Applied General Topology
© Universidad Politécnica de Valencia
Volume 5, No. 2, 2004
pp. 231-242

Which topologies can have immediate successors in the lattice of T_1-topologies?

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Abstract. We give a new characterization of those topologies which have an immediate successor or *cover* in the lattice of T_1-topologies on a set and show that certain classes of compact and countably compact topologies do not have covers.

Keywords: lattice of T_1-topologies, maximal point, KC-space, cover of a topology, upper topology, lower topology

2000 AMS Classification: Primary 54A10; Secondary 06A06

1. Introduction and Preliminary Results

The lattice $\mathcal{L}_1(X)$ of T_1-topologies on a set X has been studied, with differing emphasis, by many authors. Most articles on the subject have considered complementation in the lattice and properties of mutually complementary spaces (see for example, [12, 13, 15]. Here we consider a facet of the order structure of the lattice $\mathcal{L}_1(X)$, specifically the problem of when a jump can occur in the order; that is to say, when there exist topologies τ and τ^+ on a set X such that whenever μ is a topology on X such that $\tau \subseteq \mu \subseteq \tau^+$ then $\mu = \tau$ or $\mu = \tau^+$. The existence of jumps in $\mathcal{L}_1(X)$ has been studied in [10] and [16] where the immediate successor τ^+ was said to be *a cover of* (or simply *to cover*) τ. We prefer order-theoretic terminology and then call τ a *lower topology* for X; the topology τ^+ will then be termed an *upper topology*. That such topologies exist was noted in both of the previously mentioned papers - the best known example is the topology of the space Σ of Problem 4M of [7], called an *ultratopology* in [10] and [16], which is an anti-atom of $\mathcal{L}_1(\omega)$ (for lattice-theoretic terminology we refer the reader to the survey paper [9]). However, topologies which are dense-in-themselves can also be lower, a Tychonoff example is the maximal space constructed in [4]. A characterization of lower topologies was given in

*Research supported by Consejo Nacional de Ciencia y Tecnología (México), grant 38164-E and Fundação de Amparo a Pesquisa do Estado de São Paulo (Brasil)

Theorem 1 of [10] in terms of equivalence classes of locally equal sets. Here we give a somewhat different characterization of lower topologies which we find much simpler and easier to work with, but we note that Lemma 1.2 and Corollary 1.3 can, with a little effort, be derived from Section 2 of [10]. Corollary 1 to Theorem 10 in [16] states that no first countable Hausdorff topology on X has a cover in $\mathcal{L}_1(X)$, thus generalizing Example 2 of [10] where this same result was proved for the space of real numbers with the usual metric topology. However, these papers do not address further the problem: Which topologies can have, and which classes of topologies do not have, covers in $\mathcal{L}_1(X)$? This is the problem we study below.

All spaces considered below are (at least) T_1 and undefined topological notation and terminology can be found in [6]. The closure (respectively, interior) of a set A in a topological space (X,τ) will be denoted by $\mathrm{cl}_\tau(A)$ (respectively, $\mathrm{int}_\tau(A)$) or simply by $\mathrm{cl}(A)$ (respectively $\mathrm{int}(A)$) if no confusion is possible. If $\mathcal{S}$ is a family of sets, then we use the notation $<\mathcal{S}>$ to denote the topology generated by $\mathcal{S}$ as a subbase and the symbol $\subsetneq$ is used to denote proper containment.

We call a point p a *maximal point* of a topological space (X,τ) if it is not isolated and whenever U is open in X and $p \in \mathrm{cl}_\tau(U)$, then $U \cup \{p\} \in \tau$. It is clear that if p is a maximal point of X and p is an accumulation point of $A \subseteq X$, then p is a maximal point of $A \cup \{p\}$. Using the well-known fact (see for example Problem 12G of [18]) that an open filter $\mathcal{F}$ is an ultrafilter if and only if whenever $U \in \tau$, then $U \in \mathcal{F}$ or $X \setminus \mathrm{cl}_\tau(U) \in \mathcal{F}$, it is an easy exercise to check that p is a maximal point of X if and only if the trace of the open neighbourhood system at p on $X \setminus \{p\}$ is an open ultrafilter in the open sets of this latter space. We formulate this fact as:

Proposition 1.1. *A point $p \in X$ is a maximal point if and only if the trace of the open neighbourhood filter at p on the subspace $X \setminus \{p\}$ is an open ultrafilter.*

We note that a maximal point is also a *1-point* in the sense of [17] but not vice versa. It is also true that in a space which is maximal in the sense of [4], that is to say, is maximal with respect to being dense-in-itself, every point is a maximal point, but again the converse is false (the cofinite topology on a countable set is the requisite counterexample). Our first task is to give a new characterization of lower topologies in terms of maximal points.

