Applied General Topology
© Universidad Politécnica de Valencia
Volume 4, No. 1, 2003
pp. 157–192

Di-uniform texture spaces

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Abstract. Textures were introduced by the second author as a point-based setting for the study of fuzzy sets, and have since proved to be an appropriate framework for the development of complement-free mathematical concepts. In this paper the authors lay the foundation for a theory of uniformities in a textural context. Analogues are given for both the diagonal and covering approaches to the classical theory of uniform structures, the notion of uniform topology is generalized and an analogue given for the well known result that a topological space is uniformizable if and only if it is completely regular. Finally a textural analogue of the classical interplay between uniformities and families of pseudo-metrics is presented.

2000 AMS Classification: Primary: 54E15, 54A05. Secondary: 06D10, 03E20, 54A40, 54D10, 54D15, 54E35.

Keywords: Uniformity, texturing, direlation, dicover, difunction, di-uniform texture space, uniform ditopology, uniform bicontinuity, initial di-uniformity, separation, dimetric, complement-free, point-free, fuzzy set.

1. Introduction

Textures were introduced by the second author as a point-based setting for the study of fuzzy sets, and have since proved to be an appropriate setting for the development of complement-free mathematical concepts. In this paper the authors lay the foundation for a theory of uniformities imposed on textures. Analogues are given for both the diagonal and covering approaches to the classical theory of uniform structures, the notion of uniform topology is generalized and an analogue given for the well known result that a topological space is uniformizable if and only if it is completely regular. Finally the notion of pseudo dimetric is given and a pseudo metrization theorem for di-uniformities and for ditopologies is presented.

*Dedicated to the memory of Professor Doğan Çoker.

Let S be a non-empty set. We recall [2] that a *texturing* on S is a point separating, complete, completely distributive lattice $\mathcal{S}$ of subsets of S with respect to inclusion, which contains S, $\varnothing$, and for which arbitrary meet $\bigwedge$ coincides with intersection $\bigcap$ and finite joins $\vee$ coincide with unions $\cup$. The pair $(S, \mathcal{S})$ is then called a *texture*.

In general a texturing of S need not be closed under set complementation. The sets

$$P_s = \bigcap\{A \in \mathcal{S} \mid s \in A\},\ Q_s = \bigvee\{P_u \mid u \in S,\ s \notin P_u\},\ s \in S,$$

are important in the study of textures, and the following facts concerning these so called p–sets and q–sets will be used extensively below.

Lemma 1.1. [1]

(1) *$s \notin A \implies A \subseteq Q_s \implies s \notin A^\flat$ for all $s \in S$, $A \in \mathcal{S}$.*
(2) *$A^\flat = \{s \mid A \not\subseteq Q_s\}$ for all $A \in \mathcal{S}$.*
(3) *For $A_i \in \mathcal{S}$, $i \in I$ we have $(\bigvee_{i\in I} A_i)^\flat = \bigcup_{i\in I} A_i^\flat$.*
(4) *A is the smallest element of $\mathcal{S}$ containing $A^\flat$ for all $A \in \mathcal{S}$.*
(5) *For $A, B \in \mathcal{S}$, if $A \not\subseteq B$ then there exists $s \in S$ with $A \not\subseteq Q_s$ and $P_s \not\subseteq B$.*
(6) *$A = \bigcap\{Q_s \mid P_s \not\subseteq A\}$ for all $A \in \mathcal{S}$.*
(7) *$A = \bigvee\{P_s \mid A \not\subseteq Q_s\}$ for all $A \in \mathcal{S}$.*

Here $A^\flat$ is defined by

$$A^\flat = \bigcap\Big\{\bigcup\{A_i \mid i \in I\} \mid \{A_i \mid i \in I\} \subseteq \mathcal{S},\ A = \bigvee\{A_i \mid i \in I\}\Big\}$$

and known as the *core* of $A \in \mathcal{S}$. The above lemma exposes an important formal duality in $(S, \mathcal{S})$, namely that between $\bigcap$ and $\bigvee$, Q_s and P_s, and $P_s \not\subseteq A$ and $A \not\subseteq Q_s$. Indeed, it is to emphasize this duality that we normally write $P_s \not\subseteq A$ in preference to $s \notin A$.

Lemma 1.1 (5) is particularly useful in establishing inclusion by *reductio ad absurdum*, and will be used without comment in the sequel.

The simplest example of a texture is $(X, \mathcal{P}(X))$, for which $P_x = \{x\}$ and $Q_x = X \setminus \{x\}$, $x \in X$. A natural texturing of the unit interval $\mathbb{I} = [0, 1]$ is defined by

$$\mathcal{I} = \{[0, r) \mid r \in \mathbb{I}\} \cup \{[0, r] \mid r \in \mathbb{I}\}.$$

For the texture $(\mathbb{I}, \mathcal{I})$ we have $P_r = [0, r]$ and $Q_r = [0, r)$, $r \in \mathbb{I}$. This texture will prove useful in the later sections. Both $(X, \mathcal{P}(X))$ and $(\mathbb{I}, \mathcal{I})$ have the property that join coincides with union (equivalently, that $P_s \not\subseteq Q_s$ for all s), but certainly this is not the case in general.

The definition of a diagonal uniformity on a set S involves binary relations on S, but the standard theory of binary relations and functions is largely inappropriate for general textures $(S, \mathcal{S})$ because of their lack of symmetry. With this in mind, the second author has recently introduced notions of *relation* and *corelation* [1] for textures, based on the duality mentioned above. It is shown in [1] that, working in terms of *direlations*, which are pairs consisting of a relation and a corelation, a theory is obtained which resembles in many important respects that of classical binary relations and functions. It will be appropriate, therefore, to base our textural analogue of a diagonal uniformity on the concept of direlation and for the convenience of the reader we recall some basic definitions and results from [1]. The reader is referred to [1] for more details, motivation and examples.

For textures $(S,\mathcal{S})$, $(T,\mathcal{T})$ we denote by $\mathcal{S}\otimes\mathcal{T}$ the product texturing of $S\times T$ [4]. Thus, $\mathcal{S}\otimes\mathcal{T}$ consists of arbitrary intersections of sets of the form $(A\times T)\cup(S\times B)$, $A\in\mathcal{S}$, $B\in\mathcal{T}$. For $s\in S$, P_s and Q_s will always denote the p-sets and q-sets for the texture $(S,\mathcal{S})$, while for $t\in T$, P_t and Q_t will denote the p-sets and q-sets for $(T,\mathcal{T})$. We reserve the notation $P_{(s,t)}$, $Q_{(s,t)}$, $s\in S$, $t\in T$, for the p-sets, q-sets in $(S\times T,\mathcal{S}\otimes\mathcal{T})$. On the other hand, $\overline{P}_{(s,t)}$ and $\overline{Q}_{(s,t)}$ will denote the p-sets and q-sets for the texture $(S\times T,\ \mathcal{P}(S)\otimes\mathcal{T})$. Hence (see [1]) we have $\overline{P}_{(s,t)}=\{s\}\times P_t$ and $\overline{Q}_{(s,t)}=[(S\setminus\{s\})\times T]\cup[S\times Q_t]$. Likewise, $\overline{P}_{(t,s)}$ and $\overline{Q}_{(t,s)}$ are the p-sets and q-sets for $(T\times S,\ \mathcal{P}(T)\otimes\mathcal{S})$. It is easy to verify that $\overline{P}_{(s,t)}\not\subseteq\overline{Q}_{(s',t')}\iff s=s'$ and $P_t\not\subseteq Q_{t'}$. Again, we will use this fact, and its companion $\overline{P}_{(t,s)}\not\subseteq\overline{Q}_{(t',s')}\iff t=t'$ and $P_s\not\subseteq Q_{s'}$, without comment in what follows. Now let us recall:

Definition 1.2. [1] Let $(S,\mathcal{S})$, $(T,\mathcal{T})$ be textures. Then

(1) $r\in\mathcal{P}(S)\otimes\mathcal{T}$ is called a *relation on* $(S,\mathcal{S})$ *to* $(T,\mathcal{T})$ if it satisfies
R1 $r\not\subseteq\overline{Q}_{(s,t)}, P_{s'}\not\subseteq Q_s\implies r\not\subseteq\overline{Q}_{(s',t)}$.
R2 $r\not\subseteq\overline{Q}_{(s,t)}\implies\exists s'\in S$ such that $P_s\not\subseteq Q_{s'}$ and $r\not\subseteq\overline{Q}_{(s',t)}$.

(2) $R\in\mathcal{P}(S)\otimes\mathcal{T}$ is called a *co-relation on* $(S,\mathcal{S})$ *to* $(T,\mathcal{T})$ if it satisfies
CR1 $\overline{P}_{(s,t)}\not\subseteq R, P_s\not\subseteq Q_{s'}\implies\overline{P}_{(s',t)}\not\subseteq R$.
CR2 $\overline{P}_{(s,t)}\not\subseteq R\implies\exists s'\in S$ such that $P_{s'}\not\subseteq Q_s$ and $\overline{P}_{(s',t)}\not\subseteq R$.

(3) A pair (r,R), where r is a relation and R a co-relation on $(S,\mathcal{S})$ to $(T,\mathcal{T})$ is called a *direlation on* $(S,\mathcal{S})$ *to* $(T,\mathcal{T})$.

Normally, relations will be denoted by lower case and co-relations by upper case letters, as in the above definition.

For direlations (p,P), (q,Q) on $(S,\mathcal{S})$ to $(T,\mathcal{T})$ we write $(p,P)\sqsubseteq(q,Q)$ if and only if $p\subseteq q$ and $Q\subseteq P$.

For a general texture $(S,\mathcal{S})$ we define

$$i=i_S=\bigvee\{\overline{P}_{(s,s)}\mid s\in S\}\text{ and }I=I_S=\bigcap\{\overline{Q}_{(s,s)}\mid s\in S\}.$$

If we note that $i\not\subseteq\overline{Q}_{(s,t)}\iff P_s\not\subseteq Q_t$ and $\overline{P}_{(s,t)}\not\subseteq I\iff P_t\not\subseteq Q_s$ then it is trivial to verify that i is a relation and I a co-relation on $(S,\mathcal{S})$ to $(S,\mathcal{S})$. We refer to (i,I) as the *identity direlation* on $(S,\mathcal{S})$.

A direlation (r,R) on $(S,\mathcal{S})$ (that is, on $(S,\mathcal{S})$ to $(S,\mathcal{S})$) is *reflexive* if r and R are reflexive, that is if $(i,I)\sqsubseteq(r,R)$.

If (r,R) is a direlation on $(S,\mathcal{S})$ to $(T,\mathcal{T})$, the *inverse* $(r,R)^{\leftarrow} = (R^{\leftarrow}, r^{\leftarrow})$ of (r,R) is the direlation on $(T,\mathcal{T})$ to $(S,\mathcal{S})$ defined by

$$\begin{aligned} r^{\leftarrow} &= \bigcap\{\overline{Q}_{(t,s)} \mid r \not\subseteq \overline{Q}_{(s,t)}\}, \\ R^{\leftarrow} &= \bigvee\{\overline{P}_{(t,s)} \mid \overline{P}_{(s,t)} \not\subseteq R\}. \end{aligned}$$

A direlation (r,R) on $(S,\mathcal{S})$ is called *symmetric* if $(r,R) = (r,R)^{\leftarrow}$, that is if and only if $R = r^{\leftarrow}$. This notion of symmetry is quite different from the classical notion of symmetry for relations. However, as we will see, it will play the same role in the theory of textural uniformities as does classical symmetry in the theory of uniformities.

Definition 1.3. Let $(S,\mathcal{S})$, $(T,\mathcal{T})$ be textures, r a relation and R a co-relation on $(S,\mathcal{S})$ to $(T,\mathcal{T})$.

(1) For $A \subseteq S$ the *A–section of* r is the element $r(A)$ of $\mathcal{T}$ defined by

$$r(A) = \bigcap\{Q_t \mid \forall\, s,\ r \not\subseteq \overline{Q}_{(s,t)} \implies A \subseteq Q_s\} \in \mathcal{T}.$$

(2) For $A \subseteq S$ the *A–section of* R is the element $R(A)$ of $\mathcal{T}$ defined by

$$R(A) = \bigvee\{P_t \mid \forall\, s, \overline{P}_{(s,t)} \not\subseteq R \implies P_s \subseteq A\} \in \mathcal{T}.$$

(3) For $B \subseteq T$ the *B–presection of* r (*B–presection of* R) is the B–section $r^{\leftarrow}(B) \in \mathcal{S}$ of the co-relation $r^{\leftarrow}$ (respectively, the B–section $R^{\leftarrow}(B) \in \mathcal{S}$ of the relation $R^{\leftarrow}$) on $(T,\mathcal{T})$ to $(S,\mathcal{S})$.

The following lemma gives formulae for directly calculating the presections.

Lemma 1.4. *For a relation r, a co-relation R and $B \subseteq T$ we have:*

(1) $r^{\leftarrow}(B) = \bigvee\{P_s \mid \forall\, t,\ r \not\subseteq \overline{Q}_{(s,t)} \implies P_t \subseteq B\} \in \mathcal{S}$.
(2) $R^{\leftarrow}(B) = \bigcap\{Q_s \mid \forall\, t,\ \overline{P}_{(s,t)} \not\subseteq R \implies B \subseteq Q_t\} \in \mathcal{S}$.

The following results from [1] will prove useful later on.

Lemma 1.5. *For a direlation (r,R) on $(S,\mathcal{S})$ to $(T,\mathcal{T})$ we have*

(1) $r \not\subseteq \overline{Q}_{(s,t)} \iff \overline{P}_{(t,s)} \not\subseteq r^{\leftarrow}$ *and* $\overline{P}_{(s,t)} \not\subseteq R \iff R^{\leftarrow} \not\subseteq \overline{Q}_{(t,s)}$.
(2) $r \not\subseteq \overline{Q}_{(s,t)} \iff r(P_s) \not\subseteq Q_t$ *and* $\overline{P}_{(s,t)} \not\subseteq R \iff P_t \not\subseteq R(Q_s)$.

Proposition 1.6. *With the notation as in Definition 1.3:*

(1) *For relations r_1, r_2 with $r_1 \subseteq r_2$, co-relations R_1, R_2 with $R_1 \subseteq R_2$, A_1, A_2 in $\mathcal{S}$ with $A_1 \subseteq A_2$ and B_1, B_2 in $\mathcal{T}$ with $B_1 \subseteq B_2$ we have $r_1(A_1) \subseteq r_2(A_2)$, $R_1(A_1) \subseteq R_2(A_2)$, $r_2^{\leftarrow}(B_1) \subseteq r_1^{\leftarrow}(B_2)$ and $R_2^{\leftarrow}(B_1) \subseteq R_1^{\leftarrow}(B_2)$.*
(2) *For any relation r we have $r(\varnothing) = \varnothing$, $A \subseteq r^{\leftarrow}(r(A))$ for $A \in \mathcal{S}$ and $r(r^{\leftarrow}(B)) \subseteq B$ for $B \in \mathcal{T}$.*
(3) *For any co-relation R we have $R(S) = T$, $R^{\leftarrow}(R(A)) \subseteq A$ for $A \in \mathcal{S}$ and $B \subseteq R(R^{\leftarrow}(B))$ for $B \in \mathcal{T}$.*
(4) *For the identity direlation (i,I) on $(S,\mathcal{S})$ and $A \in \mathcal{S}$ we have $i(A) = I(A) = A$ and hence $i^{\leftarrow}(A) = I^{\leftarrow}(A) = A$.*

(5) *If a relation* r *(co-relation* R*) on* $(S,\mathcal{S})$ *is reflexive then for all* $A \in \mathcal{S}$ *we have* $A \subseteq r(A)$ $(R(A) \subseteq A)$.

(6) *For a relation* r *and co-relation* R *on* $(S,\mathcal{S})$ *to* $(T,\mathcal{T})$ *we have*

$$r(\bigvee_{j\in J} A_j) = \bigvee_{j\in J} r(A_j) \text{ and } R(\bigcap_{j\in J} A_j) = \bigcap_{j\in J} R(A_j)$$

for any $A_j \in \mathcal{S}$, $j \in J$.

(7) *For a relation* r *and co-relation* R *on* $(S,\mathcal{S})$ *to* $(T,\mathcal{T})$ *we have*

$$r^{\leftarrow}(\bigcap_{j\in J} B_j) = \bigcap_{j\in J} r^{\leftarrow}(B_j) \text{ and } R^{\leftarrow}(\bigvee_{j\in J} B_j) = \bigvee_{j\in J} R^{\leftarrow}(B_j)$$

for any $B_j \in \mathcal{T}$, $j \in J$.

Another important concept for direlations is that of composition. We recall the following:

Definition 1.7. [1] Let $(S,\mathcal{S})$, $(T,\mathcal{T})$, $(U,\mathcal{U})$ be textures.

(1) If p is a relation on $(S,\mathcal{S})$ to $(T,\mathcal{T})$ and q a relation on $(T,\mathcal{T})$ to $(U,\mathcal{U})$ then their *composition* is the relation $q \circ p$ on $(S,\mathcal{S})$ to $(U,\mathcal{U})$ defined by

$$q \circ p = \bigvee\{\overline{P}_{(s,u)} \mid \exists\, t \in T \text{ with } p \not\subseteq \overline{Q}_{(s,t)} \text{ and } q \not\subseteq \overline{Q}_{(t,u)}\}.$$

(2) If P is a co-relation on $(S,\mathcal{S})$ to $(T,\mathcal{T})$ and Q a co-relation on $(T,\mathcal{T})$ to $(U,\mathcal{U})$ then their *composition* is the co-relation $Q \circ P$ on $(S,\mathcal{S})$ to $(U,\mathcal{U})$ defined by

$$Q \circ P = \bigcap\{\overline{Q}_{(s,u)} \mid \exists\, t \in T \text{ with } \overline{P}_{(s,t)} \not\subseteq P \text{ and } \overline{P}_{(t,u)} \not\subseteq Q\}.$$

(3) With p, q; P, Q as above, the *composition* of the direlations (p,P), (q,Q) is the direlation

$$(q,Q) \circ (p,P) = (q \circ p, Q \circ P).$$

It is shown in [1] that the operation of taking the composition of direlations is associative, and that the identity direlations are identities for this operation.

If (r,R) is a direlation on $(S,\mathcal{S})$ then $(r,R) \circ (r,R) = (r \circ r, R \circ R)$ is also a direlation on $(S,\mathcal{S})$, which we denote by $(r,R)^2$. We give the obvious meaning to $(r,R)^n$ for any $n = 3,4,\ldots$. The direlation (r,R) on $(S,\mathcal{S})$ is called *transitive* if $(r,R)^2 \sqsubseteq (r,R)$.

We will also have occasion to consider the greatest lower bound of direlations. We give the definition for two direlations, but it may be extended in the obvious way to any family of direlations.

