \geq n_0$ *and* $A_n q U$. *In other cases we set* $F - pre - \overline{\lim_N}(A_n) = \overline{0}$.

Example 3.2. Let (X, τ) be a fuzzy space such that $X = \{x, y\}$ and $\tau = \{\bar{0}, \bar{1}, p_x^{\frac{1}{2}}\}$. Also let $\{A_n, n \in N\}$ be a net of fuzzy sets of X such that $A_n(X) = \{0.5\}$ for every $n \in N$.

The fuzzy point $p_x^{\frac{1}{2}} \in F - \overline{\lim_N}(A_n)$. Indeed, for every $n_0 \in N$ and for the only fuzzy open Q-neighborhood $U = \bar{1}$ of $p_x^{\frac{1}{2}}$ there exists an element $n \in N$ for which $n \geq n_0$ and $A_n q U$.

The fuzzy point $p_x^{\frac{1}{2}} \notin F - pre - \overline{\lim_N}(A_n)$. Indeed, for every $n_0 \in N$ and for the fuzzy preopen Q-pre-neighborhood $U = p_x^{\frac{2}{3}}$ of $p_x^{\frac{1}{2}}$ does not exist any element $n \in N$ such that $n \geq n_0$ and $A_n q U$.

By the above we have

$$F - \overline{\lim_N}(A_n) \neq F - pre - \overline{\lim_N}(A_n).$$

Definition 3.3. *Let* $\{A_n, n \in N\}$ *be a net of fuzzy sets in a fuzzy space* X. *Then by* $F - pre - \underline{\lim_N}(A_n)$, *we denote the fuzzy pre-lower limit of the net* $\{A_n, n \in N\}$ *in* I^X, *that is, the fuzzy set which is the union of all fuzzy points* p_x^λ *in* X *such that for every fuzzy preopen* $Q-$*pre-neighborhood* U *of* p_x^λ *in* X *there exists an element* $n_0 \in N$ *such that* $A_n q U$, *for every* $n \in N$, $n \geq n_0$. *In other cases we set* $F - pre - \underline{\lim_N}(A_n) = \overline{0}$.

Definition 3.4. *A net* $\{A_n, n \in N\}$ *of fuzzy sets in a fuzzy topological space* X *is said to be fuzzy pre-convergent to the fuzzy set* A *if* $F - pre - \underline{\lim_N}(A_n) = F - pre - \overline{\lim_N}(A_n) = A$. *We then write* $F - pre - \lim_N(A_n) = A$.

Proposition 3.5. *Let* $\{A_n, n \in N\}$ *and* $\{B_n, n \in N\}$ *be two nets of fuzzy sets in* X. *Then the following statements are true:*

(1) *The fuzzy pre-upper limit is preclosed.*
(2) $F - pre - \overline{\lim_N}(A_n) = F - \overline{\lim_N}(pCl(A_n))$.
(3) *If* $A_n = A$ *for every* $n \in N$, *then* $F - pre - \overline{\lim_N}(A_n) = pCl(A)$
(4) *The fuzzy upper limit is not affected by changing a finite number of the* A_n.
(5) $F - pre - \overline{\lim_N}(A_n) \leq pCl(\vee\{A_n : n \in N\})$.
(6) *If* $A_n \leq B_n$ *for every* $n \in N$, *then* $F - pre - \overline{\lim_N}(A_n) \leq F - pre - \overline{\lim_N}(B_n)$.
(7) $F - pre - \overline{\lim_N}(A_n \vee B_n) = F - pre - \overline{\lim_N}(A_n) \vee F - pre - \overline{\lim_N}(B_n)$.
(8) $F - pre - \overline{\lim_N}(A_n \wedge B_n) \leq F - pre - \overline{\lim_N}(A_n) \wedge F - pre - \overline{\lim_N}(B_n)$.

Proof. We prove only the statements (1)-(5).