Lemma 1.2. *Let (X,τ) be a T_1-space; if τ has an immediate successor, which we denote by τ^+, then there is $p \in X$ and $U \in \tau$ such that $U \cup \{p\} \notin \tau$ and $\tau^+ = <\tau \cup \{U \cup \{p\}\}>$*

Proof. First note that if $\tau \subsetneq \sigma$, then we can find $V \in \sigma \setminus \tau$ and hence the topology τ^* generated by $\tau \cup \{V\}$ is such that $\tau \subsetneq \tau^* \subseteq \sigma$. Thus if τ^+ exists it must have the form $<\tau \cup \{V\}>$ for some $V \notin \tau$.

We now claim that the set V defined in the previous paragraph must have the form $U \cup \{p\}$ for some $p \in X \setminus U$ and $U \in \tau$; again let $\sigma = <\tau \cup \{V\}>$. To prove our claim, let $U = \mathrm{int}_\tau(V)$; then if $U = V$, it follows that $V \in \tau$ and hence

$\sigma = \tau$. Thus we suppose $U \subsetneq V$. If $|V \setminus U| \geq 2$, then we can choose distinct points $p, q \in V \setminus U$ and since $p \notin \mathrm{int}_\tau(V)$ it is clear that $V \setminus \{q\} \notin \tau$. We consider the topology $\tau^* =< \tau \cup \{V \setminus \{q\}\} >$; since X is T_1 and $V \setminus \{q\} = V \cap (X \setminus \{q\})$, it follows that $V \setminus \{q\} \in \sigma$. However, since $V \notin \tau^*$ we have that $\tau \subsetneq \tau^* \subsetneq \sigma$, showing that σ is not an immediate successor of τ. Thus $|V \setminus U| = 1$ and we are done. □

To simplify the notation somewhat, we denote the topology $< \tau \cup \{U \cup \{p\}\} >$ by $\tau_{U(p)}$ or simply by τ_U whenever p is understood to be fixed.

Corollary 1.3. *A topology τ on X is a lower topology if and only if there is $p \in X$ and $U \in \tau$ such that whenever $V \in \tau$ is such that $\tau_U = \tau_{U \cap V}$ then either $\tau_V = \tau_U$ or $\tau_V = \tau$.*

Proof. Suppose that τ is a lower topology, whose immediate successor we again denote by τ^+. By Lemma 1.2, there exist $p \in X$ and $U \in \tau$ such that $\tau^+ = \tau_{U(p)} (= \tau_U)$. But then, if $V \in \tau$ is such that $\tau_{U \cap V} = \tau_V$, then $\tau \subseteq \tau_V$ and since $V \cup \{p\} = V \cup [(U \cap V) \cup \{p\}]$ it follows that $\tau_V \subseteq \tau_{U \cap V} = \tau_U$ and hence $\tau_V = \tau_U$ or $\tau_V = \tau$.

Conversely, suppose that τ is not a lower topology and let $p \in X$ be fixed. If $U \in \tau$, then there is some topology σ on X such that $\tau \subsetneq \sigma \subsetneq \tau_U$. It is clear that there is then some $V \in \tau$ such that $\sigma = \tau_V$ and then since $V \cup \{p\} \in \tau_U$, it follows that $(U \cap V) \cup \{p\} \in \tau_U$ and hence $\tau_{U \cap V} = \tau_U$, a contradiction. □

Let D be a directed set and $f : D \to X$ a net which is finally in $U \cup V$. If f is cofinally in both U and V and $D_U = \{\alpha \in D : f(\alpha) \in U\}$ and $D_V = \{\alpha \in D : f(\alpha) \in V\}$ then both D_U and D_V are directed sets and if the nets $f_U = f|D_U$ and $f_V = f|D_V$ both converge to the same point p, then f also converges to p. We will make use of this simple result in the next theorem.

Theorem 1.4. *A topology τ is a lower topology on X if and only if (X, τ) has a closed subspace with a maximal point.*

Proof. For the necessity, we suppose that no closed subspace of (X, τ) has a maximal point. Then for all $W \in \tau$ and $p \in X \setminus W$, p is not a maximal point of $X \setminus W$. However, if τ were a lower topology, then there would exist $U \in \tau$ and $p \in X$ such that $\tau^+ = \tau_{U(p)} \neq \tau$. For the rest of this paragraph, we consider p fixed and write $\tau_{U(p)} = \tau_U$. Since p is not a maximal point of $X \setminus U$, there is some open set $V' \in \tau|(X \setminus U)$ such that $V' \cup \{p\} \notin \tau|(X \setminus U)$ and $p \in \mathrm{cl}_\tau(V')$; then $p \in \mathrm{cl}_\tau[(X \setminus U) \setminus (V' \cup \{p\})]$. Let $V \in \tau$ be such that $V \cap (X \setminus U) = V'$. We claim that $\tau \subsetneq \tau_{U \cup V} \subsetneq \tau_U$. That $\tau \subseteq \tau_{U \cup V} \subseteq \tau_U$ is clear and to see that
($\imath$) $\tau \neq \tau_{U \cup V}$, note that $p \notin \mathrm{cl}_{\tau_{U \cup V}}[(X \setminus U) \setminus (V \cup \{p\})]$.
($\imath\imath$) $\tau_{U \cup V} \neq \tau_U$, note that p is not isolated in $\tau_{U \cup V}|(X \setminus U)$.