Definition 1.8. [1] Let (p,P), (q,Q) be direlations on $(S,\mathcal{S})$ to $(T,\mathcal{T})$. Then

$$p \sqcap q = \bigvee\{\overline{P}_{(s,t)} \mid \exists\, v \in S \text{ with } P_s \not\subseteq Q_v \text{ and } p, q \not\subseteq \overline{Q}_{(v,t)}\},$$

$$P \sqcup Q = \bigcap\{\overline{Q}_{(s,t)} \mid \exists\, v \in S \text{ with } P_v \not\subseteq Q_s \text{ and } \overline{P}_{(v,t)} \not\subseteq P, Q\}, \text{ and}$$

$$(p,P) \sqcap (q,Q) = (p \sqcap q, P \sqcup Q).$$

Proposition 1.9. *With the notation as in Definition 1.8,*

(1) $p \sqcap q$ *is a relation on* $(S, \mathcal{S})$ *to* $(T, \mathcal{T})$*. It is the greatest lower bound of* p *and* q *in the set of all relations on* $(S, \mathcal{S})$ *to* $(T, \mathcal{T})$*, ordered by inclusion.*
(2) $P \sqcup Q$ *is a co-relation on* $(S, \mathcal{S})$ *to* $(T, \mathcal{T})$*. It is the least upper bound of* P *and* Q *in the set of all co-relations on* $(S, \mathcal{S})$ *to* $(T, \mathcal{T})$*, ordered by inclusion.*
(3) *The direlation* $(p, P) \sqcap (q, Q)$ *is the greatest lower bound of* (p, P) *and* (q, Q) *on the set of all direlations on* $(S, \mathcal{S})$ *to* $(T, \mathcal{T})$*, ordered by the relation* $\sqsubseteq$*.*
(4) $(p \sqcap q)^{\leftarrow} = p^{\leftarrow} \sqcup q^{\leftarrow}$ *and* $(P \sqcup Q)^{\leftarrow} = P^{\leftarrow} \sqcap Q^{\leftarrow}$*.*
(5) *For* $A \in \mathcal{S}$, $(p \sqcap q)(A) \subseteq p(A) \cap q(A)$ *and* $P(A) \cup Q(A) \subseteq (P \sqcup Q)(A)$*.*
(6) *For* $B \in \mathcal{T}$, $p^{\leftarrow}(B) \cup q^{\leftarrow}(B) \subseteq (p \sqcap q)^{\leftarrow}(B)$ *and* $(P \sqcup Q)^{\leftarrow}(B) \subseteq P^{\leftarrow}(B) \cap Q^{\leftarrow}(B)$*.*
(7) *Let* (p_1, P_1), (p_2, P_2) *be direlations on* $(S, \mathcal{S})$ *to* $(T, \mathcal{T})$ *and* (q_1, Q_1), (q_2, Q_2) *direlations on* $(T, \mathcal{T})$ *to* $(U, \mathcal{U})$*. Then* $((q_1, Q_1) \sqcap (q_2, Q_2)) \circ ((p_1, P_1) \sqcap (q_2, Q_2)) \sqsubseteq ((q_1, Q_1) \circ (p_1, P_1)) \sqcap ((q_2, Q_2) \circ (p_2, P_2))$*.*

The notion of difunction is derived from that of direlation as follows.

Definition 1.10. [1] Let (f, F) be a direlation on $(S, \mathcal{S})$ to $(T, \mathcal{T})$. Then (f, F) is called a *difunction on* $(S, \mathcal{S})$ *to* $(T, \mathcal{T})$ if it satisfies the following two conditions.

DF1 For $s, s' \in S$, $P_s \not\subseteq Q_{s'} \implies \exists\, t \in T$ with $f \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s',t)} \not\subseteq F$.
DF2 For $t, t' \in T$ and $s \in S$, $f \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s,t')} \not\subseteq F \implies P_{t'} \not\subseteq Q_t$.

Difunctions are preserved under composition. It is easy to see that the identity direlation (i_S, I_S) on $(S, \mathcal{S})$ is in fact a difunction on $(S, \mathcal{S})$ to $(S, \mathcal{S})$. In this context we refer to (i_S, I_S) as the *identity difunction on* $(S, \mathcal{S})$.

If $(f, F) : (S, \mathcal{S}) \to (T, \mathcal{T})$ is a difunction, $A \in \mathcal{S}$, then $f(A)$ is called the *image* and $F(A)$ the *co-image* of A. Likewise, for $B \in \mathcal{T}$, $f^{\leftarrow}(B)$ is called the *inverse image* and $F^{\leftarrow}(B)$ the *inverse co-image* of B. It is shown in [1] that $f^{\leftarrow}(B) = F^{\leftarrow}(B)$ for all $B \in \mathcal{T}$, that is the inverse image and inverse co-image coincide.

Since a texturing is generally not closed under set complementation, when discussing topological concepts we cannot insist that closed sets should be the complement of open sets. This leads to the notion of a *dichotomous topology*, or *ditopology* for short [2]. This is a pair (τ, κ) of subsets of $\mathcal{S}$, where the set of *open sets* τ satisfies

(1) $S, \varnothing \in \tau$,
(2) $G_1, G_2 \in \tau \implies G_1 \cap G_2 \in \tau$ and
(3) $G_i \in \tau,\ i \in I \implies \bigvee_i G_i \in \tau$,

and the set of *closed sets* κ satisfies

(1) $S, \varnothing \in \kappa$,
(2) $K_1, K_2 \in \kappa \implies K_1 \cup K_2 \in \kappa$ and
(3) $K_i \in \kappa,\ i \in I \implies \bigcap_i K_i \in \kappa$.

The reader is referred to [2, 3, 6, 7] for some results on ditopological texture spaces and their relation with fuzzy topologies.

A subset β of τ is called a *base of* τ if every set in τ can be written as a join of sets in β, while a subset β of κ is a *base of* κ if every set in κ can be written as an intersection of sets in β.

For the unit interval texture $(\mathbb{I}, \mathcal{I})$ mentioned above, we may define a natural ditopology $(\tau_{\mathbb{I}}, \kappa_{\mathbb{I}})$ by

$$\tau_{\mathbb{I}} = \{[0, s) \mid s \in \mathbb{I}\} \cup \{\mathbb{I}\},\ \kappa_{\mathbb{I}} = \{[0, s] \mid s \in \mathbb{I}\} \cup \{\varnothing\}.$$

Continuity of difunctions is the subject of the following definition.

Definition 1.11. [6] Let $(S_k, \mathcal{S}_k, \tau_k, \kappa_k)$, $k = 1, 2$, be ditopological texture spaces and (f, F) a difunction on $(S_1, \mathcal{S}_1)$ to $(S_2, \mathcal{S}_2)$. Then

(1) (f, F) is *continuous* if $G \in \tau_2 \implies F^{\leftarrow}(G) \in \tau_1$.
(2) (f, F) is *cocontinuous* if $K \in \kappa_2 \implies f^{\leftarrow}(K) \in \kappa_1$.
(3) (f, F) is *bicontinuous* if it is continuous and cocontinuous.

The reader is referred to [9] for general terms related to lattice theory.

This paper comprises part of the first author's research towards her PhD thesis to be submitted to Hacettepe University.

We pause here to mention our motivation for introducing textures as a substrate for topology.

Ditopological texture spaces were conceived as a point-set setting for the study of fuzzy topology, and provide a unified setting for the study of topology, bitopology and fuzzy topology. Some of the links with Hutton spaces, $\mathbb{L}$–fuzzy sets and topologies are expressed in a categorical setting in [6]. Here it is the choice of bicontinuous difunctions for the morphisms on the textural side which makes possible a correspondence with the point-free concept of Hutton space.

Despite the close links with fuzzy sets and topologies, the development of the theory of ditopological texture spaces has proceeded largely independently, and has concentrated on the development of concepts which help to compensate for the possible lack of complementation. One such is that of direlation and difunction, another that of dicover ([2,3], see §2 below). Both play a crucial role in this paper. If one takes the view that a texturing $\mathcal{S}$ can provide a much more economic computational model than $\mathcal{P}(S)$, it is important that we do not lose power in other directions. For example $\mathcal{I}$ is certainly much simpler than $\mathcal{P}(\mathbb{I})$, but if we consider only ordinary open covers (and closed cocovers) it is trivial that for the usual ditopology, every closed subset is compact (and every open set cocompact). However this is non-trivially equivalent [2] to the fact that every open, coclosed dicover has a finite, cofinite subcover and this, via a bitopological argument, can be shown to be equivalent to the compactness of $\mathbb{I}$ under its usual topology. Hence this compactness property of $(\mathbb{I}, \mathcal{I})$ in its dicovering form is as powerful as that of $\mathbb{I}$, and we will see later that with an appropriate di-uniformity $(\mathbb{I}, \mathcal{I})$ can again play the same role as $\mathbb{I}$ does in the usual theory of uniformities.

Duality is an important element in defining such concepts. When applied to ditopologies it often gives rise to pairs of properties, such as compact – cocompact, regular – coregular. In the case of uniform ditopologies, as we will see, it actually links the open and closed sets via symmetry, and this causes the ditopology to be simultaneously completely regular and completely coregular.

A form of duality also plays a role in Giovanni Sambin's basic picture for formal topology [11]. There are clear parallels here which warrant further study. Likewise, links with the theory of locales and with domain theory have yet to be worked out. Finally, complement free textural concepts can be expected to find applications in negation free logics, and indeed (ditopological) textures themselves could well prove to be useful models for certain classes of such logics.

2. Direlations and Dicovers

As mentioned in the introduction, the entourages of a diagonal uniformity in the classical sense [13] will be replaced by direlations in the textural setting. A second important formulation of the theory of uniformities is that of the covering uniformity [12], so we will require an appropriate notion of cover in order to obtain an analogous description for textures. In this section we show that the notion of dicover, used in [2] to characterize the important form of compactness mentioned above and in [3] to describe various covering properties of ditopological texture spaces, is associated in a natural way with direlations. Hence this notion will form the basis for our description of covering uniformities in the textural sense.

Let us recall [2,3] that by a *dicover* of the texture $(S, \mathcal{S})$ we mean a family $\mathcal{C} = \{(A_i, B_i) \mid i \in I\}$ of elements of $\mathcal{S} \times \mathcal{S}$ which satisfies $\bigcap_{i \in I_1} B_i \subseteq \bigvee_{i \in I_2} A_i$ for all partitions (I_1, I_2) of I, including the trivial partitions. An important example is the family $\mathcal{P} = \{(P_s, Q_s) \mid s \in S^\flat\}$, which is shown in [3] to be a dicover for any texture $(S, \mathcal{S})$. If $\mathcal{D}$ is a dicover we often write $L\,\mathcal{D}\,M$ in place of $(L, M) \in \mathcal{D}$. We recall the following notions for dicovers given in [3].

(1) $\mathcal{C}$ is a *refinement* of $\mathcal{D}$ if given $i \in I$ we have $L\,\mathcal{D}\,M$ so that $A_i \subseteq L$ and $M \subseteq B_i$. In this case we write $\mathcal{C} \prec \mathcal{D}$.

(2) The *star* and *co-star* of $C \in \mathcal{S}$ with respect to $\mathcal{C}$ are respectively the sets

$$\mathrm{St}(\mathcal{C}, C) = \bigvee \{A_i \mid i \in I,\ C \not\subseteq B_i\} \in \mathcal{S}, \text{ and}$$

$$\mathrm{CSt}(\mathcal{C}, C) = \bigcap \{B_i \mid i \in I,\ A_i \not\subseteq C\} \in \mathcal{S}.$$

We say that $\mathcal{C}$ is a *delta refinement* of $\mathcal{D}$, and write $\mathcal{C} \prec_{(\Delta)} \mathcal{D}$, if

$$\mathcal{C}^\Delta = \{(\mathrm{St}(\mathcal{C}, P_s), \mathrm{CSt}(\mathcal{C}, Q_s)) \mid s \in S^\flat\} \prec \mathcal{D}.$$

We say that $\mathcal{C}$ is a *star refinement* of $\mathcal{D}$, and write $\mathcal{C} \prec_{(\star)} \mathcal{D}$, if

$$\mathcal{C}^\star = \{(\mathrm{St}(\mathcal{C}, A_i), \mathrm{CSt}(\mathcal{C}, B_i)) \mid i \in I\} \prec \mathcal{D}.$$

Before describing the link between direlations and dicovers, it will be appropriate for us to define a particular class of dicovers that will arise naturally in this connection.

Definition 2.1. A family $\mathcal{C} \subseteq \mathcal{S} \times \mathcal{S}$ is called an *anchored dicover* if it satisfies:

(1) $\mathcal{P} \subseteq \mathcal{C}$, and

(2) Given $A \,\mathcal{C}\, B$ there exists $s \in S$ satisfying

  (a) $A \not\subseteq Q_u \implies \exists\, A' \,\mathcal{C}\, B'$ with $A' \not\subseteq Q_u$ and $P_s \not\subseteq B'$, and

  (b) $P_v \not\subseteq B \implies \exists\, A'' \,\mathcal{C}\, B''$ with $P_v \not\subseteq B''$ and $A'' \not\subseteq Q_s$.

Since $\mathcal{P}$ is a dicover, we see by (1) that an anchored dicover is a dicover. It is straightforward to verify that $\mathcal{P}$ itself is anchored. The notion of anchored dicover enables us to improve ([3], Lemma 4.7 (3)). Since these results will be useful later on we present the modified lemma in full.

Lemma 2.2. *Let $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ be dicovers on $(S, \mathcal{S})$.*

(1) $\mathcal{C} \prec_{(\star)} \mathcal{D} \implies \mathcal{C} \prec \mathcal{D}$.

(2) *If $\mathcal{P} \prec \mathcal{C}$ then $\mathcal{C} \prec_{(\star)} \mathcal{D} \implies \mathcal{C} \prec_{(\Delta)} \mathcal{D}$*

(3) *If $\mathcal{C}$ is anchored then*

  (i) $\mathcal{C} \prec_{(\Delta)} \mathcal{D} \implies \mathcal{C} \prec \mathcal{D}$.

  (ii) $\mathcal{C} \prec_{(\Delta)} \mathcal{D} \prec_{(\Delta)} \mathcal{E} \implies \mathcal{C} \prec_{(\star)} \mathcal{E}$.

Proof. (1) and (2) are proved in [3] so we concentrate on (3).

(i). Take $A \,\mathcal{C}\, B$ and $s \in S$ as in Definition 2.1 (2). It will suffice to show $A \subseteq \mathrm{St}(\mathcal{C}, P_s)$ and $\mathrm{CSt}(\mathcal{C}, Q_s) \subseteq B$. If $A \not\subseteq \mathrm{St}(\mathcal{C}, P_s)$ then we have $u \in S$ with $A \not\subseteq Q_u$ and $P_u \not\subseteq \mathrm{St}(\mathcal{C}, P_s)$. By (2)(a) there exists $A' \,\mathcal{C}\, B'$ with $A' \not\subseteq Q_u$ and $P_s \not\subseteq B'$. But then $A' \subseteq \mathrm{St}(\mathcal{C}, P_s)$ so $P_u \not\subseteq A'$, which gives the contradiction $A' \subseteq Q_u$. The inclusion $\mathrm{CSt}(\mathcal{C}, Q_s) \subseteq B$ is proved likewise.

(ii). Take $A \,\mathcal{C}\, B$ and for $s \in S$ satisfying Definition 2.1 (2), choose $L \,\mathcal{D}\, M$ so that $\mathrm{St}(\mathcal{D}, P_s) \subseteq L$ and $M \subseteq \mathrm{CSt}(\mathcal{D}, Q_s)$. It will suffice to show $\mathrm{St}(\mathcal{C}, A) \subseteq L$ and $M \subseteq \mathrm{CSt}(\mathcal{C}, B)$. We prove the first inclusion, the second being dual. Suppose $\mathrm{St}(\mathcal{C}, A) \not\subseteq L$ and take $w \in S$ with $\mathrm{St}(\mathcal{C}, A) \not\subseteq Q_w$ and $P_w \not\subseteq L$. Now we have $A_1 \,\mathcal{C}\, B_1$ satisfying $A_1 \not\subseteq Q_w$ and $A \not\subseteq B_1$. Let us choose $u \in S$ with $A \not\subseteq Q_u$ and $P_u \not\subseteq B_1$. By condition (2)(a) we have $A' \,\mathcal{C}\, B'$ with $A' \not\subseteq Q_u$ and $P_s \not\subseteq B'$. Choose $U \,\mathcal{D}\, V$ with $\mathrm{St}(\mathcal{C}, P_u) \subseteq U$ and $V \subseteq \mathrm{CSt}(\mathcal{C}, Q_u)$. Then $A_1 \subseteq \mathrm{St}(\mathcal{C}, P_u) \subseteq U$ and $V \subseteq \mathrm{CSt}(\mathcal{C}, Q_u) \subseteq B'$. Since $P_s \not\subseteq B'$ we now have $P_s \not\subseteq V$, and so $U \subseteq \mathrm{St}(\mathcal{D}, P_s) \subseteq L$, whence $A_1 \subseteq L$. $A_1 \not\subseteq Q_w$ and $P_w \not\subseteq L$ now give a contradiction, and the proof is complete. □

Let us now show that we may associate an anchored dicover with each reflexive direlation (d, D) on $(S, \mathcal{S})$.

Proposition 2.3. *Let (d, D) be a reflexive direlation on $(S, \mathcal{S})$ and for $s \in S$ let $d[s] = d(P_s)$ and $D[s] = D(Q_s)$. Then*

$$\gamma(d, D) = \{(d[s], D[s]) \mid s \in S^\flat\}$$

is an anchored dicover of $(S, \mathcal{S})$.

Proof. Set $\mathcal{C} = \gamma(d, D)$. Since (d, D) is reflexive, $P_s \subseteq d(P_s) = d[s]$ and $D[s] = D(Q_s) \subseteq Q_s$ by Proposition 1.6 (5)). Hence, $\mathcal{P} \prec \mathcal{C}$.

Let us associate s with $d[s] \,\mathcal{C}\, D[s]$ and take $d[s] \not\subseteq Q_u$. Now $d[s] = d(P_s) = d(\bigvee\{P_{s'} \mid P_s \not\subseteq Q_{s'}\}) = \bigvee\{d(P_{s'}) \mid P_s \not\subseteq Q_{s'}\}$ by Proposition 1.6 (6), so there exists $s' \in S$ with $P_s \not\subseteq Q_{s'}$ and $d[s'] \not\subseteq Q_u$. Since $D[s'] \subseteq Q_{s'}$ we also have $P_s \not\subseteq D[s']$, whence $d[s'] \,\mathcal{C}\, D[s']$ satisfies the condition in Definition 2.1 (2a). The proof of (2b) is dual to this. $\square$

Let us denote by $\mathcal{RDR}$ the family of reflexive direlations and by $\mathcal{ADC}$ the family of anchored dicovers on $(S, \mathcal{S})$. The above proposition can now be seen as giving us a mapping $\gamma : \mathcal{RDR} \to \mathcal{ADC}$.

Proposition 2.4. *For $(d, D), (e, E) \in \mathcal{RDR}$ we have $(d, D) \circ (d, D)^{\leftarrow} \sqsubseteq (e, E)$ $\implies \gamma(d, D) \prec_{(\Delta)} \gamma(e, E)$.*

Proof. We establish $\mathrm{St}(\gamma(d, D), P_s) \subseteq e[s]$ for $s \in S^\flat$. Suppose this is not so. Then we have $z \in S$ with $d[z] \not\subseteq e[s]$ and $P_s \not\subseteq D[z]$. Take $t \in S$ with $d[z] \not\subseteq Q_t$ and $P_t \not\subseteq e[s]$, and then $t' \in S$ satisfying $d[z] \not\subseteq Q_{t'}$ and $P_{t'} \not\subseteq Q_t$. From $d[z] \not\subseteq Q_{t'}$ we obtain $d \not\subseteq \overline{Q}_{(z,t')}$, while $P_s \not\subseteq D[z]$ implies $\overline{P}_{(z,s)} \not\subseteq D$, and hence $D^{\leftarrow} \not\subseteq \overline{Q}_{(s,z)}$. Thus $\overline{P}_{(s,t')} \subseteq D^{\leftarrow} \circ d \subseteq e$, so $e \not\subseteq \overline{Q}_{(s,t)}$ which gives the contradiction $P_t \subseteq e[s]$.

In just the same way $E[s] \subseteq \mathrm{CSt}(\gamma(d, D), Q_s)$, and the proof is complete. $\square$

Now let us show that a dicover gives rise to a reflexive, symmetric direlation in a natural way.