(1) It is sufficient to prove that

$$pCl(F - pre - \overline{\lim_N}(A_n)) \leq F - pre - \overline{\lim_N}(A_n).$$

Let $p_x^r \in pCl(F - pre - \overline{\lim_N}(A_n))$ and let U be an arbitrary fuzzy preopen $Q-$pre-neighborhood of p_y^r. Then, we have:

$$UqF - pre - \overline{\lim_N}(A_n).$$

Hence, there exists an element $x' \in X$ such that

$$U(x') + F - pre - \overline{\lim_N}(A_n)(x') > 1.$$

Let $F - pre - \overline{\lim_N}(A_n)(y') = k$. Then, for the fuzzy point $p_{x'}^k$ in X we have

$$p_{x'}^k \ q \ U \text{ and } p_{x'}^k \in F - pre - \overline{\lim_N}(A_n).$$

Thus, for every element $n_0 \in N$ there exists $n \geq n_0$, $n \in N$ such that $A_n qU$. This means that $p_x^r \in F - pre - \overline{\lim_N}(A_n)$.

(2) Clearly, it is sufficient to prove that for every fuzzy preopen set U the condition UqA_n is equivalent to $UqpCl(A_n)$.

Let UqA_n. Then there exists an element $x \in X$ such that $U(y) + A_n(x) > 1$. Since $A_n \leq pCl(A_n)$ we have $U(x) + pCl(A_n)(x) > 1$ and therefore $UqpCl(A_n)$.

Conversely, let $UqpCl(A_n)$. Then there exists an element $x \in X$ such that $U(x) + pCl(A_n)(x) > 1$. Let $pCl(A_n)(x) = r$. Then $p_x^r \in pCl(A_n)$ and the fuzzy preopen set U is a fuzzy preopen $Q-$pre-neighborhood of p_x^r. Thus UqA_n.

(3) It follows by Proposition 2.4 and the definition of the fuzzy pre-upper limit.

(4) It follows by definition of the fuzzy pre-upper limit.

(5) Let $p_x^r \in F - pre - \overline{\lim_N}(A_n)$ and U be a fuzzy preopen $Q-$pre-neighborhood of p_x^r in X. Then for every $n_0 \in N$ there exists $n \in N$, $n \geq n_0$ such that $A_n qU$ and therefore $\vee\{A_n : n \in N\}qU$. Thus, $p_x^r \in pCl(\vee\{A_n : n \in N\})$. □

Proposition 3.6. *Let $\{A_n, n \in N\}$ and $\{B_n, n \in N\}$ be two nets of fuzzy sets in Y. Then the following statements are true:*

(1) *The fuzzy pre-lower limit is preclosed.*
(2) $F - pre - \underline{\lim}_N(A_n) = F - pre - \underline{\lim}_N(pCl(A_n))$.
(3) *If $A_n = A$ for every $n \in N$, then $F - pre - \underline{\lim}_N(A_n) = pCl(A)$*
(4) *The fuzzy upper limit is not affected by changing a finite number of the A_n.*
(5) $\wedge\{A_n : n \in N\} \leq F - pre - \underline{\lim}_N(A_n)$.
(6) $F - pre - \underline{\lim}_N(A_n) \leq pCl(\vee\{A_n : n \in N\})$.
(7) *If $A_n \leq B_n$ for every $n \in N$, then $F - pre - \underline{\lim}_N(A_n) \leq F - pre - \underline{\lim}_N(B_n)$.*
(8) $F - pre - \underline{\lim}_N(A_n \vee B_n) \geq F - pre - \underline{\lim}_N(A_n) \vee F - pre - \underline{\lim}_N(B_n)$.
(9) $F - pre - \underline{\lim}_N(A_n \wedge B_n) \leq F - pre - \underline{\lim}_N(A_n) \wedge F - pre - \underline{\lim}_N(B_n)$.

Proof. The proof is similar to the proof of Proposition 3.5. □

Proposition 3.7. *For the fuzzy upper and lower limit we have the relation* $F - pre - \underline{\lim}_{N}(A_n) \leq F - pre - \overline{\lim}_{N}(A_n)$.

Proof. It is a consequence of definitions of fuzzy pre-upper and fuzzy pre-lower limits. □

Proposition 3.8. *Let* $\{A_n, n \in N\}$ *and* $\{B_n, n \in N\}$ *be two nets of fuzzy sets in a fuzzy space* Y*. Then the following propositions are true (in the following properties the nets* $\{A_n, n \in N\}$ *and* $\{B_n, n \in N\}$ *are supposed to be fuzzy pre-convergent):*

(1) $pCl(F - pre - \lim_{N}(A_n)) = F - pre - \lim_{N}(A_n) = F - pre - \lim_{N}(pCl(A_n))$.