For the converse, suppose that p is a maximal point of the closed subspace $C \subseteq X$ and let $X \setminus C = U \in \tau$. Let $V \in \tau$ be such that $\tau_{U \cap V} = \tau_U$, and hence $\tau_U = \tau_{U \cap V} \supseteq \tau_V \supseteq \tau$; then by Corollary 1.3, it suffices to show that $\tau_V = \tau$ or $\tau_V = \tau_U$. There are two cases to consider.

(1) If $p \notin \mathrm{cl}_\tau(V \cap C)$, then a net f that converges to p in the topology τ_V is finally in each τ-open set containing p and
(a) Finally in $V \cup \{p\}$, and
(b) Finally outside of $V \cap C = V \setminus U$ since $C \setminus \mathrm{cl}_\tau(V \cap C)$ is an open set in C which contains p.

Thus the net f is finally in $(V \cap U) \cup \{p\}$ and in each τ-neighbourhood of p and hence converges to p in $\tau_{U \cap V} = \tau_U$. Since τ_U and τ_V coincide on $X \setminus \{p\}$, it follows that $\tau_U = \tau_V$.

(2) If $p \in \mathrm{cl}_\tau(V \cap C)$, then since p is a maximal point of C, $(V \cap C) \cup \{p\} \in \tau|C$ and hence there is $W \in \tau$ such that $p \in W$ and $W \cap C = (V \cap C) \cup \{p\}$, that is, $(V \setminus U) \cup \{p\} = W \setminus U$. But then, $p \in W \subseteq (U \cup V) \cup \{p\}$ and hence $(U \cup V) \cup \{p\} \in \tau$. Since τ and $\tau_{U \cup V}$ coincide on $X \setminus \{p\}$, it follows that $\tau = \tau_{U \cup V}$. Thus a net f which converges to p in τ is finally in each τ-neighbourhood of p, (including $(U \cup V) \cup \{p\}$) and then either,
(c) It is finally in $U \cup \{p\}$ and hence converges in τ_U and then also in τ_V since $\tau_V \subseteq \tau_U$, or
(d) It is finally in $V \cup \{p\}$ and hence converges in τ_V, or
(e) It is cofinally in both $U \cup \{p\}$ and $V \cup \{p\}$. Using the notation introduced in the paragraph prior to this theorem, the net f_U converges to p in τ_U and hence also in τ_V and the net f_V converges to p in τ_V. Thus the net f converges to p in τ_V and so $\tau = \tau_V$. □

On the other hand, τ can be a lower topology on X but have no maximal point. Let p be a free ultrafilter on ω and let $\Sigma = \omega \cup \{p\}$ be the space of Problem 4M of [7]. Let (X, τ) be the quotient space obtained by identifying the point $p \in \Sigma$ with the unique accumulation point of a convergent sequence. That τ is a lower topology follows from Theorem 1.4 and the fact that p is a maximal point of the closed subspace Σ, but the space (X, τ) has no 1-point.

Corollary 1.5. *If a topology σ on a set X has the property that for some infinite closed subspace $D \subseteq X$, $\sigma|D$ is a lower (respectively, upper) topology on D, then σ is a lower (respectively upper) topology on X.*

At this point, having mentioned upper topologies for the first time, the reader might wonder why we have fixed our attention so exclusively on *lower* topologies since to every lower topology there corresponds at least one *upper* topology. However, as we shall see, unlike lower topologies, upper topologies are abundant. Recall from [11] that a space is *hyperconnected* if it contains no disjoint non-empty open sets, or equivalently, if every non-empty open set is dense. Clearly no Hausdorff space with at least two points is hyperconnected. If σ is the cofinite topology on a set X, then (X, σ) is hyperconnected, but other examples of hyperconnected spaces are easy to construct. However, we have the following simple result which we feel sure must be known.

Lemma 1.6. *A T_1-topology σ on a set X is the cofinite topology if and only if every infinite closed subspace of (X, σ) is hyperconnected.*

Proof. The necessity is clear. For the sufficiency, suppose that σ is not the cofinite topology on X; then there is some infinite closed proper subset $C \subsetneq X$. Let $p \in X \setminus C$, then the infinite closed subspace $C \cup \{p\}$ is not hyperconnected. □

Proposition 1.7. *A topology σ is not an upper topology on an infinite set X if and only if σ is the cofinite topology.*

Proof. The sufficiency is clear, since the cofinite topology is the minimal T_1-topology on a set.

For the necessity, suppose first that there is a non-empty regular closed set C in (X, σ) whose complement is infinite and let $p \in \mathrm{int}_\sigma(C)$. Let $\mathcal{U}$ be an open ultrafilter on the infinite open subspace $X \setminus C$ such that $\cap\mathcal{U} = \varnothing$ and define a topology τ on X by

$$\begin{aligned} &V \in \tau \text{ if and only if } V \in \sigma \text{ and whenever } p \in V, \\ &\qquad\qquad \text{then there exists } U \in \mathcal{U} \text{ such that } U \subseteq V. \end{aligned}$$

Clearly $\sigma|(X \setminus \{p\}) = \tau|(X \setminus \{p\})$ and p is a maximal point (in the topology τ) of the τ-closed set $(X \setminus \mathrm{int}_\sigma(C)) \cup \{p\}$ and hence it follows from Theorem 1.4 that τ is a lower topology. Furthermore, if we denote by W the set $\mathrm{int}_\sigma(C) \setminus \{p\}$, then $W \in \tau$ and we see from the proof of Theorem 1.4 that $\sigma = \tau^+ = \tau_{W(p)}$.