Proposition 2.5. *Let $\mathcal{C} = \{(A_j, B_j) \mid j \in J\}$ be a dicover on $(S, \mathcal{S})$ and define $\delta(\mathcal{C}) = (d(\mathcal{C}), D(\mathcal{C}))$ by*

$$\begin{aligned} d(\mathcal{C}) &= \bigvee\{\overline{P}_{(s,t)} \mid \exists\, j \in J \text{ with } A_j \not\subseteq Q_t \text{ and } P_s \not\subseteq B_j\}, \\ D(\mathcal{C}) &= \bigcap\{\overline{Q}_{(s,t)} \mid \exists\, j \in J \text{ with } P_t \not\subseteq B_j \text{ and } A_j \not\subseteq Q_s\}. \end{aligned}$$

Then $\delta(\mathcal{C})$ is a reflexive and symmetric direlation on $(S, \mathcal{S})$.

Proof. Write $d = d(\mathcal{C})$, $D = D(\mathcal{C})$ for short. First we verify that d is a relation on $(S, \mathcal{S})$, leaving the proof that D is a co-relation to the reader. Take $s, t \in S$ with $d \not\subseteq \overline{Q}_{(s,t)}$. Then we have $t' \in S$ and $j \in J$ satisfying $\overline{P}_{(s,t')} \not\subseteq \overline{Q}_{(s,t)}$, $A_j \not\subseteq Q_{t'}$ and $P_s \not\subseteq B_j$. If $P_{s'} \not\subseteq Q_s$ then $P_s \subseteq P_{s'}$, whence $P_{s'} \not\subseteq B_j$ and so $\overline{P}_{(s',t')} \subseteq d$, which gives $d \not\subseteq \overline{Q}_{(s',t)}$. This establishes *R1*. On the other hand, since $P_s \not\subseteq B_j$, we have $s' \in S$ satisfying $P_s \not\subseteq Q_{s'}$ and $P_{s'} \not\subseteq B_j$. As before, $d \not\subseteq \overline{Q}_{(s',t)}$, which verifies *R2*.

To show d is reflexive, suppose $i \not\subseteq d$ and take $s, t \in S$ with $i \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s,t)} \not\subseteq d$. Then $P_s \not\subseteq Q_t$ and for all $j \in J$ we have $A_j \subseteq Q_t$ or $P_s \subseteq B_j$. Put $J_1 = \{j \in J \mid P_s \subseteq B_j\}$ and let $J_2 = J \setminus J_1$. Then (J_1, J_2) is a partition of J, so $P_s \subseteq \bigcap_{j \in J_1} B_j \subseteq \bigvee_{j \in J_2} A_j \subseteq Q_t$, since $\mathcal{C}$ is a dicover. This gives the contradiction $P_s \subseteq Q_t$, so d is reflexive. The proof that D is reflexive is dual to this. Hence, (d, D) is reflexive.

To show (d, D) is symmetric it will suffice to verify that $d^{\leftarrow} = D$. Suppose that $d^{\leftarrow} \not\subseteq D$ and take $u, v \in S$ satisfying $d^{\leftarrow} \not\subseteq \overline{Q}_{(u,v)}$ and $\overline{P}_{(u,v)} \not\subseteq D$. We have $u' \in S$ with $P_{u'} \not\subseteq Q_u$ and $\overline{P}_{(u',v)} \not\subseteq D$ by *CR2*. There exists $t \in S$ with $\overline{P}_{(u',v)} \not\subseteq \overline{Q}_{(u',t)}$ and $j \in J$ for which $P_t \not\subseteq B_j$ and $A_j \not\subseteq Q_{u'}$, whence $\overline{P}_{(t,u')} \subseteq d$

and so $d \not\subseteq \overline{Q}_{(t,u)}$. This is easily seen to be equivalent to $\overline{P}_{(u,t)} \not\subseteq d^{\leftarrow}$ and so $d^{\leftarrow} \subseteq \overline{Q}_{(u,t)}$. Finally $Q_t \subseteq Q_v$, which gives the contradiction $d^{\leftarrow} \subseteq \overline{Q}_{(u,v)}$.

Finally, suppose $D \not\subseteq d^{\leftarrow}$ and take $u, v \in S$ with $D \not\subseteq \overline{Q}_{(u,v)}$ and $\overline{P}_{(u,v)} \not\subseteq d^{\leftarrow}$. As above we have $d \not\subseteq \overline{Q}_{(v,u)}$, whence $t \in S$ and $j \in J$ so that $\overline{P}_{(v,t)} \not\subseteq \overline{Q}_{(v,u)}$, $A_j \not\subseteq Q_t$ and $P_v \not\subseteq B_j$. Since $Q_u \subseteq Q_t$ we also have $A_j \not\subseteq Q_u$, whence we have the contradiction $D \subseteq \overline{Q}_{(u,v)}$ from the definition of D. $\square$

If we denote by $\mathcal{DC}$ the set of dicovers and by $\mathcal{SRDR}$ the set of symmetric reflexive direlations on $(S, \mathcal{S})$, this proposition defines a mapping $\delta : \mathcal{DC} \to \mathcal{SRDR}$.

Proposition 2.6. *For $\mathcal{C}, \mathcal{D} \in \mathcal{DC}$, $\mathcal{C} \prec_{(\star)} \mathcal{D} \implies \delta(\mathcal{C}) \circ \delta(\mathcal{C}) \sqsubseteq \delta(\mathcal{D})$.*

Proof. Suppose $d(\mathcal{C}) \circ d(\mathcal{C}) \not\subseteq d(\mathcal{D})$. Then we have $s, u \in S$ so that $\overline{P}_{(s,u)} \not\subseteq d(\mathcal{D})$ and there exists $t \in S$ satisfying $d(\mathcal{C}) \not\subseteq \overline{Q}_{(s,t)}$ and $d(\mathcal{C}) \not\subseteq \overline{Q}_{(t,u)}$. Now we have $t' \in S$ so that $P_{t'} \not\subseteq Q_t$ and there exists $A_1 \mathcal{C} B_1$ for which $A_1 \not\subseteq Q_{t'}$, $P_s \not\subseteq B_1$. Also we have $u' \in S$ so that $P_{u'} \not\subseteq Q_u$ and there exists $A_2 \mathcal{C} B_2$ for which $A_2 \not\subseteq Q_{u'}$, $P_t \not\subseteq B_2$. Since $\mathcal{C} \prec_{(\star)} \mathcal{D}$ we may choose $C \mathcal{D} E$ with $\mathrm{St}(\mathcal{C}, A_1) \subseteq C$ and $E \subseteq \mathrm{CSt}(\mathcal{C}, B_1)$. Hence, since we clearly have $A_1 \not\subseteq B_2$, $A_2 \subseteq \mathrm{St}(\mathcal{C}, A_1) \subseteq C$ and $E \subseteq \mathrm{CSt}(\mathcal{C}, B_1) \subseteq B_1$. Thus $C \not\subseteq Q_{u'}$ and $P_s \not\subseteq E$, and we obtain the contradiction $\overline{P}_{(s,u)} \subseteq \overline{P}_{(s,u')} \subseteq d(\mathcal{D})$.

This establishes $d(\mathcal{C}) \circ d(\mathcal{C}) \subseteq d(\mathcal{D})$, and the proof of $D(\mathcal{D}) \subseteq D(\mathcal{C}) \circ D(\mathcal{C})$ is dual to this. $\square$

Let us now discuss the relation between the mappings γ and δ.

Theorem 2.7. *Let $(S, \mathcal{S})$ be a texture. With the notation above,*

(1) *$\delta(\gamma(d, D)) = (d, D) \circ (d, D)$ for all $(d, D) \in \mathcal{SRDR}$.*
(2) *$\gamma(\delta(\mathcal{C})) = \mathcal{C}^{\Delta}$ for all $\mathcal{C} \in \mathcal{DC}$.*

Proof. (1). Take $(d, D) \in \mathcal{SRDR}$ and suppose $d(\gamma(d, D)) \not\subseteq d \circ d$. Then we have $s, t \in S$ satisfying $\overline{P}_{(s,t)} \not\subseteq d \circ d$ for which we have $z \in S^{\flat}$ satisfying $d[z] \not\subseteq Q_t$ and $P_s \not\subseteq D[z]$. However, $d[z] \not\subseteq Q_t \iff d \not\subseteq \overline{Q}_{(z,t)}$ and $P_s \not\subseteq D[z] \iff \overline{P}_{(z,s)} \not\subseteq D = d^{\leftarrow} \iff d \not\subseteq \overline{Q}_{(s,z)}$ by Lemma 1.5, since (d, D) is symmetric, and we obtain the contradiction $\overline{P}_{(s,t)} \subseteq d \circ d$.

Conversely, suppose $d \circ d \not\subseteq d(\gamma(d, D))$. Then we have $s, u \in S$ with $\overline{P}_{(s,u)} \not\subseteq d(\gamma(d, D))$ for which we have $t \in S$ satisfying $d \not\subseteq \overline{Q}_{(s,t)}$ and $d \not\subseteq \overline{Q}_{(t,u)}$. Firstly, $d \not\subseteq \overline{Q}_{(t,u)}$ gives us $d[t] \not\subseteq Q_u$. Secondly, $d \not\subseteq \overline{Q}_{(s,t)}$ implies $t \in S^{\flat}$ and also gives $\overline{P}_{(t,s)} \not\subseteq d^{\leftarrow} = D$ since (d, D) is symmetric. Hence we obtain $P_s \not\subseteq D[t]$. We deduce that $\overline{P}_{(s,u)} \subseteq d(\gamma(d, D))$, which is a contradiction.

This completes the proof that $d(\gamma(d, D)) = d \circ d$, and the proof of $D(\gamma(d, D)) = D \circ D$ is dual to this, so $\delta(\gamma(d, D)) = (d, D) \circ (d, D)$.

(2). Let $\mathcal{C} = \{(A_i, B_i) \mid i \in I\}$ and take $s \in S^{\flat}$. Suppose that $d(\mathcal{C})[s] \not\subseteq \mathrm{St}(\mathcal{C}, P_s)$. Then we have $t \in S$ with $d(\mathcal{C})[s] \not\subseteq Q_t$ and $P_t \not\subseteq \mathrm{St}(\mathcal{C}, P_s)$. Now we have $w \in S$ with $d(\mathcal{C}) \not\subseteq \overline{Q}_{(w,t)}$ and $P_s \not\subseteq Q_w$. Thus for some $t' \in S$ and $i \in I$,

$\overline{P}_{(w,t')} \not\subseteq \overline{Q}_{(w,t)}$, $A_i \not\subseteq Q_{t'}$ and $P_w \not\subseteq B_i$. We deduce that $P_s \not\subseteq B_i$, and hence $P_t \subseteq P_{t'} \subseteq A_i \subseteq \mathrm{St}(\mathcal{C}, P_s)$, which is a contradiction.

Conversely, suppose $\mathrm{St}(\mathcal{C}, P_s) \not\subseteq d(\mathcal{C})[s]$. Then we have $i \in I$ with $A_i \not\subseteq d(\mathcal{C})[s]$ and $P_s \not\subseteq B_i$. Hence for some $t \in S$ we have $A_i \not\subseteq Q_t$ and $d(\mathcal{C}) \not\subseteq \overline{Q}_{(z,t)} \implies P_s \subseteq Q_z$ for all $z \in S$. Take $s', t' \in S$ satisfying $P_s \not\subseteq Q_{s'}$, $P_{s'} \not\subseteq B_i$ and $A_i \not\subseteq Q_{t'}$, $P_{t'} \not\subseteq Q_t$. Now $A_i \not\subseteq Q_{t'}$, $P_{s'} \not\subseteq B_i$ gives $\overline{P}_{(s',t')} \subseteq d(\mathcal{C})$, so $d(\mathcal{C}) \not\subseteq \overline{Q}_{(s',t)}$ and putting $z = s'$ in the above implication gives the contradiction $P_s \subseteq Q_{s'}$.

This completes the proof that $d(\mathcal{C})[s] = \mathrm{St}(\mathcal{C}, P_s)$. likewise, $D(\mathcal{C})[s] = \mathrm{CSt}(\mathcal{C}, Q_s)$, so $\gamma(\delta(\mathcal{C})) = \mathcal{C}^{\Delta}$. $\square$

Corollary 2.8. *With the notation above:*

1. $(d, D) \sqsubseteq \delta(\gamma(d, D))$ *for all* $(d, D) \in \mathcal{RSDR}$.
2. $\mathcal{C}^{\Delta}$ *is anchored for all dicovers* $\mathcal{C}$. *If* $\mathcal{C}$ *is anchored then* $\mathcal{C} \prec \gamma(\delta(\mathcal{C}))$.

Proof. (1). Clear by Theorem 2.7 (1) since $(d, D) \sqsubseteq (d, D) \circ (d, D)$ when (d, D) is reflexive.

(2). The first statement is clear from Theorem 2.7 (2) and Proposition 2.3. For the second we need only note that $\mathcal{C} \prec_{(\Delta)} \mathcal{C}^{\Delta}$ and apply Lemma 2.2 (1) when $\mathcal{C}$ is anchored to give $\mathcal{C} \prec \mathcal{C}^{\Delta}$. Hence $\mathcal{C} \prec \gamma(\delta(\mathcal{C}))$ by Theorem 2.7 (2). $\square$

Let us recall from [3] that the meet of two dicovers $\mathcal{C}$ and $\mathcal{D}$ is the dicover $\mathcal{C} \wedge \mathcal{D} = \{(A \cap C, B \cup D) \mid A\, \mathcal{C}\, B,\ C\, \mathcal{D}\, D\}$. As might be expected, this notion is closely related to that of the greatest lower bound for direlations.

Proposition 2.9. *Let* $(S, \mathcal{S})$ *be a texture. With the notation above,*

1. *For* $(d, D), (e, E) \in \mathcal{RDR}$ *we have* $\gamma((d, D) \sqcap (e, E)) \prec \gamma(d, D) \wedge \gamma(e, E)$.
2. *For* $\mathcal{C}, \mathcal{D} \in \mathcal{DC}$ *we have* $\delta(\mathcal{C} \wedge \mathcal{D}) \sqsubseteq \delta(\mathcal{C}) \sqcap \delta(\mathcal{D})$.

Proof. (1). Since $\gamma((d, D) \sqcap (e, E)) = \{((d \sqcap e)[s], (D \sqcup E)[s]) \mid s \in S^{\flat}\}$, the result follows trivially from Proposition 1.9 (5).

(2). If $d(\mathcal{C} \wedge \mathcal{D}) \not\subseteq d(\mathcal{C}) \sqcap d(\mathcal{D})$ then we have $s, t \in S$ satisfying $\overline{P}_{(s,t)} \not\subseteq d(\mathcal{C}) \sqcap d(\mathcal{D})$ for which we have $A\mathcal{C}B$, $C\mathcal{D}E$ satisfying $A \cap C \not\subseteq Q_t$ and $P_s \not\subseteq B \cup E$. Take $t' \in S$ satisfying $A \cap C \not\subseteq Q_{t'}$ and $P_{t'} \not\subseteq Q_t$. Now $\overline{P}_{(s,t')} \subseteq d(\mathcal{C}),\ d(\mathcal{D})$, so $d(\mathcal{C}),\ d(\mathcal{D}) \not\subseteq \overline{P}_{(s,t)}$, which leads to the contradiction $\overline{P}_{(s,t)} \subseteq d(\mathcal{C}) \sqcap d(\mathcal{D})$.

This verifies $d(\mathcal{C} \wedge \mathcal{D}) \subseteq d(\mathcal{C}) \sqcap d(\mathcal{D})$, and the proof of $D(\mathcal{C}) \sqcup D(\mathcal{D}) \subseteq D(\mathcal{C} \wedge \mathcal{D})$ is dual to this, so (2) is proved. $\square$

3. Direlational and Dicover Uniformities

We now have the tools necessary to define direlational and dicover uniformities on a texture, and to prove their equivalence.

Definition 3.1. Let $(S, \mathcal{S})$ be a texture and $\mathcal{U}$ a family of direlations on $(S, \mathcal{S})$. If $\mathcal{U}$ satisfies the conditions

1. $(i, I) \sqsubseteq (d, D)$ for all $(d, D) \in \mathcal{U}$. That is, $\mathcal{U} \subseteq \mathcal{RDR}$.
2. $(d, D) \in \mathcal{U}$, $(e, E) \in \mathcal{DR}$ and $(d, D) \sqsubseteq (e, E)$ implies $(e, E) \in \mathcal{U}$.

(3) $(d, D), (e, E) \in \mathcal{U}$ implies $(d, D) \sqcap (e, E) \in \mathcal{U}$.
(4) Given $(d, D) \in \mathcal{U}$ there exists $(e, E) \in \mathcal{U}$ satisfying $(e, E) \circ (e, E) \sqsubseteq (d, D)$.
(5) Given $(d, D) \in \mathcal{U}$ there exists $(c, C) \in \mathcal{U}$ satisfying $(c, C)^{\leftarrow} \sqsubseteq (d, D)$.

then $\mathcal{U}$ is called a *direlational uniformity* on $(S, \mathcal{S})$, and $(S, \mathcal{S}, \mathcal{U})$ is known as a *direlational uniform texture space.*

It will be noted that this definition is formally the same as the usual definition of a diagonal uniformity, and the notions of *base* and *subbase* may be defined in the obvious way. Exactly as for diagonal uniformities we have the following lemma.

Lemma 3.2. *A direlational uniformity* $\mathcal{U}$ *on* $(S, \mathcal{S})$ *has a base of symmetric direlations.*

Proof. Take $(d, D) \in \mathcal{U}$. By condition (5) we have $(e, E) \in \mathcal{U}$ with $(e, E)^{\leftarrow} \sqsubseteq (d, D)$, so $(e, E) \sqsubseteq (d, D)^{\leftarrow}$ and $(d, D)^{\leftarrow} \in \mathcal{U}$ by condition (2). But now $(f, F) = (d, D) \sqcap (d, D)^{\leftarrow} \in \mathcal{U}$ by condition (3), and clearly (f, F) is symmetric and satisfies $(f, F) \sqsubseteq (d, D)$. $\square$

The following example of a direlational uniformity will prove important later on.

Example 3.3. Let $(\mathbb{I}, \mathcal{I})$ be the unit interval texture and for $\epsilon > 0$ define $d_\epsilon = \{(r, s) \mid r, s \in \mathbb{I},\ s < r + \epsilon\}$, $D_\epsilon = \{(r, s) \mid r, s \in \mathbb{I},\ s \le r - \epsilon\}$. Clearly (d_ϵ, D_ϵ) is a reflexive, symmetric direlation on $(\mathbb{I}, \mathcal{I})$. Moreover, $(d_\epsilon, D_\epsilon)^2 \sqsubseteq (d_{2\epsilon}, D_{2\epsilon})$, while for $\epsilon \le \delta$, $(d_\epsilon, D_\epsilon) \sqsubseteq (d_\delta, D_\delta)$ and so $(d_\epsilon, D_\epsilon) \sqcap (d_\delta, D_\delta) = (d_\epsilon, D_\epsilon)$. Hence

$$\mathcal{U}_{\mathbb{I}} = \{(d, D) \mid (d, D) \in \mathcal{DR} \text{ and there exists } \epsilon > 0 \text{ with } (d_\epsilon, D_\epsilon) \sqsubseteq (d, D)\}$$

is a direlational uniformity on $(\mathbb{I}, \mathcal{I})$. We will call $\mathcal{U}_{\mathbb{I}}$ the *usual direlational uniformity on* $(\mathbb{I}, \mathcal{I})$.

Definition 3.4. Let $(S, \mathcal{S}, \mathcal{U})$ be a direlational uniform texture space and $\mathcal{C}$ a dicover of S. Then $\mathcal{C}$ is called *uniform* if $\gamma(c, C) \prec \mathcal{C}$ for some $(c, C) \in \mathcal{U}$.