(2) If $A_n = A$ *for every* $n \in N$*, then* $F - pre - \lim_{N}(A_n) = pCl(A)$

(3) If $A_n \leq B_n$ *for every* $n \in N$*, then* $F - pre - \lim_{N}(A_n) \leq F - pre - \lim_{N}(B_n)$.

(4) $F - pre - \lim_{N}(A_n \vee B_n) = F - pre - \lim_{N}(A_n) \vee F - pre - \lim_{N}(B_n)$.

Proof. The proof of this proposition follows by Propositions 3.5 and 3.6. □

4. Fuzzy pre-continuous functions, fuzzy pre-continuous convergence and fuzzy points

Definition 4.1. *A function* f *from a fuzzy space* Y *into a fuzzy space* Z *is called fuzzy pre-continuous if for every fuzzy point* p_x^λ *in* Y *and every fuzzy preopen* $Q-$*pre-neighborhood* V *of* $f(p_x^\lambda)$*, there exists a fuzzy preopen* $Q-$*pre-neighborhood* U *of* p_x^λ *such that* $f(U) \leq V$.

Let Y *and* Z *be two fuzzy spaces. Then by* $FPC(Y, Z)$ *we denote the set of all fuzzy pre-continuous maps of* Y *into* Z.

Example 4.2. Let (Y, τ_1) and (Y, τ_2) be two fuzzy spaces such that $Y = \{x, y\}$, $\tau_1 = \{\bar{0}, \bar{1}\}$ and $\tau_2 = \{\bar{0}, \bar{1}, p_x^{\frac{1}{2}}\}$.

We consider the map $i : (Y, \tau_1) \to (Y, \tau_2)$ for which $i(z) = z$ for every $z \in Y$.

We prove that the map i is not fuzzy continuous at the fuzzy point $p_x^{0.8}$ but it is fuzzy pre-continuous at the fuzzy point $p_x^{0.8}$.

Indeed, for the fuzzy open Q-neighborhood $V = p_x^{\frac{1}{2}}$ of $i(p_x^{0.8}) = p_x^{0.8}$ does not exist a fuzzy open Q-neighborhood U of $p_x^{0.8}$ such that $i(U) \leq V$. The only fuzzy open Q-neighborhood U of $p_x^{0.8}$ in (Y, τ_1) is the fuzzy set $\bar{1}$ and $i(\bar{1}) \not\leq V$.

Now, we prove that the map i is fuzzy pre-continuous at the fuzzy point $p_x^{0.8}$. Let V be an arbitrary fuzzy preopen $Q-$pre-neighborhood V of $i(p_x^{0.8}) = p_x^{0.8}$.

The family of all fuzzy preopen sets of (Y, τ_2) are the following fuzzy sets V of Y:

i) $V \in I^Y$ such that $V(x) \in [0, \frac{1}{2}]$ and $V(y) = 0$.

ii) $V \in I^Y$ such that $V(x) \in (\frac{1}{2}, 1]$ and $V(y) \in [0, 1]$.

The above fuzzy sets V (cases i) and ii)) are also fuzzy preopen sets of (Y, τ_1). So for every fuzzy preopen $Q-$pre-neighborhood V of $i(p_x^{0.8})$ in (Y, τ_2) there exists the fuzzy preopen $Q-$pre-neighborhood $U = V$ of $p_x^{0.8}$ in (Y, τ_1) such that $i(U) \leq V$.

Definition 4.3. *A fuzzy net $S = \{s_\lambda, \lambda \in \Lambda\}$ in a fuzzy space (X, τ) is said to be p-convergent to a fuzzy point e in X relative to τ and write $p\lim s_\lambda = e$ if for every fuzzy preopen Q-pre-neighborhood U of e and for every $\lambda \in \Lambda$ there exists $m \in \Lambda$ such that Uqs_m and $m \geq \lambda$.*

Proposition 4.4. *A function f from a fuzzy space X into a fuzzy space Y is fuzzy pre-continuous if and only if for every fuzzy net $S = \{s_\lambda, \lambda \in \Lambda\}$, S p-converges to p, then $f \circ S = \{f(s_\lambda), \lambda \in \Lambda\}$ is a fuzzy net in Y and p-converges to $f(p)$.*