Thus if σ is not an upper topology, then the closure of every non-empty element of σ has finite complement. Now suppose that $\varnothing \neq U \in \sigma$; if the finite open set $X \setminus \mathrm{cl}_\sigma(U) \neq \varnothing$ for some $U \in \sigma$, each point $q \in X \setminus \mathrm{cl}_\sigma(U)$ is isolated and so $\{q\}$ is a regular closed set with infinite complement, a contradiction. Thus $\mathrm{cl}_\sigma(U) = X$, that is U is dense in X and so X is hyperconnected.

However, from Corollary 1.5, we have that if σ is not an upper topology on X, then it is not an upper topology on any infinite closed subset of X. Hence every infinite closed subspace of X is hyperconnected. The result now follows from Lemma 1.6. □

It is not hard to see that in Proposition 1.7, if (X, σ) is Hausdorff but not H-closed, then τ can be chosen to be Hausdorff as well.

2. Topologies which cannot be lower

It is now clear that if P is a topological property which is inherited by closed subspaces, then the class of topologies with property P can contain a lower topology if and only if there is a member of the class with a maximal point. We now consider the problem of which classes of topologies do not contain a lower topology. As mentioned earlier, the best known examples of lower topologies, Σ and van Douwen's maximal space are far from being compact. That this is not a coincidence is shown in the next theorem. Recall that a topological space is a *KC-space* if all compact subsets are closed. Hausdorff spaces are KC and KC-spaces are necessarily T_1. Our main results below show that KC-spaces with "nice" covering and convergence properties cannot be lower topologies. However, these results fail in case the KC separation axiom is weakened to T_1.

Theorem 2.1. *A compact KC-space cannot have a maximal point.*

Proof. Let (X, τ) be a compact KC-space; we assume to begin with that X is dense-in-itself. Suppose to the contrary that p is a maximal point of X and let

$$K = \cap\{\mathrm{cl}(V) : p \in V \in \tau\}.$$

Clearly, K is compact and $p \in K$. Let λ be the minimal cardinal such that $K = \cap\{\mathrm{cl}(V_\alpha) : \alpha \in \lambda,\ p \in V_\alpha \in \tau\}$ and let $F_\beta = \cap\{\mathrm{cl}(V_\alpha) : \alpha \leq \beta\}$.

Proceeding as in Lemma 2.3 of [1], we can find a discrete subset D in X such that $p \notin D$ and $\mathrm{cl}(D) \cap F_\alpha \neq \varnothing$ for each $\alpha \in \lambda$. Since X is compact, it follows that $\mathrm{cl}(D) \cap K \neq \varnothing$.

There are now two possibilities:

($\imath$) $K = \{p\}$ and hence $p \in \mathrm{cl}(D)$. Since X has no isolated points it follows that $\mathrm{cl}(D)$ is nowhere dense and hence $U = X \setminus \mathrm{cl}(D)$ is a dense open set. Since p is maximal, $U \cup \{p\}$ is open, which contradicts the fact that $p \in \mathrm{cl}(D)$.

($\imath\imath$) If $K \setminus \{p\} \neq \varnothing$, then choose $q \in K \setminus \{p\}$. If $\mathcal{F}$ is an open covering of $X \setminus \{p\}$, then there is $U \in \mathcal{F}$ such that $q \in U$ and since p and q do not have disjoint neighbourhoods, $p \in \mathrm{cl}(U)$. Since p is maximal, $U \cup \{p\}$ is open and hence $\mathcal{F} \cup \{U \cup \{p\}\}$ is an open covering of the compact space X, which then has a finite subcovering. Since $U \in \mathcal{F}$, this clearly induces a finite subcovering of $\mathcal{F}$ of $X \setminus \{p\}$, showing that this latter space is compact. Since X is KC, p must be an isolated point of X, a contradiction.

Now suppose that X is an arbitrary compact KC-space, $p \in X$ and let D be the set of isolated points of X. If $p \notin \mathrm{cl}(D)$, then there is some neighbourhood U of p which is dense-in-itself and hence p is a maximal point of $\mathrm{cl}(U)$ which is a dense-in-itself compact KC-space. That p is not maximal now follows from the previous argument.

If $p \in \mathrm{cl}(D)$, then if p were a maximal point of X it would also be a maximal point of $\mathrm{cl}(D)$ and hence without loss of generality we assume that $X = \mathrm{cl}(D)$. Since p is maximal, it follows that $D \cup \{p\}$ is open. Let $K = \cap\{\mathrm{cl}(V) : V$ is an open neighbourhood of $p\}$.