Lemma 3.5. *Let* $(S, \mathcal{S}, \mathcal{U})$ *be a direlational uniform texture space and* υ *the family of uniform dicovers. Then* υ *has the following properties:*

(1) *Given* $\mathcal{C} \in \upsilon$ *there exists* $\mathcal{D} \in \upsilon \cap \mathcal{ADC}$ *with* $\mathcal{D} \prec \mathcal{C}$.
(2) $\mathcal{C} \in \upsilon$, $\mathcal{D} \in \mathcal{DC}$ *and* $\mathcal{C} \prec \mathcal{D}$ *implies* $\mathcal{D} \in \upsilon$.
(3) $\mathcal{C}, \mathcal{D} \in \upsilon$ *implies* $\mathcal{C} \wedge \mathcal{D} \in \upsilon$.
(4) *Given* $\mathcal{C} \in \upsilon$ *there exists* $\mathcal{D} \in \upsilon$ *with* $\mathcal{D} \prec_{(\star)} \mathcal{C}$.

Proof. (1). By hypothesis there exists $(c, C) \in \mathcal{U}$ with $\gamma(c, C) \prec \mathcal{C}$. But $\mathcal{D} = \gamma(c, C) \in \upsilon \cap \mathcal{ADC}$ and $\mathcal{D} \prec \mathcal{C}$.

(2). Immediate.

(3). Take $\mathcal{C}, \mathcal{D} \in \upsilon$ and $(c, C), (d, D) \in \mathcal{U}$ with $\gamma(c, C) \prec \mathcal{C}$, $\gamma(d, D) \prec \mathcal{D}$. Then $(c, C) \sqcap (d, D) \in \mathcal{U}$ by Definition 3.1 (3), and $\gamma((c, C) \sqcap (d, D)) \prec \mathcal{C} \wedge \mathcal{D}$ by Proposition 2.9 (1), so $\mathcal{C} \wedge \mathcal{D} \in \upsilon$.

(4). Take $\mathcal{C} \in \upsilon$ and $(c, C) \in \mathcal{U}$ with $\gamma(c, C) \prec \mathcal{C}$. By Definition 3.1 (4) we have $(d, D) \in \mathcal{U}$ with $(d, D) \circ (d, D) \sqsubseteq (c, C)$, and then by Definition 3.1 (5) we have $(e, E) \in \mathcal{U}$ with $(e, E)^{\leftarrow} \sqsubseteq (d, D)$. If we let $(f, F) = (d, D) \sqcap (e, E)$ then $(f, F) \in \mathcal{U}$ and $(f, F) \circ (f, F)^{\leftarrow} \sqsubseteq (c, C)$, so $\gamma(f, F) \prec_{(\Delta)} \gamma(c, C) \prec \mathcal{C}$ by Proposition 2.4. In exactly the same way we may find $(g, G) \in \mathcal{U}$ with $\gamma(g, G) \prec_{(\Delta)} \gamma(f, F)$. If we let $\mathcal{D} = \gamma(g, G)$ then $\mathcal{D} \in \upsilon$ is anchored by Proposition 2.3, so by Lemma 2.2 (3 ii) we have $\mathcal{D} \prec_{(\star)} \mathcal{C}$. □

This leads to the following definition.

Definition 3.6. Let $(S, \mathcal{S})$ be a texture. If υ is a family of dicovers of S satisfying conditions (1)–(4) of Lemma 3.5 we say υ is a *dicovering uniformity* on $(S, \mathcal{S})$, and call $(S, \mathcal{S}, \upsilon)$ a *dicovering uniform texture space*.

We can now see Lemma 3.5 as associating a dicovering uniformity with a given direlational uniformity. The following theorem expresses the equivalence of these two concepts.

Theorem 3.7. *Let $(S, \mathcal{S})$ be a texture.*

1. *To each direlational uniformity $\mathcal{U}$ on $(S, \mathcal{S})$ we may associate a dicovering uniformity $\upsilon = \Gamma(\mathcal{U}) = \{\mathcal{C} \in \mathcal{DC} \mid \exists\, (c, C) \in \mathcal{U}$ with $\gamma(c, C) \prec \mathcal{C}\}$.*
2. *To each dicovering uniformity υ on $(S, \mathcal{S})$ we may associate a direlational uniformity $\mathcal{U} = \Delta(\upsilon) = \{(d, D) \in \mathcal{RDR} \mid \exists\, \mathcal{C} \in \upsilon$ with $\delta(\mathcal{C}) \sqsubseteq (d, D)\}$.*
3. *$\Delta(\Gamma(\mathcal{U})) = \mathcal{U}$ for every direlational uniformity $\mathcal{U}$ on $(S, \mathcal{S})$.*
4. *$\Gamma(\Delta(\upsilon)) = \upsilon$ for every dicovering uniformity υ on $(S, \mathcal{S})$.*

Proof. (1). This is just Lemma 3.5.

(2). We need to establish the conditions (1)–(5) of Definition 3.1 for $\mathcal{U} = \Delta(\upsilon)$. Conditions (1) and (2) are an immediate consequence of the definition of $\Delta(\upsilon)$, and (3) follows trivially from Proposition 2.9 (2). Take $(d, D) \in \Delta(\upsilon)$. Then we have $\mathcal{C} \in \upsilon$ satisfying $\delta(\mathcal{C}) \sqsubseteq (d, D)$. Now $(c, C) = \delta(\mathcal{C}) \in \Delta(\upsilon)$, and since (c, C) is symmetric by Proposition 2.5 we have $(c, C)^{\leftarrow} = (c, C) \sqsubseteq (d, D)$, which proves (5). Finally we have $\mathcal{E} \in \upsilon$ satisfying $\mathcal{E} \prec_{(\star)} \mathcal{C}$, and then $(e, E) = \delta(\mathcal{E}) \in \Delta(\upsilon)$ and $(e, E) \circ (e, E) \sqsubseteq (d, D)$ by Proposition 2.6, so (4) is established also.

(3). First take $(d, D) \in \Delta(\Gamma(\mathcal{U}))$. Then we have $\mathcal{C} \in \Gamma(\mathcal{U})$ with $\delta(\mathcal{C}) \sqsubseteq (d, D)$, and then $(c, C) \in \mathcal{U}$ with $\gamma(c, C) \prec \mathcal{C}$. Without loss of generality we may take $(c, C) \in \mathcal{RSDR}$ since the symmetric elements of $\mathcal{U}$ form a base, so by Corollary 2.8,

$$(c, C) \sqsubseteq \delta(\gamma(c, C)) \sqsubseteq \delta(\mathcal{C}) \sqsubseteq (d, D),$$

which shows $(d, D) \in \mathcal{U}$. Conversely, take $(d, D) \in \mathcal{U}$ and choose $(e, E) \in \mathcal{U}$ with (e, E) symmetric so that $(e, E) \circ (e, E) \sqsubseteq (d, D)$. Then $\delta(\gamma(e, E)) \sqsubseteq (d, D)$ by Theorem 2.7 (1), and we have established $(d, D) \in \Delta(\Gamma(\mathcal{U}))$.

(4). First take $\mathcal{C} \in \Gamma(\Delta(\upsilon))$. Then we have $(c, C) \in \Delta(\upsilon)$ with $\gamma(c, C) \prec \mathcal{C}$, and then $\mathcal{D} \in \upsilon$ with $\delta(\mathcal{D}) \sqsubseteq (c, C)$. Without loss of generality we may take

$\mathcal{D} \in \mathcal{ADC}$ since the anchored elements of υ form a base, so by Corollary 2.8 (2).

$$\mathcal{D} \prec \gamma(\delta(\mathcal{D})) \prec \gamma(c, C) \prec \mathcal{C},$$

whence $\mathcal{C} \in \upsilon$. Conversely, take $\mathcal{C} \in \upsilon$ and choose $\mathcal{E} \in \upsilon$ with $\mathcal{E} \prec_{(\star)} \mathcal{C}$. Without loss of generality we may assume $\mathcal{E}$ is anchored, so by Lemma 2.2 (2) we have $\mathcal{E} \prec_{(\Delta)} \mathcal{C}$, whence $\mathcal{E}^{\Delta} \prec \mathcal{C}$. Now Theorem 2.7 (2) gives $\gamma(\delta(\mathcal{E})) \prec \mathcal{C}$, so $\mathcal{C} \in \Gamma(\Delta(\upsilon))$, as required. $\square$

We will use the term *di-uniformity* to refer to direlational and dicovering uniformities in general.

Example 3.8. Consider the texture $(\mathbb{I}, \mathcal{I})$. The dicovering uniformity $\upsilon_{\mathbb{I}}$ corresponding to the direlational uniformity $\mathcal{U}_{\mathbb{I}}$ of Example 3.3 has a base consisting of the dicovers $\mathcal{D}_\epsilon$, $\epsilon > 0$, where

$$\mathcal{D}_\epsilon = \{([0, r+\epsilon), [0, r-\epsilon]) \mid r \in I\},$$

and $[0, r+\epsilon)$ is understood to be $[0,1]$ when $r + \epsilon > 1$ and $[0, r-\epsilon]$ is $\varnothing$ if $r - \epsilon < 0$.

4. The Uniform Ditopology

We begin by associating a ditopology with a direlational uniformity.

Proposition 4.1. *Let $(S, \mathcal{S}, \mathcal{U})$ be a direlational uniform texture space. Then the family $(\eta_{\mathcal{U}}(s), \mu_{\mathcal{U}}(s))$, $s \in S^\flat$, defined by*

$$\begin{aligned}
\eta_{\mathcal{U}}(s) &= \{N \in \mathcal{S} \mid N \not\subseteq Q_s,\ P_s \not\subseteq Q_t \implies \exists\, (d, D) \in \mathcal{U},\ d[t] \subseteq N\},\\
\mu_{\mathcal{U}}(s) &= \{M \in \mathcal{S} \mid P_s \not\subseteq M,\ P_t \not\subseteq Q_s \implies \exists\, (d, D) \in \mathcal{U},\ M \subseteq D[t]\},
\end{aligned}$$

is the dineighbourhood system for a ditopology on $(S, \mathcal{S})$.

Proof. We must verify that the family $\eta_{\mathcal{U}}(s)$, $s \in S^\flat$, satisfies the following conditions [6]:

(1) $N \in \eta_{\mathcal{U}}(s) \implies N \not\subseteq Q_s$.
(2) $N \in \eta_{\mathcal{U}}(s),\ N \subseteq N' \in \mathcal{S} \implies N' \in \eta_{\mathcal{U}}(s)$.
(3) $N_1, N_2 \in \eta_{\mathcal{U}}(s),\ N_1 \cap N_2 \not\subseteq Q_s \implies N_1 \cap N_2 \in \eta_{\mathcal{U}}(s)$.
(4) (a) $N \in \eta_{\mathcal{U}}(s) \implies \exists\, N^\star \in \mathcal{S},\ P_s \subseteq N^\star \subseteq N$ so that $N^\star \not\subseteq Q_t \implies N^\star \in \eta_{\mathcal{U}}(t),\ t \in S^\flat$.
(b) For $N \in \mathcal{S}$ and $N \not\subseteq Q_s$, if there exists $N^\star \in \mathcal{S}, P_s \subseteq N^\star \subseteq N$ which satisfies $N^\star \not\subseteq Q_t \implies N^\star \in \eta_{\mathcal{U}}(t), t \in S^\flat$, then $N \in \eta_{\mathcal{U}}(s)$.

Conditions (1) and (2) are immediate from the definitions, and (3) follows at once from the inclusion $(d \sqcap e)(P_t) \subseteq d(P_t) \cap e(P_t)$ (Proposition 1.9 (5)).

(4) (a). Take $N \in \eta_{\mathcal{U}}(s)$ and define

$$N^\star = \bigvee \{P_z \mid P_z \not\subseteq Q_t \implies \exists\, (d, D) \in \mathcal{U} \text{ with } d[t] \subseteq N\}.$$

Clearly $P_s \subseteq N^\star \subseteq N$ and if $N^\star \not\subseteq Q_t$ it is easy to show that $N \in \eta_{\mathcal{U}}(t)$.

(4) (b). Take $N \in \mathcal{S}$ with $N \not\subseteq Q_s$ and $N^\star \in \mathcal{S}$ with $P_s \subseteq N^\star \subseteq N$ and satisfying $N^\star \not\subseteq Q_t \implies N^\star \in \eta_{\mathcal{U}}(t)$. To show $N \in \eta_{\mathcal{U}}(s)$ take $t \in S$ with

$P_s \not\subseteq Q_t$. Since $P_s \subseteq N^\star$ we have $N^\star \not\subseteq Q_t$. Choose $t' \in S$ with $N^\star \not\subseteq Q_{t'}$ and $P_{t'} \not\subseteq Q_t$. By hypothesis $N \in \eta_{\mathcal{U}}(t')$ so $P_{t'} \not\subseteq Q_t$ now gives $(d, D) \in \mathcal{U}$ with $d[t] \subseteq N$. Since $N \not\subseteq Q_s$ this shows that $N \in \eta_{\mathcal{U}}(s)$, as required.

In just the same way the sets $\mu_{\mathcal{U}}(s)$ satisfy the dual of conditions (1)–(4) above, and this completes the proof (cf. [6]). $\square$

Definition 4.2. Let $(S, \mathcal{S}, \mathcal{U})$ be a direlational uniform texture space and $\eta_{\mathcal{U}}(s)$, $\mu_{\mathcal{U}}(s)$ defined as above. The ditopology with dineighbourhood system $\{(\eta_{\mathcal{U}}(s), \mu_{\mathcal{U}}(s)) \mid s \in S^\flat\}$ is called the *uniform ditopology of* $\mathcal{U}$ and denoted by $(\tau_{\mathcal{U}}, \kappa_{\mathcal{U}})$.

Lemma 4.3. *Let* $(S, \mathcal{S}, \mathcal{U})$ *be a direlational uniform texture space with uniform ditopology* $(\tau_{\mathcal{U}}, \kappa_{\mathcal{U}})$.

(i) $G \in \tau_{\mathcal{U}} \iff (G \not\subseteq Q_s \implies \exists\, (d, D) \in \mathcal{U}$ *with* $d[s] \subseteq G)$.

(ii) $K \in \kappa_{\mathcal{U}} \iff (P_s \not\subseteq K \implies \exists\, (d, D) \in \mathcal{U}$ *with* $K \subseteq D[s])$.

Proof. We prove (i), leaving (ii) to the reader.

It is shown in [6] that the open sets are characterized by the property that $G \not\subseteq Q_s \implies G \in \eta_{\mathcal{U}}(s)$. Take $G \in \tau_{\mathcal{U}}$ and $s \in S$ with $G \not\subseteq Q_s$. Now we have $s' \in S$ with $G \not\subseteq Q_{s'}$ and $P_{s'} \not\subseteq Q_s$. By the above $G \in \eta_{\mathcal{U}}(s')$ and now $P_{s'} \not\subseteq Q_s$ implies there exists $(d, D) \in \mathcal{U}$ with $d[s] \subseteq G$.

Conversely suppose G has the property stated in (i). Then if $G \not\subseteq Q_s$ we have $(d, D) \in \mathcal{U}$ with $d[s] \subseteq G$. Now if $P_s \not\subseteq Q_t$ we have $P_t \subseteq P_s$ and so $d[t] \subseteq d[s] \subseteq G$, which shows that $G \in \eta_{\mathcal{U}}(s)$. Thus $G \in \tau_{\mathcal{U}}$. $\square$

Proposition 4.4. *Let* υ *be a dicovering uniformity on* $(S, \mathcal{S})$. *Denote by* (τ, κ) *the uniform ditopology of the direlational uniformity* $\Delta(\upsilon)$. *Then:*

(i) $G \in \tau \iff (G \not\subseteq Q_s \implies \exists\, \mathcal{C} \in \upsilon$ *with* $\mathrm{St}(\mathcal{C}, P_s) \subseteq G)$.

(ii) $K \in \kappa \iff (P_s \not\subseteq K \implies \exists\, \mathcal{C} \in \upsilon$ *with* $K \subseteq \mathrm{CSt}(\mathcal{C}, Q_s))$.

Proof. (i). Take $G \in \tau$ and $G \not\subseteq Q_s$. Then by Lemma 4.3 we have $(d, D) \in \Delta(\upsilon)$ with $d[s] \subseteq G$. We may take $(e, E) \in \Delta(\upsilon)$ with $(e, E) \circ (e, E)^{\leftarrow} \sqsubseteq (d, D)$ and as in the proof of Proposition 2.4 we have $\mathrm{St}(\gamma(e, E), P_s) \subseteq d[s]$. There exists $\mathcal{C} \in \upsilon$ with $\delta(\mathcal{C}) \sqsubseteq (e, E)$ and without loss of generality we may assume $\mathcal{C}$ is anchored. Hence by Corollary 2.8 (2), $\mathcal{C} \prec \gamma(\delta(\mathcal{C})) \prec \gamma(e, E)$, so $\mathrm{St}(\mathcal{C}, P_s) \subseteq d[s] \subseteq G$.

Conversely, suppose that given $G \not\subseteq Q_s$ there exists $\mathcal{C} \in \upsilon$ with $\mathrm{St}(\mathcal{C}, P_s) \subseteq G$. If we set $(d, D) = \delta(\mathcal{C})$ then $(d, D) \in \Delta(\upsilon)$ and by Theorem 2.7 (2) we have $d[s] = \mathrm{St}(\mathcal{C}, P_s) \subseteq G$. Hence $G \in \tau$.

(ii). The proof is dual to (i), and is omitted. $\square$

This justifies the following definition.

Definition 4.5. Let υ be a dicovering uniformity on $(S, \mathcal{S})$. Then the ditopology $(\tau_\upsilon, \kappa_\upsilon)$ defined by

$$\begin{aligned} \tau_\upsilon &= \{G \in \mathcal{S} \mid G \not\subseteq Q_s \implies \exists\, \mathcal{C} \in \upsilon,\ \mathrm{St}(\mathcal{C}, P_s) \subseteq G\}, \\ \kappa_\upsilon &= \{K \in \mathcal{S} \mid P_s \not\subseteq K \implies \exists\, \mathcal{C} \in \upsilon,\ K \subseteq \mathrm{CSt}(\mathcal{C}, Q_s)\}, \end{aligned}$$

is called the *uniform ditopology of* υ.

In just the same way the dineighbourhood system $(\eta_\upsilon(s), \mu_\upsilon(s))$, $s \in S^\flat$, for $(\tau_\upsilon, \kappa_\upsilon)$ is given by

$$\eta_\upsilon(s) = \{N \in \mathcal{S} \mid N \not\subseteq Q_s,\ P_s \not\subseteq Q_t \implies \exists\, \mathcal{C} \in \upsilon,\ \mathrm{St}(\mathcal{C}, P_t) \subseteq N\},$$
$$\mu_\upsilon(s) = \{M \in \mathcal{S} \mid P_s \not\subseteq M,\ P_t \not\subseteq Q_s \implies \exists\, \mathcal{C} \in \upsilon,\ M \subseteq \mathrm{CSt}(\mathcal{C}, Q_t\}.$$

We omit the details.

The following lemma enables us to generate open sets and closed sets for the uniform ditopology of a dicovering uniformity.

Lemma 4.6. *Let υ be a dicovering uniformity on $(S, \mathcal{S})$ and take $L \in \mathcal{S}$.*

(1) *The set*

$$G = G(L) = \bigvee\{P_u \mid \exists\, \mathcal{D} \in \upsilon,\ \mathrm{St}(\mathcal{D}, P_u) \subseteq L\}$$

is open for the uniform ditopology.

(2) *The set*

$$K = K(L) = \bigcap\{Q_u \mid \exists\, \mathcal{D} \in \upsilon,\ L \subseteq \mathrm{CSt}(\mathcal{D}, Q_u)\}$$

is closed for the uniform ditopology.

Proof. We establish (1), leaving the dual proof of (2) to the reader.