Proof. It is obvious. □

Proposition 4.5. *Let $f : Y \to Z$ be a fuzzy pre-continuous map, p be a fuzzy point in Y and U, V be fuzzy preopen $Q-$neighborhoods of p and $f(p)$, respectively such that $f(U) \not\leq V$. Then there exists a fuzzy point p_1 in Y such that $p_1 qU$ and $f(p_1) \not{q} V$.*

Proof. Since $f(U) \not\leq V$. We have $U \not\leq f^{-1}(V)$. Thus there exists $x \in Y$ such that $U(x) > f^{-1}(V)(x)$ or $U(x) - f^{-1}(V)(x) > 0$ and therefore $U(x) + 1 - f^{-1}(V)(x) > 1$ or $U(x) + (f^{-1}(V))^c(x) > 1$. Let $(f^{-1}(V))^c(x) = r$. Clearly, for the fuzzy point p_x^r we have $p_x^r qU$ and $p_x^r \in (f^{-1}(V))^c$. Hence for the fuzzy point $p_1 \equiv p_x^r$ we have $p_1 qU$ and $f(p_1) \not{q} V$. □

Definition 4.6. *A net $\{f_\mu, \mu \in M\}$ in $FPC(Y, Z)$ fuzzy pre-continuously converges to $f \in FPC(Y, Z)$ if for every fuzzy net $\{p_\lambda, \lambda \in \Lambda\}$ in Y which p-converges to a fuzzy point p in Y we have that the fuzzy net $\{f_\mu(p_\lambda), (\lambda, \mu) \in \Lambda \times M\}$ p-converges to the fuzzy point $f(p)$ in Z.*

Proposition 4.7. *A net $\{f_\mu, \mu \in M\}$ in $FPC(Y, Z)$ fuzzy pre-continuously converges to $f \in FC(Y, Z)$ if and only if for every fuzzy point p in Y and for every fuzzy preopen $Q-$pre-neighborhood V of $f(p)$ in Z there exist an element $\mu_0 \in M$ and a fuzzy preopen $Q-$pre-neighborhood U of p in Y such that*

$$f_\mu(U) \leq V,$$

for every $\mu \geq \mu_0$, $\mu \in M$.

Proof. Let p be a fuzzy point in Y and V be a fuzzy preopen $Q-$pre-neighborhood of $f(p)$ in Z such that for every $\mu \in M$ and for every fuzzy preopen $Q-$pre-neighborhood U of p in Y there exists $\mu' \geq \mu$ such that

$$f_{\mu'}(U) \not\leq V.$$

Then for every fuzzy preopen $Q-$neighborhood U of p in Y we can choose a fuzzy point p_U in Y (see Proposition 4.5) such that

$$p_U \ q \ U \text{ and } f_{\mu'}(p_U) \not{q} V.$$

Clearly, the fuzzy net $\{p_U, U \in \mathcal{N}_{Q-p-n}(p)\}$ p-converges to p, but the fuzzy net $\{f_\mu(p_U), (U, \mu) \in \mathcal{N}_{Q-p-n}(p) \times M\}$ does not p-converge to $f(p)$ in Z.

Conversely, let $\{p_\lambda, \lambda \in \Lambda\}$ be a fuzzy net in $FPC(Y,Z)$ which p-converges to the fuzzy point p in Y and let V be an arbitrary fuzzy preopen $Q-$pre-neighborhood of $f(p)$ in Z. By assumption there exist a fuzzy preopen $Q-$pre-neighborhood U of p in Y and an element $\mu_0 \in M$ such that $f_\mu(U) \leq V$, for every $\mu \geq \mu_0$, $\mu \in M$. Since the fuzzy net $\{p_\lambda, \lambda \in \Lambda\}$ p-converges to p in Y. There exists $\lambda_0 \in \Lambda$ such that $p_\lambda q U$, for every $\lambda \in \Lambda$, $\lambda \geq \lambda_0$. Let $(\lambda_0, \mu_0) \in \Lambda \times M$. Then for every $(\lambda, \mu) \in \Lambda \times M$, $(\lambda, \mu) \geq (\lambda_0, \mu_0)$ we have $f_\mu(p_\lambda)\ q\ f_\mu(U) \leq V$, that is $f_\mu(p_\lambda)\ q\ V$. Thus the net $\{f_\mu(p_\lambda), (\lambda, \mu) \in \Lambda \times M\}$ p-converges to $f(p)$ and the net $\{f_\mu, \mu \in M\}$ fuzzy pre-continuously converges to f. □