If $K \setminus \{p\} \neq \varnothing$, then as in ($\imath\imath$) above, p is an isolated point of X, a contradiction.

If on the other hand $K = \{p\}$, then p can be separated from every point $q \in X$ by disjoint open sets. Since p is an isolated point of $X \setminus D$, it follows that $(X \setminus D) \setminus \{p\}$ is compact and a standard argument now shows that p and $(X \setminus D) \setminus \{p\}$ can be separated by open sets. Thus there is some closed, hence compact neighbourhood V of p which misses $(X \setminus D) \setminus \{p\}$. Thus V is a compact KC-space whose only accumulation point is a maximal point. This is clearly seen to be impossible and we are done. □

Corollary 2.2. *A compact KC-topology is not a lower topology.*

In contrast with Corollary 2.2, the topology τ of the one-point compactification X of the space Σ is a lower topology and the cofinite topology on ω is even a first countable compact T_1-topology in which every point is a maximal point.

If (X,τ) is a Hausdorff space, then for each $p \in X$, set $\psi_c(p,X) = \min\{|\mathcal{U}| : \mathcal{U} \subseteq \tau,\ p \in U$ for all $U \in \mathcal{U}$ and $\bigcap\{\mathrm{cl}(U) : U \in \mathcal{U}\} = \{p\}\}$; as in [8], we then define the *closed pseudocharacter* of X, $\psi_c(X) = \sup\{\psi_c(p,X) : p \in X\}$.

Recall that if λ is a cardinal, then a space X is said to be *initially λ-compact* if every open cover of size at most λ has a finite subcover. It is a trivial exercise to show that if X is initially λ-compact, then every decreasing sequence of length $\leq \lambda$ of closed sets in X has non-empty intersection. Furthermore, an initially λ-compact Hausdorff space X with $\psi_c(X) \leq \lambda$ is regular (see Theorems 2.2 and 3.4 of [14]). The next theorem has a proof very similar to that of Theorem 2.1, we mention only the minor points of difference.

Theorem 2.3. *An initially λ-compact Hausdorff space X with $\psi_c(X) \leq \lambda$ cannot have a maximal point.*

Proof. Again we begin by assuming that X is dense-in-itself, p is a maximal point of X and $\kappa \leq \lambda$ is the minimal cardinal such that there exists a family $\{V_\alpha : \alpha \in \kappa\}$ of open sets such that $\{p\} = \cap\{\mathrm{cl}(V_\alpha) : \alpha \in \kappa\}$. Using exactly the same notation as that of Theorem 2.1 and applying Lemma 2.3 of [1], we can find a discrete set $D \subseteq X \setminus \{p\}$ such that $\mathrm{cl}(D) \cap F_\alpha \neq \varnothing$ for each $\alpha \in \kappa$. We note that since X is initially λ-compact, it is κ-compact and hence $\varnothing \neq \bigcap\{\mathrm{cl}(D) \cap F_\alpha : \alpha \in \kappa\} \subseteq \{p\}$ and so $p \in \mathrm{cl}(D)$. The rest of the proof proceeds exactly as in Theorem 2.1, after noting that X is, in fact, a T_3-space. $\square$

A space X is said to be *weakly discretely generated* if whenever $A \subsetneq X$ is not closed, then there is some discrete subset $D \subseteq A$ such that $\mathrm{cl}(D) \setminus A \neq \varnothing$. The space X is *discretely generated* if whenever $x \in \mathrm{cl}(A)$ there is a discrete subspace $D \subseteq A$ such that $x \in \mathrm{cl}(D)$; being discretely generated is a hereditary property. Clearly, a discretely generated space is weakly discretely generated and it was shown in Proposition 3.1 of [5] that every compact Hausdorff space is weakly discretely generated. A similar proof applying Lemma 2.3 of [1] can be used to show that compact KC-spaces are weakly discretely generated. It was further shown in [2] (and implicitly earlier in [3]) that a regular countably compact space with countable tightness is discretely generated. Our next lemma generalizes this last result and has a similar proof. We say that a space X is *locally countably compact* if each point $p \in X$ has a local base of countably compact neighbourhoods. A countably compact T_3-space is locally countably compact, but this is not the case for Hausdorff spaces as we shall see later. It is easy to see that being locally countably compact is both open and closed hereditary.

Lemma 2.4. *A locally countably compact Hausdorff space X with countable tightness is discretely generated.*

Proof. Let $A \subseteq X$ be a set which is not closed and $y \in \mathrm{cl}(A) \setminus A$. Since X has countable tightness, we may assume that $A = \{a_n : n \in \omega\}$ is countable. Our aim is to construct a discrete subset $D \subseteq A$ such that $y \in \mathrm{cl}(D)$. Since