Take $G \not\subseteq Q_s$. Then we have $u \in S$ and $\mathcal{D} \in \upsilon$ satisfying $P_u \not\subseteq Q_s$ and $\mathrm{St}(\mathcal{D}, P_u) \subseteq L$. Take $\mathcal{E} \in \upsilon$ with $\mathcal{E} \prec_{(\star)} \mathcal{D}$. By Definition 4.5 it will be sufficient to show that $\mathrm{St}(\mathcal{E}, P_s) \subseteq G$. If this is not so then we have $A_0\, \mathcal{E}\, B_0$ with $P_s \not\subseteq B_0$ and $A_0 \not\subseteq G$ so we may take $v \in S$ with $A_0 \not\subseteq Q_v$ and $P_v \not\subseteq G$. If we can show that $\mathrm{St}(\mathcal{E}, P_v) \subseteq \mathrm{St}(\mathcal{D}, P_u)$ we will obtain an immediate contradiction to the definition of G, so take $A_1\, \mathcal{E}\, B_1$ with $P_v \not\subseteq B_1$, and choose $A_0'\, \mathcal{D}\, B_0'$ satisfying $\mathrm{St}(\mathcal{E}, A_0) \subseteq A_0'$ and $B_0' \subseteq \mathrm{CSt}(\mathcal{E}, B_0)$. Since $\mathrm{CSt}(\mathcal{E}, B_0) \subseteq B_0$, $P_s \not\subseteq B_0$ and $P_u \not\subseteq Q_s$ we see that $P_u \not\subseteq B_0'$, whence $A_1 \subseteq \mathrm{St}(\mathcal{E}, A_0) \subseteq A_0' \subseteq \mathrm{St}(\mathcal{D}, P_u)$, using the evident fact that $A_0 \not\subseteq B_1$. This establishes the required inclusion and completes the proof. □

Corresponding results for direlational uniformities may easily be formulated and the details are left to the interested reader.

It is well known that a classical uniformity has a base of open members and a base of closed members. We now establish an analogous result for di-uniformities. We confine our attention to the dicovering case since there is a well established meaning to the notions of openness and closedness for dicovers [3]. Namely, a dicover $\mathcal{C}$ of the ditopological texture space $(S, \mathcal{S}, \tau, \kappa)$ is *open* (respectively, *closed*, *co-open*, *coclosed*) if $A\, \mathcal{C}\, B \implies A \in \tau$ ($A \in \kappa$, $B \in \tau$, $B \in \kappa$). First we require the following lemma.

Lemma 4.7. *Let υ be a dicovering uniformity on $(S, \mathcal{S})$, $\mathcal{C} \in \upsilon$ and $L \in \mathcal{S}$. Consider the uniform ditopology on $(S, \mathcal{S})$. Then:*

(1) $L \subseteq \,]\mathrm{St}(\mathcal{C}, L)[$ *and* $[\mathrm{CSt}(\mathcal{C}, L)] \subseteq L$.

(2) $[L] \subseteq \mathrm{St}(\mathcal{C}, L)$ *and* $\mathrm{CSt}(\mathcal{C}, L) \subseteq \,]L[$.

Proof. 1. If $H = H(\mathrm{St}(\mathcal{C}, L))$ is the open set defined in Lemma 4.6 (1) it is trivial to verify that $L \subseteq H \subseteq \mathrm{St}(\mathcal{C}, L)$, whence $L \subseteq\,]\mathrm{St}(\mathcal{C}, L)[$. The second inclusion follows in the same way from Lemma 4.6 (2).

2. If $K = K(L)$ is the closed set defined in Lemma 4.6 (2) it is trivial to verify that $L \subseteq K \subseteq \mathrm{St}(\mathcal{C}, L)$, whence $[L] \subseteq \mathrm{St}(\mathcal{C}, L)$. The second inclusion follows in the same way from Lemma 4.6 (1). □

Proposition 4.8. *A dicovering uniformity has a base of open, coclosed dicovers and a base of closed, co-open dicovers.*

Proof. Trivial from Lemma 4.7. □

Definition 4.9. A ditopological texture space $(S, \mathcal{S}, \tau, \kappa)$ is called *di-uniformizable* if there exists a di-uniformity on $(S, \mathcal{S})$ whose uniform ditopology coincides with (τ, κ).

We recall the following regularity axioms for ditopological texture spaces.

Definition 4.10. [3] Let (τ, κ) be a ditopology on $(S, \mathcal{S})$. Then (τ, κ) is called

(1) *Regular* if $G \in \tau$, $G \not\subseteq Q_s \implies \exists H \in \tau$ with $H \not\subseteq Q_s$, $[H] \subseteq G$.
(2) *Coregular* if $F \in \kappa$, $P_s \not\subseteq F \implies \exists K \in \kappa$ with $P_s \not\subseteq K$, $F \subseteq\,]K[$.
(3) *Biregular* if it is regular and coregular.

Using Proposition 4.8 it is straightforward to verify that a di-uniformizable ditopology is biregular. However we will shortly prove a more powerful result, and so omit the details.

Definition 4.11. [7] Let (τ, κ) be a ditopology on $(S, \mathcal{S})$. Then (τ, κ) is called

(1) *Completely regular* if given $G \in \tau$, $G \not\subseteq Q_s$, there exists a bicontinuous difunction $(f, F) : (S, \mathcal{S}) \to (\mathbb{I}, \mathcal{I})$ satisfying $P_s \subseteq f^{\leftarrow}(P_0)$ and $F^{\leftarrow}(Q_1) \subseteq G$.
(2) *Completely coregular* if given $K \in \tau$, $P_s \not\subseteq K$, there exists a bicontinuous difunction $(f, F) : (S, \mathcal{S}) \to (\mathbb{I}, \mathcal{I})$ satisfying $K \subseteq f^{\leftarrow}(P_0)$ and $F^{\leftarrow}(Q_1) \subseteq Q_s$.
(3) *Completely biregular* if it is completely regular and completely coregular.

We end this section by showing that a di-uniformizable ditopology is completely biregular. Since it is easy to see that the complete regularity conditions imply the corresponding regularity conditions it will follow that a di-uniformizable ditopology is biregular. We choose to work with direlational uniformities. First we require the following lemma, which is the textural analogue of the Metrization Lemma ([10], Page 185).

Lemma 4.12. *Let $(S, \mathcal{S})$ be a texture and r_n, $n \in \mathbb{N}$, a sequence of reflexive relations satisfying $r_{n+1}^3 \subseteq r_n$, $n \in \mathbb{N}$. Define the function $\varphi : S \times S \to [0, 1]$ by*

$$\varphi(u,v) = \begin{cases} 0 & \textit{if } r_n \not\subseteq \overline{Q}_{(u,v)}\ \forall\, n \in \mathbb{N}, \\ 1 & \textit{if } r_n \subseteq \overline{Q}_{(u,v)}\ \forall\, n \in \mathbb{N}, \\ 2^{-n} & \textit{if } \exists\, n \in \mathbb{N},\ r_n \not\subseteq \overline{Q}_{(u,v)},\ r_{n+1} \subseteq \overline{Q}_{(u,v)}. \end{cases}$$

Then there exists a function $q : S \times S \to [0, \infty)$ *satisfying*

(1) $\frac{1}{2}\varphi(u,v) \leq q(u,v) \leq \varphi(u,v)$, $\forall\, u,v \in S$.
(2) $P_u \not\subseteq Q_v \implies q(u,v) = 0\ \forall\, u,v \in S$.
(3) $q(u,v) \leq q(u,w) + q(w,v)\ \forall\, u,v,w \in S$.

Proof. We consider chains $u_0, u_1, \dots, u_n$ of elements of S and write

$$s(u_0, u_1, \dots, u_n) = \sum_{i=1}^{n-1} \varphi(u_i, u_{i+1}),\ n > 0,\ s(u_0, u_0) = \varphi(u_0, u_0) = 0.$$

Consider the function $q : S \times S \to [0, \infty)$ defined by

$$q(u,v) = \inf\{s(u_0, \dots, u_n) \mid u = u_0 \text{ and } v = u_n,\ n \in \mathbb{N}\}.$$

(1) It is clearly sufficient to prove that $\varphi(u,v) \leq 2s(u_0, \dots, u_n)$ for any chain $u_0, u_1, \dots, u_n$ with $u = u_0$ and $v = u_n$. The proof is by induction on $n \in \mathbb{N}$ and follows essentially the same steps as the proof of the Metrizaton Lemma. We therefore omit the details.

(2) For each $n \in \mathbb{N}$ we have $i_S \subseteq r_n$ so $P_u \not\subseteq Q_v$ implies $i_S \not\subseteq \overline{Q}_{(u,v)}$, and hence $r_n \not\subseteq \overline{Q}_{(u,v)}$. By (1) we now have $0 \leq q(u,v) \leq \varphi(u,v) = 0$, whence $q(u,v) = 0$.

(3) Immediate from the definition of q. □

Lemma 4.13. *If we consider a sequence of corelations* R_n*,* $n \in \mathbb{N}$*, satisfying* $R_n \subseteq R^3_{n+1}$ *and define*

$$\varphi^*(u,v) = \begin{cases} 0 & \text{if } \overline{P}_{(u,v)} \not\subseteq R_n\ \forall\, n \in \mathbb{N}, \\ 1 & \text{if } \overline{P}_{(u,v)} \subseteq R_n\ \forall\, n \in \mathbb{N}, \\ 2^{-n} & \text{if } \exists\, n \in \mathbb{N},\ \overline{P}_{(u,v)} \not\subseteq R_n,\ \overline{P}_{(u,v)} \subseteq R_{n+1}, \end{cases}$$

we obtain $q^* : S \times S \to [0, \infty)$ *satisfying*

(1) $\frac{1}{2}\varphi^*(u,v) \leq q^*(u,v) \leq \varphi^*(u,v)$, $\forall\, u,v \in S$.
(2) $P_v \not\subseteq Q_u \implies q^*(u,v) = 0\ \forall\, u,v \in S$.
(3) $q^*(u,v) \leq q^*(u,w) + q^*(w,v)\ \forall\, u,v,w \in S$.

In case $R_n = r_n^{\leftarrow}$ *then we clearly have* $\varphi^*(u,v) = \varphi(v,u)$ *and* $q^*(u,v) = q(v,u)$ *for all* $u,v \in S$.

Now we may give:

Theorem 4.14. *A diuniformizable ditopological texture space is completely biregular.*

Proof. Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space and $\mathcal{U}$ a compatible direlational uniformity.

To show that (τ, κ) is completely regular take $G \in \tau$ and $a \in S$ with $G \not\subseteq Q_a$. Then there exists $(r, R) \in \mathcal{U}$ with $r(P_a) \subseteq G$. Let $(r_0, R_0) = (r, R)$. By Definition 3.1 there exists $(r_1, R_1) \in \mathcal{U}$ such that $(r_1, R_1)^3 \sqsubseteq (r_0, R_0)$, $(r_2, R_2) \in \mathcal{U}$ with $(r_2, R_2)^3 \sqsubseteq (r_1, R_1)$, and so on. Hence we obtain a sequence (r_n, R_n) of reflexive direlations satisfying $(r_{n+1}, R_{n+1})^3 \sqsubseteq (r_n, R_n)$, and by

Lemma 3.2 there is no loss of generality in assuming that the (r_n, R_n) are symmetric, i.e. $R_n = r_n^{\leftarrow}$ for each $n \in \mathbb{N}$. Let φ and q be the functions given in Lemma 4.12 for the sequence r_n, $n \in \mathbb{N}$ of reflexive relations and define $\theta : S \to [0,1]$ by

$$\theta(s) = 2q(a,s) \wedge 1.$$

We take the texture $\mathcal{I}$ on $\mathbb{I} = [0,1]$ and verify that the point function θ satisfies the condition $P_u \not\subseteq Q_v \implies P_{\theta(u)} \not\subseteq Q_{\theta(v)}$ of ([1], Theorem 3.14). However if $P_u \not\subseteq Q_v$ then $q(a,v) \leq q(a,u) + q(u,v) = q(a,u)$ by Lemma 4.12 (2),(3), so $\theta(v) \leq \theta(u)$, which is equivalent to $P_{\theta(u)} \not\subseteq Q_{\theta(v)}$ in $(\mathbb{I}, \mathcal{I})$. It follows that

$$f = \bigvee \{\overline{P}_{(s,t)} \mid \exists\, v \in S \text{ with } P_s \not\subseteq Q_v \text{ and } t \leq \theta(v)\},$$

$$F = \bigcap \{\overline{Q}_{(s,t)} \mid \exists\, v \in S \text{ with } P_v \not\subseteq Q_s \text{ and } \theta(v) \leq t\},$$

define a difunction $(f,F) : (S, \mathcal{S}) \to (\mathbb{I}, \mathcal{I})$. If we take the usual ditopology $(\tau_{\mathbb{I}}, \kappa_{\mathbb{I}})$ on $(\mathbb{I}, \mathcal{I})$ then (f,F) is bicontinuous. We prove continuity, leaving the dual proof of cocontinuity to the reader.

For $s \in S$ suppose $F^{\leftarrow}([0,r)) \not\subseteq Q_s$. Then we have $t \in \mathbb{I}$ with $\overline{P}_{(s,t)} \not\subseteq F$ and $t < r$. From the definition of F we have $v \in S$ and $t' \in \mathbb{I}$ with $\overline{P}_{(s,t)} \not\subseteq \overline{Q}_{(s,t')}$, $P_v \not\subseteq Q_s$ and $\theta(v) \leq t'$. From $P_t \not\subseteq Q_{t'}$ we have $t' \leq t$ and so $\theta(v) < r$. Clearly $\theta(v) < 1$ and so $\theta(v) = 2q(a,v) < r$, whence there exists n with $2(q(a,v) + 2^{-n}) < r$. We verify that $r_n(P_s) \subseteq F^{\leftarrow}([0,r))$. Suppose the contrary and take $w \in S$ with $r_n(P_s) \not\subseteq Q_w$ and $P_w \not\subseteq F^{-1}([0,r))$. Now we have $z \in S$ with $r_n \not\subseteq \overline{Q}_{(z,w)}$ and $P_v \not\subseteq Q_z$, whence $r_n \not\subseteq \overline{Q}_{(v,w)}$ and so $q(v,w) \leq \varphi(v,w) \leq 2^{-n}$ by Lemma 4.12. Hence we have

$$\theta(w) \leq 2q(a,w) \leq 2(q(a,v) + q(v,w)) \leq 2(q(a,v) + 2^{-n}) < r.$$

On the other hand, from $P_w \not\subseteq F^{\leftarrow}([0,r))$ we have $w' \in S$ with $P_w \not\subseteq Q_{w'}$ for which

$$\overline{P}_{(w',u)} \not\subseteq F \implies u \leq r. \tag{4.1}$$

Choose $r' \in \mathbb{I}$ satisfying $\theta(w) < r' < r$. Then $P_{r'} \not\subseteq Q_{\theta(w)}$ and $P_w \not\subseteq Q_{w'}$, so by the definition of F we have $F \subseteq \overline{Q}_{(w',r')}$, which is equivalent to $\overline{P}_{(w',r')} \not\subseteq F$. Applying implication (4.1) with $u = r'$ now gives the contradiction $r \leq r'$, and we have proved $F^{\leftarrow}([0,r)) \not\subseteq Q_s \implies r_n(P_s) \subseteq F^{\leftarrow}([0,r))$. Hence $F^{\leftarrow}([0,r)) \in \tau$ since $(r_n, R_n) \in \mathcal{U}$ and $\tau = \tau_{\mathcal{U}}$. This proves continuity since $F^{\leftarrow}(\mathbb{I}) = S \in \tau$.

It remains to show that $P_a \subseteq f^{\leftarrow}(P_0)$ and $F^{\leftarrow}(Q_1) \subseteq G$. Suppose first that $P_a \not\subseteq f^{\leftarrow}(P_0)$. Then we have $b \in \mathbb{I}$ with $f \not\subseteq \overline{Q}_{(a,b)}$ and $P_b \not\subseteq P_0$, that is $b > 0$. By the definition of f we have $b' \in \mathbb{I}$ with $\overline{P}_{(a,b')} \not\subseteq \overline{Q}_{(a,b)}$ and $v \in S$ with $P_a \not\subseteq Q_v$ satisfying $b' \leq \theta(v)$. Hence $0 < b \leq b' \leq \theta(v)$ whence $q(a,v) > 0$. However $P_a \not\subseteq Q_v$ implies $q(a,v) = 0$ by Lemma 4.12, which is a contradiction.

If now we suppose $F^{\leftarrow}([0,1)) \not\subseteq G$ then we have $s \in S$ satisfying $F^{\leftarrow}([0,1)) \not\subseteq Q_s$ and $P_s \not\subseteq G$. Hence we have $t \in \mathbb{I}$ with $\overline{P}_{(s,t)} \not\subseteq F$ and $[0,1) \not\subseteq Q_t$, that is $t < 1$. From the definition of F we now have $t' \in \mathbb{I}$ with $\overline{P}_{(s,t)} \not\subseteq \overline{Q}_{(s,t')}$

and $v \in S$ with $P_v \not\subseteq Q_s$ and $\theta(v) \le t'$. Hence $\theta(v) \le t' \le t < 1$, whence $2q(a,v) < 1$ and so $\varphi(a,v) < 1$ by Lemma 4.12. Hence there exists $n \in \mathbb{N}$ with $r_n \not\subseteq \overline{Q}_{(a,v)}$ and so $r = r_0 \not\subseteq \overline{Q}_{(a,v)}$. This leads to $r(P_a) \not\subseteq Q_a$ and so $G \not\subseteq Q_v$. On the other hand $P_v \not\subseteq Q_s$ and $P_s \not\subseteq G$ give $P_v \not\subseteq G$, and we have the contradiction $G \subseteq Q_v$.

This completes the proof of complete regularity, and complete coregularity can be proved in a similar way using the conjugate functions φ^* and q^* of Lemma 4.13. □

The converse of the above proposition is also true, but we postpone the proof until we have discussed initial di-uniformities in the next section.

Definition 4.15. A direlational uniformity $\mathcal{U}$ satisfying $\bigsqcap \mathcal{U} = (i, I)$ is called *separated.*

We recall from [7] the following characteristic property of T_0 ditopological spaces:

$$(\tau, \kappa) \text{ is } T_0 \text{ if and only if } Q_s \not\subseteq Q_t \implies \exists\, B \in \tau \cup \kappa \text{ with } P_s \not\subseteq B \not\subseteq Q_t.$$

Theorem 4.16. *Let $\mathcal{U}$ be a direlational uniformity. Then the uniform ditopology $(\tau_{\mathcal{U}}, \kappa_{\mathcal{U}})$ is T_0 if and only if $\mathcal{U}$ is separated.*

Proof. $\implies$. We know that $i \subseteq \bigsqcap\{d \mid (d,D) \in \mathcal{U}\}$, so suppose $\bigsqcap\{d \mid (d,D) \in \mathcal{U}\} \not\subseteq i$. Then we have $s, t \in S$ with $\overline{P}_{(s,t)} \not\subseteq i$ for which we have $s' \in S$ satisfying $d \not\subseteq \overline{Q}_{(s',t)}$ for all $(d,D) \in \mathcal{U}$. Now $\overline{P}_{(s,t)} \not\subseteq i$ implies $P_t \not\subseteq P_s$, which with $P_s \not\subseteq Q_{s'}$ gives $Q_t \not\subseteq Q_s$. Since $(\tau_{\mathcal{U}}, \kappa_{\mathcal{U}})$ is T_0 we have $B \in \tau_{\mathcal{U}} \cup \kappa_{\mathcal{U}}$ satisfying $P_t \not\subseteq B \not\subseteq Q_s$. There are two cases to consider:

(a) $B \in \tau_{\mathcal{U}}$. Now $B \not\subseteq Q_{s'}$ implies $d[s'] \subseteq B$ for some $(d,D) \in \mathcal{U}$. It follows that $P_t \not\subseteq d[s'] = d(P_{s'})$. But now $d(P_{s'}) \subseteq Q_t$, so by Lemma 1.5(2) we have $d \subseteq \overline{Q}_{(s',t)}$, which is a contradiction.