Proposition 4.8. *A net $\{f_\lambda, \lambda \in \Lambda\}$ in $FPC(Y,Z)$ fuzzy pre-continuously converges to $f \in FPC(Y,Z)$ if and only if*

$$F - pre - \overline{\lim_{\Lambda}}(f_\lambda^{-1}(K)) \leq f^{-1}(K), \tag{1}$$

for every fuzzy preclosed subset K of Z.

Proof. Let $\{f_\lambda,\ \lambda \in \Lambda\}$ be a net in $FPC(Y,Z)$, which fuzzy pre-continuously converges to f and let K be an arbitrary fuzzy preclosed subset of Z. Let $p \in F- pre - \overline{\lim_{\Lambda}}(f_\lambda^{-1}(K))$ and W be an arbitrary fuzzy preopen $Q-$pre-neighborhood of $f(p)$ in Z. Since the net $\{f_\lambda,\ \lambda \in \Lambda\}$ fuzzy pre-continuously converges to f, there exist a fuzzy preopen $Q-$pre-neighborhood V of p in Y and an element $\lambda_0 \in \Lambda$ such that $f_\lambda(V) \leq W$, for every $\lambda \in \Lambda$, $\lambda \geq \lambda_0$. On the other hand, there exists an element $\lambda \geq \lambda_0$ such that $V q f_\lambda^{-1}(K)$. Hence, $f_\lambda(V) q K$ and therefore $W q K$. This means that $f(p) \in pCl(K) = K$. Thus, $p \in f^{-1}(K)$.

Conversely, let $\{f_\lambda,\ \lambda \in \Lambda\}$ be a net in $FPC(Y,Z)$ and $f \in FPC(Y,Z)$ such that the relation (1) holds for every fuzzy preclosed subset K of Z. We prove that the net $\{f_\lambda,\ \lambda \in \Lambda\}$ fuzzy continuously converges to f. Let p be a fuzzy point of Y and W be a fuzzy preopen $Q-$pre-neighborhood of $f(p)$ in Z. Since $p \notin f^{-1}(K)$, where $K = W^c$ we have $p \notin F - pre - \overline{\lim_{\Lambda}}(f_\lambda^{-1}(K))$. This means that there exist an element $\lambda_0 \in \Lambda$ and a fuzzy preopen $Q-$pre-neighborhood V of p in Y such that $f_\lambda^{-1}(K)\ q\!\!\!/\ V$, for every $\lambda \in \Lambda$, $\lambda \geq \lambda_0$. Then we have $V \leq (f_\lambda^{-1}(K))^c = f_\lambda^{-1}(K^c) = f_\lambda^{-1}(W)$. Therefore, $f_\lambda(V) \leq W$, for every $\lambda \in \Lambda$, $\lambda \geq \lambda_0$, that is the net $\{f_\lambda,\ \lambda \in \Lambda\}$ fuzzy pre-continuously converges to f. □

Proposition 4.9. *The following statements are true:*

1. *If $\{f_\lambda, \lambda \in \Lambda\}$ is a net in $FPC(Y,Z)$ such that $f_\lambda = f$, for every $\lambda \in \Lambda$, then the $\{f_\lambda, \lambda \in \Lambda\}$ fuzzy pre-continuously converges to $f \in FPC(Y,Z)$.*
2. *If $\{f_\lambda, \lambda \in \Lambda\}$ is a net in $FPC(Y,Z)$, which fuzzy pre-continuously converges to $f \in FPC(Y,Z)$ and $\{g_\mu, \mu \in M\}$ is a subnet of $\{f_\lambda, \lambda \in \Lambda\}$, then the net $\{g_\mu, \mu \in M\}$ fuzzy pre-continuously converges to f.*

Acknowledgements. S. P. Moshokoa has been supported by the South African National Research Foundation under grant number 2053847.

Also, the authors thank the referee for making several suggestions which improved the quality of this paper.

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Received September 2004

Accepted January 2005

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