X is Hausdorff, for each $n \in \omega$ there is an open neighbourhood U_n of y such that $a_n \notin \mathrm{cl}(U_n)$. We assume that $U_{n+1} \subseteq U_n$ for each $n \in \omega$ and clearly $\bigcap\{\mathrm{cl}(U_n) : n \in \omega\} \cap A = \varnothing$. Now let S be the set of all cluster points of sets of the form $\{z_n : n \in \omega\}$ where $z_n \in U_n \cap A$; we claim that $y \in \mathrm{cl}(S)$. To see this, let W be a countably compact neighbourhood of y and for each $n \in \omega$, pick $z_n \in W \cap U_n \cap A$. Each accumulation point of $\{z_n : n \in \omega\}$ lies in S, and since W is countably compact at least one of them also lies in W; hence $W \cap S \neq \varnothing$. Again since X has countable tightness, there is a countable subset $\{s_n : n \in \omega\} \subseteq S$ such that $y \in \mathrm{cl}(\{s_n : n \in \omega\})$ and for each $n \in \omega$ we choose a sequence $\{z_n^k : n \in \omega\} \subseteq A$ which witnesses the fact that $s_k \in S$, that is to say, $z_n^k \in U_n \cap A$ and $s_k \in \mathrm{cl}(\{z_n^k : n \in \omega\})$. Now let $D = \{z_n^k : k, n \in \omega \text{ and } k \leq n\}$. It is clear that $s_k \in \mathrm{cl}(D)$ for each $k \in \omega$ and hence $y \in \mathrm{cl}(D)$. To see that D is discrete, fix $k, n \in \omega$; then there is some $m \geq n$ such that $z_n^k \notin \mathrm{cl}(U_m)$ and since all but finitely many elements of D lie in U_m, z_n^k is not an accumulation point of D. □

Theorem 2.5. *A locally countably compact Hausdorff topology of countable tightness is not a lower topology.*

Proof. Let (X, τ) be a locally countably compact Hausdorff space and suppose that τ has an immediate successor which we denote by σ. Then there is some $U \in \tau$ and $p \in X$ such that $\sigma =< \tau \cup \{U \cup \{p\}\} >$. Furthermore, since the property of being locally countably compact is inherited by closed subspaces, we may assume that p is a maximal point of X. Let $S = X \setminus U$ and note that p is an accumulation point of the closed subspace S, for otherwise we have $U \cup \{p\} \in \tau$ and hence $\sigma = \tau$. Furthermore, by the previous lemma, S is discretely generated. There are two possibilities to consider.

1) The point p is an accumulation point of some dense-in-itself closed subset C of S. Since X is discretely generated, there is some discrete subset D in $C \setminus \{p\}$ such that $p \in \mathrm{cl}_\tau(D)$. Furthermore, since C is dense-in-itself, $T = \mathrm{cl}_\tau(D)$ is nowhere dense in C, hence also in S, and so $p \in \mathrm{cl}_\tau(S \setminus T)$. Now define $\tau^\# =< \tau \cup \{(X \setminus T) \cup \{p\} >$; since $p \in \mathrm{cl}_\tau(T \setminus \{p\})$ it follows immediately that $T \setminus \{p\}$ is not closed in τ, but is closed in $\tau^\#$ and hence $\tau \subsetneq \tau^\#$. Furthermore, since $(X \setminus T) \cup \{p\} = ((X \setminus S) \cup \{p\}) \cup (X \setminus T)$ it follows that $(X \setminus T) \cup \{p\} \in \sigma$, showing that $\tau^\# \subseteq \sigma$. Finally, since each $\tau^\#$-neighbourhood of p meets $S \setminus T$, but the σ-neighbourhood $U \cup \{p\}$ of p misses $S \setminus T$, we have $p \in \mathrm{cl}_{\tau^\#}(S \setminus T) \setminus \mathrm{cl}_\sigma(S \setminus T)$ and it then follows that $\tau^\# \neq \sigma$, contradicting the supposition that σ is the immediate successor of τ.

2) Suppose now that p is not in the closure of any dense-in-itself subset of S. If S is not scattered, then let $\mathcal{G}$ be the family of all dense-in-themselves subsets of S ordered by set inclusion. Clearly the union of any chain of elements of $\mathcal{G}$ is dense-in-itself, and hence $\mathcal{G}$ has a maximal element G say, which clearly must be closed and $p \notin G$. Since S is closed in X, it is locally countably compact and so there is a countably compact neighbourhood W of p in S missing G. Then p is an accumulation point of the countably compact scattered space $W \subseteq S$. Suppose that the scattering length of W is κ and for each $\alpha < \kappa$, let W_α denote