(b) $B \in \kappa_{\mathcal{U}}$. Noting that $\mathcal{U}$ has a base of symmetric direlations, a dual argument again leads to a contradiction.

This completes the proof of $\bigsqcap\{d \mid (d,D) \in \mathcal{U}\} = i$, and $\bigsqcup\{D \mid (d,D) \in \mathcal{U}\} = I$ is dual.

$\impliedby$. Take $s, t \in S$ with $Q_s \not\subseteq Q_t$. By the definition of Q_s there exists $u \in S$ with $P_s \not\subseteq P_u$ and $P_u \not\subseteq Q_t$. Take $s', t' \in S$ satisfying $P_s \not\subseteq Q_{s'}$, $P_{s'} \not\subseteq P_u$ and $P_u \not\subseteq Q_{t'}$, $P_{t'} \not\subseteq Q_t$. Then $\overline{P}_{(u,s')} \not\subseteq i = \bigsqcap\{d \mid (d,D) \in \mathcal{U}\}$ since $P_{s'} \not\subseteq P_u$, so as $P_u \not\subseteq Q_{t'}$ there exists $(e,E) \in \mathcal{U}$ with $e \subseteq \overline{Q}_{(t',s')}$, whence $e(P_{t'}) \subseteq Q_{s'}$.

Now let $G = \bigvee\{P_z \mid z \in S,\ \exists\,(d,D) \in \mathcal{U} \text{ with } d[z] \subseteq Q_{s'}\}$. It may be shown that $G \in \tau_{\mathcal{U}}$ (compare Lemma 4.6). Clearly $P_{t'} \subseteq G$, and so $G \not\subseteq Q_t$. On the other hand if $d[z] \subseteq Q_{s'}$ then $P_z \subseteq d[z] \subseteq Q_{s'}$ so $G \subseteq Q_{s'}$ and hence $P_s \not\subseteq G$. This verifies that $(\tau_{\mathcal{U}}, \kappa_{\mathcal{U}})$ is T_0. □

5. Uniform Bicontinuity and Initial Di-uniformities

In order to define uniform bicontinuity it will be necessary to say what we mean by the inverse of a direlation and of a dicover under a difunction. We begin with the following:

Definition 5.1. Let $(S,\mathcal{S})$, $(T,\mathcal{T})$ be textures, (r,R) a direlation on $(T,\mathcal{T})$ and (f,F) a difunction on $(S,\mathcal{S})$ to $(T,\mathcal{T})$. Then

$$\begin{aligned}(f,F)^{-1}(r) &= \bigvee\{\overline{P}_{(s_1,s_2)} \mid \exists P_{s_1} \not\subseteq Q_{s_1'} \text{ so that } \overline{P}_{(s_1',t_1)} \not\subseteq F,\ f \not\subseteq \overline{Q}_{(s_2,t_2)} \\ &\qquad\qquad \implies \overline{P}_{(t_1,t_2)} \subseteq r\},\\ (f,F)^{-1}(R) &= \bigcap\{\overline{Q}_{(s_1,s_2)} \mid \exists P_{s_1'} \not\subseteq Q_{s_1} \text{ so that } f \not\subseteq \overline{Q}_{(s_1',t_1)}, \overline{P}_{(s_2,t_2)} \not\subseteq F, \\ &\qquad\qquad \implies R \subseteq \overline{Q}_{(t_1,t_2)}\},\\ (f,F)^{-1}(r,R) &= ((f,F)^{-1}(r), (f,F)^{-1}(R)).\end{aligned}$$

Remark 5.2. In Definition 5.1, $\overline{P}_{(t_1,t_2)} \subseteq r$ may be replaced by $r \not\subseteq \overline{Q}_{(t_1,t_2)}$ and $R \subseteq \overline{Q}_{(t_1,t_2)}$ by $\overline{P}_{(t_1,t_2)} \not\subseteq R$. Indeed, if s_1', s_2 satisfy the conditions in the definition of $(f,F)^{-1}(r)$, and $\overline{P}_{(s_1',t_1)} \not\subseteq F$, $f \not\subseteq \overline{Q}_{(s_2,t_2)}$, then we may choose t_2' with $f \not\subseteq \overline{Q}_{(s_2,t_2')}$, $\overline{P}_{(s_2,t_2')} \not\subseteq \overline{Q}_{(s_2,t_2)}$. This gives us $\overline{P}_{(t_1,t_2')} \subseteq r$, and so $r \not\subseteq \overline{Q}_{(t_1,t_2)}$ since $P_{t_2'} \not\subseteq Q_{t_2}$. The opposite direction is trivial, and the second property is dual.

It is trivial to verify that $(f,F)^{-1}(r,R)$ is indeed a direlation on $(S,\mathcal{S})$, and we omit the proof. Let us examine the properties of this inverse mapping.

Proposition 5.3. *Let (f,F) be a difunction on $(S,\mathcal{S})$ to $(T,\mathcal{T})$. Then*

$$(f,F)^{-1}(i_T,I_T) = (i_S,I_S),$$

where (i_S,I_S), (i_T,I_T) are the identity direlations on $(S,\mathcal{S})$, $(T,\mathcal{T})$ respectively.

Proof. To establish $(f,F)^{-1}(i_T) = i_S$ we first suppose that $i_S \not\subseteq (f,F)^{-1}(i_T)$. Then $i_S \not\subseteq \overline{Q}_{(s,s')}$ and $\overline{P}_{(s,s')} \not\subseteq (f,F)^{-1}(i_T)$ for some $s,s' \in S$. We have $P_s \not\subseteq Q_{s'}$ since $i_S \not\subseteq \overline{Q}_{(s,s')}$. By Definition 5.1 there exists $w_1,w_2 \in T$ satisfying $\overline{P}_{(s',w_1)} \not\subseteq F$, $f \not\subseteq \overline{Q}_{(s',w_2)}$ and $\overline{P}_{(w_1,w_2)} \not\subseteq i_T$. On the other hand *DF2* implies $P_{w_1} \not\subseteq Q_{w_2}$. Hence $i_T \not\subseteq \overline{Q}_{(w_1,w_2)}$ which contradicts $\overline{P}_{(w_1,w_2)} \not\subseteq i_T$.

The proof of the reverse inclusion is similar and the proof of the dual equality $(f,F)^{-1}(I_T) = I_S$ is left to the reader. □

Corollary 5.4. *Let (f,F) be a difunction on $(S,\mathcal{S})$ to $(T,\mathcal{T})$. If (r,R) is a reflexive direlation on $(T,\mathcal{T})$ then $(f,F)^{-1}(r,R)$ is a reflexive direlation on $(S,\mathcal{S})$.*

Proof. Let (r,R) be reflexive. Then

$$(i_T,I_T) \sqsubseteq (r,R) \implies (i_S,I_S) = (f,F)^{-1}(i_T,I_T) \sqsubseteq (f,F)^{-1}(r,R)$$

by the proposition so $(f,F)^{-1}(r,R)$ is reflexive. □

Proposition 5.5. *Let (f,F) be a difunction on $(S,\mathcal{S})$ to $(T,\mathcal{T})$ and (r,R) a direlation on $(T,\mathcal{T})$. Then*

$$((f,F)^{-1}(r,R))^{\leftarrow} = (f,F)^{-1}((r,R)^{\leftarrow}).$$

Proof. It is clearly sufficient to establish $((f,F)^{-1}(r))^{\leftarrow} = (f,F)^{-1}(r^{\leftarrow})$, since the dual equality $((f,F)^{-1}(R))^{\leftarrow} = (f,F)^{-1}(R^{\leftarrow})$ then follows by replacing r by $R^{\leftarrow}$ and taking the inverse of both sides.

First suppose that $((f,F)^{-1}(r))^{\leftarrow} \not\subseteq (f,F)^{-1}(r^{\leftarrow})$. Then $((f,F)^{-1}(r))^{\leftarrow} \not\subseteq \overline{Q}_{(s,s')}$ and $\overline{P}_{(s,s')} \not\subseteq (f,F)^{-1}(r^{\leftarrow})$ for some $s,s' \in S$. Since $r^{\leftarrow}$ is a corelation, by Remark 5.2 we have $u,v \in S$ with $\overline{P}_{(s,s')} \not\subseteq \overline{Q}_{(s,v)}$, $P_u \not\subseteq Q_s$ so that

$$f \not\subseteq \overline{Q}_{(u,t_1)},\ \overline{P}_{(v,t_2)} \not\subseteq F \implies \overline{P}_{(t_1,t_2)} \not\subseteq r^{\leftarrow} \tag{5.2}$$

for all $t_1,t_2 \in T$. On the other hand $((f,F)^{-1}(r))^{\leftarrow} \not\subseteq \overline{Q}_{(s,s')}$ gives $(f,F)^{-1}(r) \subseteq \overline{Q}_{(s',s)}$ and so $\overline{P}_{(s',u)} \not\subseteq (f,F)^{-1}(r)$. Since $P_{s'} \not\subseteq Q_v$ we have $w_1,w_2 \in T$ for which $\overline{P}_{(v,w_1)} \not\subseteq F$, $f \not\subseteq \overline{Q}_{(u,w_2)}$ and $r \subseteq \overline{Q}_{(w_1,w_2)}$ by Remark 5.2. Putting $t_1 = w_2$, $t_2 = w_1$ in the implication (5.2) now gives $\overline{P}_{(w_2,w_1)} \not\subseteq r^{\leftarrow}$, so giving the contradiction $r \not\subseteq \overline{Q}_{(w_1,w_2)}$. Hence $((f,F)^{-1}(r))^{\leftarrow} \subseteq (f,F)^{-1}(r^{\leftarrow})$.

The proof of $(f,F)^{-1}(r^{\leftarrow}) \subseteq ((f,F)^{-1}(r))^{\leftarrow}$ is similar, and is omitted. □

Corollary 5.6. *Let (f,F) be a difunction on $(S,\mathcal{S})$ to $(T,\mathcal{T})$. If (r,R) is a symmetric direlation on $(T,\mathcal{T})$ then $(f,F)^{-1}(r,R)$ is a symmetric direlation on $(S,\mathcal{S})$.*

Proof. Immediate. □

Proposition 5.7. *Let (f,F) be a difunction on $(S,\mathcal{S})$ to $(T,\mathcal{T})$, (p,P) and (q,Q) direlations on $(T,\mathcal{T})$. Then*

$$(f,F)^{-1}(p,P) \circ (f,F)^{-1}(q,Q) \sqsubseteq (f,F)^{-1}((p,P) \circ (q,Q)).$$

Proof. Suppose that $(f,F)^{-1}(p) \circ (f,F)^{-1}(q) \not\subseteq (f,F)^{-1}(p \circ q)$. By the definition of composition of relations we have $s,u,z \in S$ with $\overline{P}_{(s,u)} \not\subseteq (f,F)^{-1}(p \circ q)$, $(f,F)^{-1}(q) \not\subseteq \overline{Q}_{(s,z)}$ and $(f,F)^{-1}(p) \not\subseteq \overline{Q}_{(z,u)}$. By *R2* there exists $s' \in S$ with $P_s \not\subseteq Q_{s'}$ and $(f,F)^{-1}(q) \not\subseteq \overline{Q}_{(s',z)}$. By Definition 5.1, $\overline{P}_{(s,u)} \not\subseteq (f,F)^{-1}(p \circ q)$ gives $w_1,w_2 \in S$ satisfying $\overline{P}_{(s',w_1)} \not\subseteq F$, $f \not\subseteq \overline{Q}_{(u,w_2)}$ and $\overline{P}_{(w_1,w_2)} \not\subseteq p \circ q$.

On the other hand from $(f,F)^{-1}(q) \not\subseteq \overline{Q}_{(s',z)}$ we have $z',s'' \in S$ with $\overline{P}_{(s',z')} \not\subseteq \overline{Q}_{(s',z)}$, $P_{s'} \not\subseteq Q_{s''}$ for which

$$\overline{P}_{(s'',t_1)} \not\subseteq F,\ f \not\subseteq \overline{Q}_{(z',t_2)} \implies q \not\subseteq \overline{Q}_{(t_1,t_2)} \tag{5.3}$$

for all $t_1,t_2 \in T$ by Remark 5.2. Likewise, $(f,F)^{-1}(p) \not\subseteq \overline{Q}_{(z,u)}$ gives $u',z'' \in S$ with $\overline{P}_{(z,u')} \not\subseteq \overline{Q}_{(z,u)}$, $P_z \not\subseteq Q_{z''}$ for which

$$\overline{P}_{(z'',v_1)} \not\subseteq F,\ f \not\subseteq \overline{Q}_{(u',v_2)} \implies p \not\subseteq \overline{Q}_{(v_1,v_2)} \tag{5.4}$$

for all $v_1,v_2 \in T$. Finally $P_{z'} \not\subseteq Q_{z''}$ so by *DF1* we have $v \in T$ for which $f \not\subseteq \overline{Q}_{(z',v)}$ and $\overline{P}_{(z'',v)} \not\subseteq F$.

By *CR1*, $\overline{P}_{(s',w_1)} \not\subseteq F$ and $P_{s'} \not\subseteq Q_{s''}$ gives $\overline{P}_{(s'',w_1)} \not\subseteq F$ so implication (5.3) may be applied with $t_1 = w_1$, $t_2 = v$ to give $q \not\subseteq \overline{Q}_{(w_1,v)}$. Likewise (5.4) may be applied with $v_1 = v$, $v_2 = w_2$ to give $p \not\subseteq \overline{Q}_{(v,w_2)}$. This gives the contradiction $\overline{P}_{(w_1,w_2)} \subseteq p \circ q$ and we have shown $(f,F)^{-1}(p) \circ (f,F)^{-1}(q) \subseteq (f,F)^{-1}(p \circ q)$.

The proof of $(f,F)^{-1}(P \circ Q) \subseteq (f,F)^{-1}(P) \circ (f,F)^{-1}(Q)$ is dual to the above, and is omitted. □

Corollary 5.8. *Let (f,F) be a difunction on $(S,\mathcal{S})$ to $(T,\mathcal{T})$. If (r,R) is a transitive direlation on $(T,\mathcal{T})$ then $(f,F)^{-1}(r,R)$ is a transitive direlation on $(S,\mathcal{S})$.*

Proof. Straightforward. □

Now let us make the following definition.

Definition 5.9. Let $\mathcal{U}$ be a direlational uniformity on $(S,\mathcal{S})$, $\mathcal{V}$ a direlational uniformity on $(T,\mathcal{T})$ and (f,F) a difunction from $(S,\mathcal{S})$ to $(T,\mathcal{T})$. If $(d,D) \in \mathcal{V} \implies (f,F)^{-1}(d,D) \in \mathcal{U}$ the difunction (f,F) is said to be $\mathcal{U}$–$\mathcal{V}$ *uniformly bicontinuous.*

Example 5.10. Let $\mathcal{U}$ be a direlational uniformity on $(S,\mathcal{S})$. Then the identity difunction (i,I) on $(S,\mathcal{S})$ is $\mathcal{U}$–$\mathcal{U}$–uniformly bicontinuous. To see this it is clearly sufficient to note that

$$(i,I)^{-1}(r,R) = (r,R)$$

for all direlations (r,R) on $(S,\mathcal{S})$. The proof of this equality is straightforward and is left to the interested reader.

Now let us consider the composition of uniformly bicontinuous difunctions. The following lemma will be useful.

Lemma 5.11. *Let $(S,\mathcal{S})$, $(T,\mathcal{T})$ and $(W,\mathcal{W})$ be textures, (f,F) a difunction on $(S,\mathcal{S})$ to $(T,\mathcal{T})$, (g,G) a difunction on $(T,\mathcal{T})$ to $(W,\mathcal{W})$ and (r,R) a direlation on $(W,\mathcal{W})$. Then*

$$(f,F)^{-1}((g,G)^{-1}(r,R)) = ((g,G) \circ (f,F))^{-1}(r,R).$$

Proof. First suppose that $(f,F)^{-1}((g,G)^{-1}(r)) \not\subseteq (g \circ f, G \circ F)^{-1}(r)$. Then we have $s, s', u \in S$ with $\overline{P}_{(s,u)} \not\subseteq (g \circ f, G \circ F)^{-1}(r)$, $P_s \not\subseteq Q_{s'}$, satisfying

$$\overline{P}_{(s',t_1)} \not\subseteq F,\ f \not\subseteq \overline{Q}_{(u,t_2)} \implies (g,G)^{-1}(r) \not\subseteq \overline{Q}_{(t_1,t_2)} \tag{5.5}$$

for all $t_1, t_2 \in T$ by Remark 5.2. Now $\overline{P}_{(s,u)} \not\subseteq (g \circ f, G \circ F)^{-1}(r)$, $P_s \not\subseteq Q_{s'}$, gives $w_1, w_2 \in W$ for which $\overline{P}_{(s',w_1)} \not\subseteq G \circ F$, $g \circ f \not\subseteq \overline{Q}_{(u,w_2)}$ and $r \subseteq \overline{Q}_{(w_1,w_2)}$.

Now we obtain $w_1' \in W$, $v_1 \in T$ with $\overline{P}_{(s',v_1)} \not\subseteq F$, $\overline{P}_{(v_1,w_1')} \not\subseteq G$, and $w_2' \in W$, $v_2 \in T$ with $f \not\subseteq \overline{Q}_{(u,v_2)}$ and $g \not\subseteq \overline{Q}_{(v_2,w_2')}$. Hence we may apply (5.5) with $t_1 = v_1$, $t_2 = v_2$ to give $(g,G)^{-1}(r) \not\subseteq \overline{Q}_{(v_1,v_2)}$. Hence we have $v_1', v_2' \in T$ satisfying $\overline{P}_{(v_1,v_2')} \not\subseteq \overline{Q}_{(v_1,v_2)}$, $P_{v_1} \not\subseteq Q_{v_1'}$, for which

$$\overline{P}_{(v_1',z_1)} \not\subseteq G,\ g \not\subseteq \overline{Q}_{(v_2',z_2)} \implies r \not\subseteq \overline{Q}_{(z_1,z_2)} \tag{5.6}$$

for all $z_1, z_2 \in W$ by Remark 5.2. Now $\overline{P}_{(v_1,w_1')} \not\subseteq G$, $P_{v_1} \not\subseteq Q_{w_1'}$ and $P_{v_1} \not\subseteq Q_{v_1'}$ gives $\overline{P}_{(v_1',w_1)} \not\subseteq G$. Also, $g \not\subseteq \overline{Q}_{(v_2,w_2')}$, $P_{v_2'} \not\subseteq Q_{v_2}$ and $P_{w_2'} \not\subseteq Q_{w_2}$ gives $f \not\subseteq \overline{Q}_{(v_2',w_2)}$. Hence we may apply (5.6) with $z_1 = w_1$, $z_2 = w_2$ to give $r \not\subseteq \overline{Q}_{(w_1,w_2)}$, which is a contradiction. Hence $(f,F)^{-1}((g,G)^{-1}(r)) \subseteq (g \circ f, G \circ F)^{-1}(r)$, and the proof of the reverse inclusion is similar and is omitted.

The proof of the dual equality $(f,F)^{-1}((g,G)^{-1}(R)) = (g \circ f, G \circ F)^{-1}(R)$ is left to the interested reader. □

The following is now immediate from the definitions:

Proposition 5.12. *Uniform bicontinuity is preserved under composition of difunctions.*

As expected we also have:

Proposition 5.13. *Let (τ_k, κ_k), $k = 1, 2$, be the uniform ditopology of the direlational uniformity $\mathcal{U}_k$ on $(S_k, \mathcal{S}_k)$, and let (f,F) be a difunction on $(S_1, \mathcal{S}_1)$ to $(S_2, \mathcal{S}_2)$. Then if (f,F) is $\mathcal{U}_1$–$\mathcal{U}_2$ uniformly bicontinuous it is (τ_1, κ_1)–(τ_2, κ_2) bicontinuous.*

Proof. Take $G \in \tau_2$ and $s \in S_1$ with $F^{\leftarrow}(G) \not\subseteq Q_s$. Then we have $t \in S_2$ with $\overline{P}_{(s,t)} \not\subseteq F$ and $G \not\subseteq Q_t$, whence by Lemma 4.3 there exists $(d,D) \in \mathcal{U}_2$ satisfying $d[t] \subseteq G$. Let $(e,E) = (f,F)^{-1}(d,D) \in \mathcal{U}_1$. It may be shown that $e[s] \subseteq F^{\leftarrow}(G)$, whence $F^{\leftarrow}(G) \in \tau_1$, again by Lemma 4.3. Hence (f,F) is τ_1–τ_2 continuous, and the proof of κ_1–κ_2 cocontinuity is dual to this. □

Now let us turn our attention to the notion of initial di-uniformity.

Theorem 5.14. *Let $(S, \mathcal{S})$ be a texture, $\mathcal{V}_i$, $i \in I$, direlational uniformities on the textures $(T_i, \mathcal{T}_i)$ and (f_i, F_i) difunctions on $(S, \mathcal{S})$ to $(T_i, \mathcal{T}_i)$, $i \in I$. Then the family*

$$(f_i, F_i)^{-1}(d_i, D_i),\ (d_i, D_i) \in \mathcal{V}_i,\ i \in I$$

is a subbase for a direlational di-uniformity $\mathcal{U}$ on $(S, \mathcal{S})$.

Proof. Let $\mathcal{U} = \{(d,D) \in \mathcal{DR} \mid \exists i_1, \dots, i_n \in I, (d_{i_k}, D_{i_k}) \in \mathcal{V}_{i_k},\ 1 \leq k \leq n$ with $\prod_{k=1}^{n} (f_{i_k}, F_{i_k})^{-1}(d_{i_k}, D_{i_k}) \sqsubseteq (d,D)\}$. We must verify conditions (1)–(5) of Definition 3.1. Clearly (1) is immediate from Proposition 5.3 and (2), (3) are trivial from the definition of $\mathcal{U}$. If $(d,D) \in \mathcal{U}$ then we have $(d_{i_k}, D_{i_k}) \in \mathcal{V}_{i_k}$, $1 \leq k \leq n$, with $\prod_{k=1}^{n} (f_{i_k}, F_{i_k})^{-1}(d_{i_k}, D_{i_k}) \sqsubseteq (d,D)$. For each k we may choose $(e_{i_k}, E_{i_k}) \in \mathcal{V}_{i_k}$ satisfying $(e_{i_k}, E_{i_k}) \circ (e_{i_k}, E_{i_k}) \sqsubseteq (d_{i_k}, D_{i_k})$. If we set

$(e,E) = \bigsqcap_{k=1}^{n} (f_{i_k}, F_{i_k})^{-1}(e_{i_k}, E_{i_k})$ then $(e,E) \in \mathcal{U}$ and

$$\begin{aligned}(e,E)\circ(e,E) &\sqsubseteq \bigsqcap_{k=1}^{n}((f_{i_k},F_{i_k})^{-1}(e_{i_k},E_{i_k})\circ(f_{i_k},F_{i_k})^{-1}(e_{i_k},E_{i_k}))\\ &\sqsubseteq \bigsqcap_{k=1}^{n}(f_{i_k},F_{i_k})^{-1}((e_{i_k},E_{i_k})\circ(e_{i_k},E_{i_k}))\\ &\sqsubseteq \bigsqcap_{k=1}^{n}(f_{i_k},F_{i_k})^{-1}(d_{i_k},D_{i_k}) \sqsubseteq (d,D),\end{aligned}$$

by Proposition 1.9 (7) and Proposition 5.7. Hence (4) is satisfied. Finally, (5) may be verified in a similar way using Proposition 1.9 (4) and Proposition 5.5. □

Definition 5.15. The di-uniformity $\mathcal{U}$ on $(S, \mathcal{S})$ defined in Theorem 5.14 is called the *initial direlational di-uniformity* on $(S, \mathcal{S})$ defined by the spaces $(T_i, \mathcal{T}_i, \mathcal{V}_i)$ and the difunctions (f_i, F_i), $i \in I$.

Clearly the initial di-uniformity is the coarsest di-uniformity on $(S, \mathcal{S})$ for which the difunctions (f_i, F_i) are $\mathcal{U}$–$\mathcal{V}_i$ uniformly bicontinuous for all $i \in I$.

We are now in a position to prove the converse of Theorem 4.14, and so complete our characterization of di-uniformizable ditopological texture spaces.

Theorem 5.16. *$(S, \mathcal{S}, \tau, \kappa)$ is di-uniformizable if and only if it is completely biregular.*

Proof. It remains to show that if (τ, κ) is completely biregular then there exists a compatible di-uniformity. Let $\mathcal{U}$ denote the initial direlational uniformity generated by the family of all bicontinuous difunctions from $(S, \mathcal{S}, \tau, \kappa)$ to $(\mathbb{I}, \mathcal{I}, \tau_{\mathbb{I}}, \kappa_{\mathbb{I}})$. We show that $(\tau, \kappa) = (\tau_{\mathcal{U}}, \kappa_{\mathcal{U}})$.

First take $G \in \tau_{\mathcal{U}}$ and $G \not\subseteq Q_s$. Then there exist $z, s', s'', s''', w \in S$ so that $G \not\subseteq Q_z$, $P_z \not\subseteq Q_{s'}$, $P_{s'} \not\subseteq Q_{s''}$, $P_{s''} \not\subseteq Q_{s'''}$, $P_{s'''} \not\subseteq Q_w$ and $P_w \not\subseteq Q_s$. Choose $(d, D) \in \mathcal{U}$ with $d[z] \subseteq G$. Now there exist (τ, κ)– $(\tau_{\mathbb{I}}, \kappa_{\mathbb{I}})$ bicontinuous difunctions $(f_1, F_1), \dots, (f_n, F_n)$ and $\epsilon > 0$ for which

$$e = (f_1, F_1)^{-1}(d_\epsilon) \sqcap \dots \sqcap (f_n, F_n)^{-1}(d_\epsilon) \subseteq d.$$

Since $P_{s'''} \not\subseteq Q_w$, by *DF1* there exists $r_i \in \mathbb{I}$ for each $i = 1, \dots, n$, so that $f_i \not\subseteq \overline{Q}_{(s''', r_i)}$ and $\overline{P}_{(w, r_i)} \not\subseteq F_i$. We show that

(a) $\bigcap_{i=1}^{n} F_i^{\leftarrow}([0, r_i + \epsilon)) \subseteq e[z] \subseteq d[z] \subseteq G$, and
(b) $\bigcap_{i=1}^{n} F_i^{\leftarrow}([0, r_i + \epsilon)) \not\subseteq Q_s$,

from which it follows at once that $G \in \tau$.

Suppose that (a) is false. Take $u, u', u'' \in S$ with $\bigcap_{i=1}^{n} F_i^{\leftarrow}([0, r_i + \epsilon)) \not\subseteq Q_{u''}$, $P_{u''} \not\subseteq Q_u$, $P_u \not\subseteq Q_{u'}$ and $P_{u'} \not\subseteq e[z]$. Now for each $i = 1, \dots, n$ we have $t_i \in \mathbb{I}$ with $\overline{P}_{(u'', t_i)} \not\subseteq F_i$ and $[0, r_i + \epsilon) \not\subseteq Q_{t_i}$, that is $t_i < r_i + \epsilon$. Take any $v_1, v_2 \in \mathbb{I}$ with

$$\overline{P}_{(s''', v_1)} \not\subseteq F_i \text{ and } f_i \not\subseteq \overline{Q}_{(u'', v_2)}.$$

By *DF1* we have $P_{v_1} \not\subseteq Q_{r_i}$ and $P_{t_i} \not\subseteq Q_{v_2}$, whence $r_i \le v_1$ and $v_2 \le t_i$. Thus $v_2 \le t_i < r_i + \epsilon \le v_1 + \epsilon$, so $\overline{P}_{(v_1,v_2)} \subseteq d_\epsilon$. Since $P_{s''} \not\subseteq Q_{s'''}$ and $P_{u''} \not\subseteq Q_u$ we deduce that

$$(f_i, F_i)^{-1}(d_\epsilon) \not\subseteq \overline{Q}_{(s'',u)} \,\forall\, i = 1, \dots, n,$$

whence $\overline{P}_{(s',u)} \subseteq (f_1, F_1)^{-1}(d_\epsilon) \sqcap \dots \sqcap (f_n, F_n)^{-1}(d_\epsilon)$. Since $P_u \not\subseteq Q_{u'}$ we now have $e \not\subseteq \overline{Q}_{(s',u')}$. On the other hand $P_{u'} \not\subseteq e[z]$ gives $v \in S$ with $P_{u'} \not\subseteq Q_v$ for which $e \not\subseteq \overline{Q}_{(x,v)} \implies P_z \subseteq Q_x \,\forall\, x \in S$. From the above we have $e \not\subseteq \overline{Q}_{(s',v)}$ so setting $x = s'$ in the above implication leads to the contradiction $P_z \subseteq Q_{s'}$.

To prove (b) it will suffice to show $P_w \subseteq \bigcap_{i=1}^{n} F_i^{\leftarrow}([0, r_i + \epsilon))$, so assume the contrary. Now we have j, $1 \le j \le n$, and $P_w \not\subseteq Q_{w'}$ for which $\overline{P}_{(w',y)} \not\subseteq F_j \implies [0, r_j + \epsilon) \subseteq Q_y$. However $\overline{P}_{(w,r_j)} \not\subseteq F_j$ and $P_w \not\subseteq Q_{w'}$ gives $\overline{P}_{(w',r_j)} \not\subseteq F_j$, and we obtain the contradiction $[0, r_j + \epsilon) \subseteq [0, r_j)$ by taking $y = r_j$ in the above implication.

Conversely, take $G \in \tau$ and $s \in S$ with $G \not\subseteq Q_s$. Since (τ, κ) is completely regular there exists a bicontinuous difunction (f, F) on $(S, \mathcal{S})$ to $(\mathbb{I}, \mathcal{I})$ for which $f^{\leftarrow}(P_0) \not\subseteq Q_s$ and $F^{\leftarrow}(Q_1) \subseteq G$. Take $\epsilon > 0$ and define $e = (f, F)^{-1}(d_\epsilon)$. Then $e \in \mathcal{U}$ and we will show that $e[s] \subseteq G$, whence $G \in \tau_{\mathcal{U}}$.

It will be sufficient to show $e[s] \subseteq F^{\leftarrow}(Q_1)$, so assume the contrary and take $v \in S$ with $e[s] \not\subseteq Q_v$ and $P_v \not\subseteq F^{\leftarrow}(Q_1)$. The latter gives us $v' \in S$ with

$$\overline{P}_{(v',w)} \not\subseteq F \implies Q_1 \subseteq Q_w \implies w = 1, \tag{5.7}$$

and the former gives $z \in S$ with $e \not\subseteq Q_v$ and $P_s \not\subseteq Q_z$. Now we have $v'' \in S$ with $\overline{P}_{(z,v'')} \not\subseteq \overline{Q}_{(z,v)}$ and $z' \in S$ with $P_z \not\subseteq Q_{z'}$ so that

$$\overline{P}_{(z',t_1)} \not\subseteq F,\ f \not\subseteq \overline{Q}_{(v',t_2)} \implies \overline{P}_{(t_1,t_2)} \subseteq d_\epsilon \implies t_2 < t_1 + \epsilon. \tag{5.8}$$

Clearly $P_{v''} \not\subseteq Q_{v'}$ so by *DF1* there exists $t' \in \mathbb{I}$ with $f \not\subseteq \overline{Q}_{(v'',t')}$ and $\overline{P}_{(v',t')} \not\subseteq F$. Now (7) with $w = t'$ gives $t' = 1$ and so $f \not\subseteq \overline{Q}_{(v'',1)}$.

On the other hand, from $f^{\leftarrow}(P_0) \not\subseteq Q_s$ we have $s' \in S$ with $f^{\leftarrow}(P_0) \not\subseteq Q_{s'}$, $P_{s'} \not\subseteq Q_s$. Hence we have $u \in S$ with $P_u \not\subseteq Q_{s'}$ so that

$$f \not\subseteq \overline{Q}_{(u,w)} \implies P_w \subseteq P_0 \implies w = 0. \tag{5.9}$$

Clearly $P_u \not\subseteq Q_z$ so by *DF1* we have $t \in \mathbb{I}$ so that $f \not\subseteq \overline{Q}_{(u,t)}$, $\overline{P}_{(z',t)} \not\subseteq F$. Now (9) with $w = t$ gives $t = 0$, so $\overline{P}_{(z',0)} \not\subseteq F$ and (8) with $t_1 = 0$, $t_2 = 1$ gives the contradiction $1 < \epsilon$.

We have now established $\tau = \tau_{\mathcal{U}}$, and a dual proof gives $\kappa = \kappa_{\mathcal{U}}$, so the proof is complete. □

Before leaving the topics of uniform bicontinuity and initial di-uniformity we must see how these should be defined for dicovering di-uniformities. The following gives a fairly obvious notion of inverse image of a dicover under a difunction.

Definition 5.17. Let $(S,\mathcal{S})$, $(T,\mathcal{T})$ be textures, (f,F) a difunction on $(S,\mathcal{S})$ to $(T,\mathcal{T})$ and $\mathcal{C}$ a dicover of $(T,\mathcal{T})$. Then

$$(f,F)^{-1}(\mathcal{C}) = \{(F^{\leftarrow}(A), f^{\leftarrow}(B)) \mid A\,\mathcal{C}\,B\}.$$

It is a straightforward matter to verify that $(f,F)^{-1}(\mathcal{C})$ is a dicover of $(S,\mathcal{S})$, but the authors do not know if this inverse operation preserves the property of being anchored, or even of being refined by the dicover $\mathcal{P}$. This will cause some technical difficulties, but will not prevent us using this operation to characterize uniform bicontinuity and initial diuniformities in terms of dicovers, as we will see. We begin by relating this inverse image with that given earlier for direlations.

Proposition 5.18. *Let $(f,F):(S,\mathcal{S})\to(T,\mathcal{T})$ be a difunction and (d,D) a reflexive direlation on $(T,\mathcal{T})$. Then*

$$\gamma((f,F)^{-1}(d,D)) \prec ((f,F)^{-1}(\gamma(d,D)))^{\Delta}.$$

Proof. Let $\mathcal{D}=\gamma(d,D)$ and $\mathcal{C}=(f,F)^{-1}(\mathcal{D})$. If we set $c=(f,F)^{-1}(d)$ and $C=(f,F)^{-1}(D)$ we must verify $c[s]\subseteq \mathrm{St}(\mathcal{C},P_s)$ and $\mathrm{CSt}(\mathcal{C},Q_s)\subseteq C[s]$. We prove the first inclusion, the second being dual. Recall that $c[s]=c(P_s)$ and suppose that $c(P_s)\not\subseteq \mathrm{St}(\mathcal{C},P_s)$. Now we have $u\in T$ with $c(P_s)\not\subseteq Q_u$, $P_u\not\subseteq \mathrm{St}(\mathcal{C},P_s)$ and hence $s'\in S$ with $c\not\subseteq \overline{Q}_{(s',u)}$, $P_s\not\subseteq Q_{s'}$. By Remark 5.2 we have $\overline{P}_{(s',u')}\not\subseteq \overline{Q}_{(s',u)}$ and $P_{s'}\not\subseteq Q_{s''}$ for which

$$\overline{P}_{(s'',t_1)}\not\subseteq F,\ f\not\subseteq \overline{Q}_{(u',t_2)} \implies d\not\subseteq \overline{Q}_{(t_1,t_2)} \tag{5.10}$$

for all $t_1,t_2\in T$. On the other hand, $P_u\not\subseteq \mathrm{St}(\mathcal{C},P_s)=\bigvee\{f^{\leftarrow}(d[z])\mid z\in T,\ P_s\not\subseteq F^{\leftarrow}(D[z])\}=f^{\leftarrow}(\bigvee\{d[z]\mid z\in T,\ P_s\not\subseteq F^{\leftarrow}(D[z])\})$ by Proposition 1.6 (7), and so we have $w\in T$ satisfying $f\not\subseteq \overline{Q}_{(u,w)}$ and $P_w\not\subseteq \bigvee\{d[z]\mid z\in T,\ P_s\not\subseteq F^{\leftarrow}(D[z])\}$.

Since $P_{u'}\not\subseteq Q_u$ we have $f\not\subseteq \overline{Q}_{(u',w)}$. On the other hand applying *DF1* to $P_{s'}\not\subseteq Q_{s''}$ gives $v\in T$ satisfying $f\not\subseteq \overline{Q}_{(s',v)}$ and $\overline{P}_{(s'',v)}\not\subseteq F$. We may now apply implication (5.10) with $t_1=v$, $t_2=w$ to give $d\not\subseteq \overline{Q}_{(v,w)}$. This is equivalent to $d[v]\not\subseteq Q_w$, and so $P_w\subseteq d[v]$. To obtain a contradiction it will therefore suffice to show that $F^{\leftarrow}(D[v])\subseteq Q_{s'}$, for then $P_s\not\subseteq F^{\leftarrow}(D[v])$ and so $P_w\not\subseteq d[v]$.

Suppose, therefore, that $F^{\leftarrow}(D[v])\not\subseteq Q_{s'}$. Then we have $t\in T$ with $\overline{P}_{(s',t)}\not\subseteq F$ and $D[v]\not\subseteq Q_t$. Using *DF2* now gives $P_t\not\subseteq Q_v$, whence $D[v]\subseteq Q_v\subseteq Q_t$ since D is reflexive. This contradiction completes the proof. □

Proposition 5.19. *Let $(f,F):(S,\mathcal{S})\to(T,\mathcal{T})$ be a difunction and (c,C), (d,D) reflexive direlations on $(T,\mathcal{T})$. Then*

$$(d,D)\circ(d,D)^{\leftarrow}\sqsubseteq(c,C) \implies \delta((f,F)^{-1}(\gamma(d,D)))\sqsubseteq(f,F)^{-1}(c,C).$$

Proof. Assume that $(d,D)\circ(d,D)^{\leftarrow}\sqsubseteq(c,C)$, i.e. $d\circ D^{\leftarrow}\subseteq c$ and $C\subseteq D\circ d^{\leftarrow}$.

Let $\mathcal{D}=\gamma(d,D)$, $\mathcal{E}=(f,F)^{-1}(\mathcal{D})$, and assume that $d(\mathcal{E})\not\subseteq(f,F)^{-1}(c)$. Now we have $s,s'\in S$ with $\overline{P}_{(s,s')}\not\subseteq(f,F)^{-1}(c)$ and $t\in T$ with $P_s\not\subseteq$

$f^{\leftarrow}(D(Q_t))$, $F^{\leftarrow}(d(P_t)) \not\subseteq Q_{s'}$. Hence we have $v \in T$ with $f \not\subseteq \overline{Q}_{(s,v)}$, $P_v \not\subseteq D(Q_t)$, and $v' \in T$ with $\overline{P}_{(s',v')} \not\subseteq F$, $d(P_t) \not\subseteq Q_{v'}$. Also, by *R2* for the relation f, we have $u \in S$ with $P_s \not\subseteq Q_u$ and $f \not\subseteq \overline{Q}_{(u,v)}$. Hence, since $\overline{P}_{(s,s')} \not\subseteq (f,F)^{-1}(c)$, there exists $t_1, t_2 \in T$ such that $\overline{P}_{(u,t_1)} \not\subseteq F$, $f \not\subseteq \overline{Q}_{(s',t_2)}$ and $\overline{P}_{(t_1,t_2)} \not\subseteq c$.