the set of points of scattering order α; since p is not an isolated point of W, $p \in W_\alpha$ for some $\alpha > 0$. There are two subcases to consider:
a) If $p \in W_\alpha$ where $\alpha > 1$, then let $T = W \setminus W_0$. It is clear that T is a closed subspace of W and hence $T = \mathrm{cl}_\tau(T) \cap W$, p is an accumulation point of T and $W \setminus T$ is dense in W. Now define $\tau^\# =< \tau \cup \{(X \setminus \mathrm{cl}_\tau(T)) \cup \{p\} >$. Obviously $\mathrm{cl}_\tau(T) \setminus \{p\}$ is closed in $\tau^\#$ but not in τ and hence $\tau \subsetneq \tau^\#$. Furthermore, $(X \setminus \mathrm{cl}_\tau(T)) \cup \{p\} = (X \setminus S) \cup \{p\} \cup (X \setminus \mathrm{cl}_\tau(T)) \in \tau$ and so $\tau^\# \subseteq \sigma$. To show that $\sigma \neq \tau^\#$ we will show that $p \in \mathrm{cl}_{\tau^\#}(S \setminus T) \setminus \mathrm{cl}_\sigma(S \setminus T)$. That $p \notin \mathrm{cl}_\sigma(S \setminus T)$ is clear since $U \cup \{p\} \in \sigma$ is a neighbourhood of p and $(S \setminus T) \cap (U \cup \{p\}) = \varnothing$. On the other hand, an open $\tau^\#$-neighbourhood of p is of the form $(V \setminus \mathrm{cl}_\tau(T)) \cup \{p\}$ where $p \in V \in \tau$ and a short calculation now shows that $(V \setminus \mathrm{cl}_\tau(T)) \cup \{p\} \cap (S \setminus T) = V \setminus (U \cup \mathrm{cl}_\tau(T))$. This latter set is nonempty since if $V \subseteq U \cup \mathrm{cl}_\tau(T)$, then $V \cap W \subseteq \mathrm{cl}_\tau(T) \cap W = T$ which is a contradiction, since each open set in W meets W_0. Thus we have again shown that σ is not the immediate successor of τ.
b) If $p \in W_1$, then $W_0 \cup W_1$ is an open neighbourhood of p in W and since W is a neighbourhood of p in S, we have $p \in \mathrm{int}_{\tau|S}(W_0 \cup W_1)$ and this latter set, being an open subset of S, is locally countably compact. Thus there is some countably compact neighbourhood V of p in which p is the only accumulation point. Clearly V is compact and it then follows from Theorem 2.1 that p is not a maximal point of V, a contradiction. □

Question 2.6. *Can a countably compact Hausdorff topology with countable tightness (or a countably compact regular topology) be a lower topology?*

We note however that a countably compact Hausdorff topology may be a lower topology and need not be locally countably compact. Let $p \in \beta\omega \setminus \omega$ and let $\mathcal{V}$ be the filter of open neighbourhoods of p. Denoting by τ the usual topology of $\beta\omega$, we define $\sigma =< \tau \cup \{(U \cap \omega) \cup \{p\} : U \in \mathcal{V}\} >$. It is straightforward to check that p is a maximal point of the countably compact Urysohn space $(\beta\omega, \sigma)$. This latter space is even H-closed (see 3.12.5 of [6]) and its semiregularization is $(\beta\omega, \tau)$.

Our next theorem generalizes Corollary 1 to Theorem 10 of [16].

Theorem 2.7. *A sequential KC-space is not a lower topology.*

Proof. Suppose (X, τ) is a sequential KC-space and $\tau \subsetneq \sigma$. Then there is some set $V \subseteq X$ which is closed in (X, σ) but not closed in (X, τ). Since this latter space is sequential, there is a sequence $S = \{b_n\}_{n \in \omega}$ in V which converges to $b \notin V$. Since (X, τ) is KC, $S \cup \{b\}$ is τ-closed and hence σ-closed and so S is σ-closed and discrete. Let $\mu =< \tau \cup \{X \setminus \{b_{2n} : n \in \omega\}\} >$. Then $\tau \subsetneq \mu \subsetneq \sigma$ since $\{b_{2n+1}\}_{n \in \omega}$ converges in (X, μ) but not in (X, σ), while $\{b_{2n}\}_{n \in \omega}$ converges in (X, τ) but not in (X, μ). □

Corollary 2.8. *Between any two distinct comparable sequential KC-topologies on a set X, there are an infinite number of topologies.*

We note in passing that all we have used in Theorem 2.7 is that sequences have unique limits. On the other hand, as with Corollary 2.2, both the theorem and its corollary are false if KC is replaced by the T_1 separation axiom: If τ is the cofinite topology on a countably infinite set X, then (X,τ) is second countable, but each point $p \in X$ is maximal and (X,τ) is a lower topology. It is also easy to see that any successor topology to τ is also second countable. However, the next theorem shows that the cofinite topology is crucial in the construction of a first countable lower topology.

Theorem 2.9. *A sequential T_1-space with a maximal point contains an infinite subspace whose relative topology is cofinite.*

Proof. Suppose (X,τ) is a sequential T_1-space with a maximal point p. Since p is not isolated, we can choose a sequence of distinct points $S = \{x_n\}_{n\in\omega} \subseteq X \setminus \{p\}$ converging to p and hence p is an accumulation point of the subspace $S \cup \{p\}$. If the set of isolated points I of $(S, \tau|S)$ is infinite, then $p \in \mathrm{cl}(I)$ and if $I = \{d_n : n \in \omega\}$ is an enumeration of I, then $p \in \mathrm{cl}(\{d_{2n} : n \in \omega\}) \cap \mathrm{cl}(\{d_{2n+1} : n \in \omega\})$ showing that p is not a 1-point, thus is not maximal in $I \cup \{p\}$, hence not maximal in X, which contradicts our hypothesis. Thus I is finite and by replacing S with $S \setminus I$ we may, without loss of generality, assume that S has no isolated points. Suppose that $U \in \tau|S$; since S has no isolated points, U is infinite and hence $p \in \mathrm{cl}(U)$. Since p is a maximal point of $S \cup \{p\}$, it follows that $U \cup \{p\} \in \tau|(S \cup \{p\})$. If $D = \{n : x_n \in S \setminus U\}$ is infinite then $\{x_n : n \in D\}$ is a subsequence of S which does not converge to p, again a contradiction. Thus $S \setminus U$ is finite, showing that S has the cofinite topology. □

Corollary 2.10. *If τ is a first countable lower T_1-topology on X then there is an infinite subset $S \subseteq X$ such that $\tau|S$ is the cofinite topology.*

As a partial converse to Theorem 2.9, it follows from Theorem 1.4, that if a space has an infinite *closed* subspace with the cofinite topology, then its topology is lower. However, a first countable T_1-space may have an infinite subset with the cofinite topology and still not be lower as the following example shows:

Let μ denote the usual metric topology on the set of reals $\mathbb{R}$ and define a new topology σ on $\mathbb{R}$ as follows:

$U \in \sigma$ if and only if $U \in \mu$ and there is some $\epsilon > 0$ such that $U \supseteq (n-\epsilon, n+\epsilon)$ for all but finitely many $n \in \mathbb{N}$.

Clearly σ is a first countable T_1 topology on $\mathbb{R}$ and $\sigma|\,\mathbb{N}$ is the cofinite topology on $\mathbb{N}$. Further note that if A is a bounded set in $\mathbb{R}$, then $\sigma|\,A = \mu|\,A$. To show that $(\mathbb{R},\sigma)$ is not a lower topology, it suffices to show that no closed subspace has a maximal point. Let C be a closed subset of $(\mathbb{R},\sigma)$ and $p \in C$; if $C \cap \mathbb{N}$ is finite, then for some $m \in \mathbb{N}$, $p \in A = C \cap (-m, m)$ and $\sigma|\,A = \mu|\,A$. Thus by Theorem 2.7, p is not a maximal point of A; now a little thought shows that p is an accumulation point of A if and only if it is an accumulation point of C and it then follows that p is not a maximal point of C. If $C \cap \mathbb{N}$ is infinite, then C is dense in $(\mathbb{R},\sigma)$ and hence $C=\mathbb{R}$. However, if $p\in\mathbb{R}$, then we define $U=(p,p+\frac{1}{2})\cup\bigcup\{(n-\frac{1}{2},n+\frac{1}{2}):n\in\mathbb{N}\setminus(p-1,p+1)\}$. That σ is not a lower topology now follows since $U\in\sigma$, $p\in \mathrm{cl}_\sigma(U)$, but $U\cup\{p\}\notin\sigma$ showing that p is not a maximal point of $(\mathbb{R},\sigma)$.

Recall that a space X is *radial* if whenever $A\subseteq X$ and $x\in \mathrm{cl}(A)$ there is a well-ordered net in A converging to x. A space X is *pseudoradial* if whenever $A\subsetneq X$ is not closed there is a well-ordered net in A which converges to a point of $X\setminus A$. Every first countable space is radial and both sequential and radial spaces are pseudoradial; clearly, being radial is a hereditary property.

Theorem 2.11. *A radial Hausdorff space cannot have a maximal point.*

Proof. Suppose that (X,τ) is a radial Hausdorff space; by Theorem 4.3 of [1], X is discretely generated. Suppose that $p\in X$ is not isolated. Since X is discretely generated, there is some discrete set $D\subseteq X\setminus\{p\}$ such that $p\in \mathrm{cl}(D)$. Since X is radial, there is a well-ordered net $\{d_\alpha\}_{\alpha\in\kappa}$ in D converging to p. Let $P=\{d_\alpha:\alpha\in\kappa\}$, $S=\{d_\alpha:\alpha=\gamma+2n$ for some limit ordinal $\gamma\in\kappa$ and $n\in\omega\}$ and $T=\{d_\alpha:\alpha=\gamma+2n+1$ for some limit ordinal $\gamma\in\kappa$ and $n\in\omega\}$. Clearly both S and T are disjoint open subsets of the subspace $P\cup\{p\}$ and $p\in \mathrm{cl}(S)\cap \mathrm{cl}(T)$ showing that p is not a 1-point, thus is not maximal in the space $P\cup\{p\}$ and hence is not maximal in X. $\square$

Theorem 2.11 cannot be extended to KC-spaces since a set of size ω_1 with the cocountable topology is a radial KC-space, each point of which is maximal. (This space even has the stronger property that each point has a nested local base.) The question then arises:

Question 2.12. *Can a pseudoradial T_2-space have a maximal point?*

Preceding Theorem 2.7, we gave an example of an H-closed space with a maximal point. Thus the following question arises:

Question 2.13. *Can a minimal Hausdorff topology be a lower topology?*

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Received May 2003

Accepted November 2003

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