On the other hand, from $d(P_t) \not\subseteq Q_{v'}$ we have $d \not\subseteq \overline{Q}_{(t,v')}$ and from $P_v \not\subseteq D(Q_t)$ we have $\overline{P}_{(t,v)} \not\subseteq D$, that is $D^{\leftarrow} \not\subseteq \overline{Q}_{(v,t)}$. Since (f,F) is a difunction, $\overline{P}_{(s',v')} \not\subseteq F$ and $f \not\subseteq \overline{Q}_{(s',t_2)}$ imply $P_{v'} \not\subseteq Q_{t_2}$ by *DF2*, so $d \not\subseteq \overline{Q}_{(t,t_2)}$. Likewise, $P_{t_1} \not\subseteq Q_v$ and so $D^{\leftarrow} \not\subseteq \overline{Q}_{(t_1,t)}$ by *R1* for the relation $D^{\leftarrow}$. We now obtain $\overline{P}_{(t_1,t_2)} \subseteq d \circ D^{\leftarrow} \subseteq c$, which is a contradiction.

The proof of $D(\mathcal{E}) \subseteq (f,F)^{-1}(C)$ is dual to the above, and is omitted. □

With the notation of Theorem 3.7 we now have:

Proposition 5.20. *Let $(f,F) : (S,\mathcal{S}) \to (T,\mathcal{T})$ be a difunction. If $\mathcal{U}$, $\mathcal{V}$ are direlational di-uniformities on $(S,\mathcal{S})$, $(T,\mathcal{T})$, respectively, then (f,F) is $\mathcal{U}$–$\mathcal{V}$ uniformly bicontinuous if and only if $\mathcal{C} \in \Gamma(\mathcal{V}) \implies (f,F)^{-1}(\mathcal{C})^{\Delta} \in \Gamma(\mathcal{U})$.*

Proof. Suppose (f,F) is $\mathcal{U}$–$\mathcal{V}$ uniformly bicontinuous and take $\mathcal{C} \in \Gamma(\mathcal{V})$. Now we have $(c,C) \in \mathcal{V}$ with $\gamma(c,C) \prec \mathcal{C}$, so $(d,D) = (f,F)^{-1}(c,C) \in \mathcal{U}$ and $\gamma(d,D) \in \Gamma(\mathcal{U})$. However $\gamma(d,D) \prec (f,F)^{-1}(\mathcal{C})^{\Delta}$ by Proposition 5.18, so $(f,F)^{-1}(\mathcal{C})^{\Delta} \in \Gamma(\mathcal{U})$.

Conversely suppose $\mathcal{C} \in \Gamma(\mathcal{V}) \implies (f,F)^{-1}(\mathcal{C})^{\Delta} \in \Gamma(\mathcal{U})$ and take $(e,E) \in \mathcal{V}$. Choose a symmetric $(c,C) \in \mathcal{V}$ with $(c,C) \circ (c,C) \sqsubseteq (e,E)$ and $(d,D) \in \mathcal{V}$ with $(d,D) \circ (d,D)^{\leftarrow} \sqsubseteq (c,C)$. By Corollary 5.6, $(f,F)^{-1}(c,C)$ is also symmetric, and it is reflexive by Corollary 5.4. Hence by Theorem 2.7 (1) and Proposition 5.7,

$$\begin{aligned}
\delta(\gamma((f,F)^{-1}(c,C))) &= (f,F)^{-1}(c,C) \circ (f,F)^{-1}(c,C) \\
&\sqsubseteq (f,F)^{-1}((c,C) \circ (c,C)) \\
&\sqsubseteq (f,F)^{-1}(e,E).
\end{aligned}$$

By Proposition 5.19, $\delta((f,F)^{-1}(\gamma(d,D)) \sqsubseteq (f,F)^{-1}(c,C)$ and so

$$\begin{aligned}
\delta((f,F)^{-1}(\gamma(d,D))^{\Delta}) &= \delta(\gamma(\delta((f,F)^{-1}(\gamma(d,D)))) \\
&\sqsubseteq \delta(\gamma((f,F)^{-1}(c,C)) \\
&\sqsubseteq (f,F)^{-1}(e,E)
\end{aligned}$$

by Theorem 2.7 (2). Since $(d,D) \in \mathcal{V}$, $\mathcal{C} = \gamma(d,D) \in \Gamma(\mathcal{V})$ and so, by hypothesis, $(f,F)^{-1}(\gamma(d,D))^{\Delta} = (f,F)^{-1}(\mathcal{C})^{\Delta} \in \Gamma(\mathcal{U})$. Hence $\delta((f,F)^{-1}(\gamma(d,D))^{\Delta}) \in \Delta(\Gamma(\mathcal{U})) = \mathcal{U}$ by Theorem 3.7 (3). It follows from the above inclusion that $(f,F)^{-1}\,(e,E) \in \mathcal{U}$, and we have shown that (f,F) is $\mathcal{U}$–$\mathcal{V}$ uniformly bicontinuous. □

This justifies the following definition.

Definition 5.21. Let υ, ν be dicovering uniformities on $(S,\mathcal{S})$, $(T,\mathcal{T})$ respectively and $(f,F):(S,\mathcal{S})\to(T,\mathcal{T})$ a difunction. Then (f,F) is called υ–ν *uniformly bicontinuous* if $\mathcal{C}\in\nu \implies (f,F)^{-1}(\mathcal{C})^{\Delta}\in\upsilon$.

Finally we have the following:

Proposition 5.22. *Let $(S,\mathcal{S})$ be a texture and for $i\in I$ let $(T_i,\mathcal{T}_i,\mathcal{V}_i)$ be a direlational di-uniform texture space and $(f_i,F_i):(S,\mathcal{S})\to(T_i,\mathcal{T}_i)$ a difunction. If $\mathcal{U}$ is the initial direlational uniformity on $(S,\mathcal{S})$ for the given system, the family*

$$\Big(\bigwedge_{k=1}^{n}(f_{i_k},F_{i_k})^{-1}(\mathcal{C}_{i_k})^{\Delta}\Big)^{\Delta},\ n\in\mathbb{N}^{+},\ i_k\in I,\ \mathcal{C}_{i_k}\in\Gamma(\mathcal{V}_{i_k}),\ 1\le k\le n,$$

is a base for the dicovering di-uniformity $\Gamma(\mathcal{U})$.

Proof. Take $\mathcal{C}\in\Gamma(\mathcal{U})$. Then there exists $(e,E)\in\mathcal{U}$ satisfying $\gamma(e,E)\prec\mathcal{C}$, and hence $i_k\in I$, $(e_{i_k},E_{i_k})\in\mathcal{V}_{i_k}$ for $1\le k\le n$ with $\bigsqcap_{k=1}^{n}(f_{i_k},F_{i_k})^{-1}(e_{i_k},E_{i_k})\sqsubseteq(e,E)$. If we choose a symmetric $(c_{i_k},C_{i_k})\in\mathcal{V}_{i_k}$ with $(c_{i_k},C_{i_k})\circ(c_{i_k},C_{i_k})\sqsubseteq(e_{i_k},E_{i_k})$, and then $(d_{i_k},D_{i_k})\in\mathcal{V}_{i_k}$ with $(d_{i_k},D_{i_k})\circ(d_{i_k},D_{i_k})^{\leftarrow}\sqsubseteq(c_{i_k},C_{i_k})$, we have $\delta((f_{i_k},F_{i_k})^{-1}(\gamma(d_{i_k},D_{i_k}))^{\Delta}\sqsubseteq(f_{i_k},F_{i_k})^{-1}(e_{i_k},E_{i_k})$, exactly as in the proof of Proposition 5.20. In view of Proposition 2.9 (2) we deduce

$$\begin{aligned}\delta\Big(\bigwedge_{k=1}^{n}(f_{i_k},F_{i_k})^{-1}(\gamma(d_{i_k},D_{i_k}))^{\Delta}\Big)&\sqsubseteq\bigsqcap_{k=1}^{n}\delta\left((f_{i_k},F_{i_k})^{-1}(\gamma(d_{i_k},D_{i_k}))^{\Delta}\right)\\&\sqsubseteq\bigsqcap_{k=1}^{n}\left((f_{i_k},F_{i_k})^{-1}(e_{i_k},E_{i_k})\right)\sqsubseteq(e,E).\end{aligned}$$

Applying γ to both sides and using Theorem 2.7 (2) now gives

$$\Big(\bigwedge_{k=1}^{n}(f_{i_k},F_{i_k})^{-1}(\mathcal{C}_{i_k})^{\Delta}\Big)^{\Delta}\prec\mathcal{C},$$

where we have set $\mathcal{C}_{i_k}=\gamma(d_{i_k},D_{i_k})\in\Gamma(\mathcal{V}_{i_k})$. It remains to show that the dicover on the left belongs to $\Gamma(\mathcal{U})$. Now $\mathcal{D}=\gamma(\bigsqcap_{k=1}^{n}(f_{i_k},F_{i_k})^{-1}(d_{i_k},D_{i_k}))\in\Gamma(\mathcal{U})$ is anchored by Proposition 2.3, and $\mathcal{D}\prec\bigwedge_{k=1}^{n}\gamma((f_{i_k},F_{i_k})^{-1}(d_{i_k},D_{i_k})$ by Proposition 2.8 (1), so by Lemma 2.2 (i) we have

$$\mathcal{D}\prec\Big(\bigwedge_{k=1}^{n}\gamma((f_{i_k},F_{i_k})^{-1}(d_{i_k},D_{i_k}))\Big)^{\Delta}\prec\Big(\bigwedge_{k=1}^{n}(f_{i_k},F_{i_k})^{-1}(\mathcal{C}_{i_k})^{\Delta}\Big)^{\Delta}$$

by Proposition 5.18, and this gives the required result. □

In view of the above proposition, the following definition is compatible with Definition 5.15.

Definition 5.23. Let $(S,\mathcal{S})$ be a texture and for each $i\in I$ let $(T_i,\mathcal{T}_i,\nu_i)$ be a dicovering di-uniform texture space and $(f_i,F_i):(S,\mathcal{S})\to(T_i,\mathcal{T}_i)$ a difunction.

Then the covering di-uniformity υ on $(S, \mathcal{S})$ with base

$$\left\{ \bigwedge_{k=1}^{n} \left((f_i, F_i)^{-1} (\mathcal{C}_{i_k})^{\Delta} \right)^{\Delta} \mid i_k \in I,\ \mathcal{C}_{i_k} \in \nu_{i_k},\ 1 \le k \le n,\ n \in \mathbb{N}^{+} \right\}$$

is called the *initial covering di-uniformity on* $(S, \mathcal{S})$ for the spaces $(T_i, \mathcal{T}_i, \nu_i)$ and difunctions (f_i, F_i), $i \in I$.

It is not known if the above results and definitions can be simplified in general.

6. Dimetrics and Diuniformities

Definition 6.1. Let $(S, \mathcal{S})$ be a texture, $\overline{\rho}$, $\underline{\rho} : S \times S \to [0, \infty)$ two point functions. Then $\rho = (\overline{\rho}, \underline{\rho})$ is called a *pseudo dimetric* on $(S, \mathcal{S})$ if

M1 $\overline{\rho}(s,t) \le \overline{\rho}(s,u) + \overline{\rho}(u,t)\ \forall\, s,u,t \in S$,
M2 $P_s \not\subseteq Q_t \implies \overline{\rho}(s,t) = 0\ \forall\, s,t \in S$,
DM $\overline{\rho}(s,t) = \underline{\rho}(t,s)\ \forall\, s,t \in S$,
CM1 $\underline{\rho}(s,t) \le \underline{\rho}(s,u) + \underline{\rho}(u,t)\ \forall\, s,u,t \in S$,
CM2 $P_t \not\subseteq Q_s \implies \underline{\rho}(s,t) = 0\ \forall\, s,t \in S$.

In this case $\overline{\rho}$ is called the *pseudo metric*, $\underline{\rho}$ the *pseudo cometric* of ρ.

If ρ is a pseudo dimetric which satisfies the conditions

M3 $P_s \not\subseteq Q_u,\ \overline{\rho}(u,v) = 0,\ P_v \not\subseteq Q_t \implies P_s \not\subseteq Q_t\ \forall\, s,t,u,v \in S$,
CM3 $P_u \not\subseteq Q_s,\ \underline{\rho}(u,v) = 0,\ P_t \not\subseteq Q_v \implies P_t \not\subseteq Q_s\ \forall\, s,t,u,v \in S$,

it is called a *dimetric*.

When giving examples it will clearly suffice to give $\overline{\rho}$ satisfying the metric conditions, since DM may then be used to define $\underline{\rho}$, which will automatically satisfy the cometric conditions. Note that for a pseudo dimetric to be a dimetric it is sufficient that $\overline{\rho}(s,t) = 0 \implies P_s \not\subseteq Q_t$, but example (4) below shows this condition is not necessary in general.

Example 6.2. (1) Let $(S, \mathcal{S})$ be any texture and define

$$\overline{\rho}(s,t) = \begin{cases} 0 & \text{if } P_s \not\subseteq Q_t, \\ 1 & \text{otherwise.} \end{cases}$$

Clearly $\overline{\rho}$ defines a dimetric ρ, which we will call the *discrete dimetric* on $(S, \mathcal{S})$.

(2) If d is a (pseudo) metric on X in the usual sense, $\rho = (d, d)$ is a (pseudo) dimetric on $(X, \mathcal{P}(X))$.

(3) Consider the texture $(\mathbb{I}, \mathcal{I})$ and set $\overline{\rho}(s,t) = (t - s) \vee 0$. Then $\overline{\rho}$ defines a dimetric ρ on $(\mathbb{I}, \mathcal{I})$, which we will call the *usual dimetric* on $(\mathbb{I}, \mathcal{I})$.

(4) Let $(L, \mathcal{L})$ be the texture $L = (0,1]$, $\mathcal{L} = \{(0,r] \mid 0 \le r \le 1\}$. Again $\overline{\rho}(s,t) = (t - s) \vee 0$ defines a dimetric, called the *usual dimetric* on $(L, \mathcal{L})$.

Note that (2) and (4) may be combined to give a rich supply of pseudo dimetrics on the product of $(X, \mathcal{P}(X))$ and $(L, \mathcal{L})$. Since this is the texture corresponding to the lattice of classic fuzzy sets on X [4,6] the connection with fuzzy topology is clear, although we will not pursue this line of enquiry here.

As expected, a (pseudo) dimetric ρ gives rise to a ditopology, which we will refer to as the *(pseudo) metric ditopology of* ρ.

Proposition 6.3. *Let ρ be a pseudo dimetric on $(S, \mathcal{S})$ and for $s \in S^\flat$, $\epsilon > 0$ define*

$$N_\epsilon^\rho(s) = \bigvee \{P_t \mid \exists\, u \in S \text{ with } P_s \not\subseteq Q_u,\ \overline{\rho}(u,t) < \epsilon\},$$
$$M_\epsilon^\rho(s) = \bigcap \{Q_t \mid \exists\, u \in S \text{ with } P_u \not\subseteq Q_s,\ \underline{\rho}(u,t) < \epsilon\}.$$

Then $\beta_\rho = \{N_\epsilon^\rho(s) \mid s \in S^\flat,\ \epsilon > 0\}$ is a base and $\gamma_\rho = \{M_\epsilon^\rho(s) \mid s \in S^\flat,\ \epsilon > 0\}$ a cobase for a ditopology (τ_ρ, κ_ρ) on $(S, \mathcal{S})$.

Proof. By M2 it is clear that $P_s \subseteq N_\epsilon^\rho(s)$ for all $s \in S^\flat$ and so $\bigvee \beta_\rho = S$. Now take $s_1, s_2, s \in S^\flat$, $\epsilon_1, \epsilon_2 > 0$ with $N_{\epsilon_1}^\rho(s_1) \cap N_{\epsilon_2}^\rho(s_2) \not\subseteq Q_s$. Choose $t \in S$ with $N_{\epsilon_1}^\rho(s_1) \cap N_{\epsilon_2}^\rho(s_2) \not\subseteq Q_t$, $P_t \not\subseteq Q_s$ and then for $k = 1, 2$ take $t_k \in S$ with $P_{t_k} \not\subseteq Q_t$ so that for some $P_{s_k} \not\subseteq Q_{s'_k}$, $s'_k \in S$, we have $\overline{\rho}(s'_k, t_k) < \epsilon_k$. Since $\overline{\rho}(t_k, t) = 0$ by M2 we deduce $\overline{\rho}(s'_k, t) < \epsilon_k$ for $k = 1, 2$ by M1, so we may choose $\epsilon \in \mathbb{R}$ satisfying $0 < \epsilon < \min(\epsilon_1 - \overline{\rho}(s'_1, t),\ \epsilon_2 - \overline{\rho}(s'_2, t))$. However it is now straightforward to verify that

$$N_\epsilon^\rho(t) \subseteq N_{\epsilon_1}^\rho(s_1) \cap N_{\epsilon_2}^\rho(s_2),\ N_\epsilon^\rho(t) \not\subseteq Q_s$$

whence by ([6], Theorem 4.3), β_ρ is a base for some topology τ_ρ on $(S, \mathcal{S})$. The proof that γ_ρ is a base for some cotopology κ_ρ on $(S, \mathcal{S})$ is dual to this, and is omitted. □

Clearly the discrete dimetric on $(S, \mathcal{S})$ gives rise to the discrete, codiscrete ditopology. Likewise, the metric ditopology of the usual dimetric on $(\mathbb{I}, \mathcal{I})$ is the usual ditopology on $(\mathbb{I}, \mathcal{I})$, while the same dimetric on $(L, \mathcal{L})$ gives the discrete, codiscrete ditopology.

Now let us verify that a pseudo dimetric also defines a di-uniformity.

Theorem 6.4. *Let ρ be a pseudo dimetric on $(S, \mathcal{S})$.*

i) *For $\epsilon > 0$ let*

$$r_\epsilon = r_\epsilon^\rho = \bigvee \{\overline{P}_{(s,t)} \mid \exists\, u \in S,\ P_s \not\subseteq Q_u \text{ and } \overline{\rho}(u,t) < \epsilon\},$$
$$R_\epsilon = R_\epsilon^\rho = \bigcap \{\overline{Q}_{(s,t)} \mid \exists\, u \in S,\ P_u \not\subseteq Q_s \text{ and } \underline{\rho}(u,t) < \epsilon\}.$$

Then the family $\{(r_\epsilon^\rho, R_\epsilon^\rho) \mid \epsilon > 0\}$ is a base for a direlational uniformity $\mathcal{U}_\rho$ on $(S, \mathcal{S})$.

ii) *The di-uniformity $\mathcal{U}_\rho$ is separated if and only if ρ is a dimetric.*

iii) *The uniform ditopology of $\mathcal{U}_\rho$ coincides with the pseudo metric ditopology of ρ.*

Proof. (i) It is trivial to verify that (r_ϵ, R_ϵ) is a direlation for all $\epsilon > 0$.

We must verify the conditions of Definition 3.1 for the family

$$\mathcal{U}_\rho = \{(r, R) \mid \exists\, \epsilon > 0, (r_\epsilon, R_\epsilon) \sqsubseteq (r, R)\}.$$

Condition (1) is trivial from M1, CM1; and (2) follows from the definition. Condition (3) is a consequence of $(r_\epsilon, R_\epsilon) \sqsubseteq (r_{\epsilon_1}, R_{\epsilon_1}) \sqcap (r_{\epsilon_2}, R_{\epsilon_2})$, where $\epsilon = \min(\epsilon_1, \epsilon_2)$, which is trivial since clearly $\epsilon \le \delta \implies (r_\epsilon, R_\epsilon) \sqsubseteq (r_\delta, R_\delta)$. To prove (4) we need only show that $(r_\epsilon, R_\epsilon)^2 \sqsubseteq (r_{2\epsilon}, R_{2\epsilon})$. If $r_\epsilon \circ r_\epsilon \not\subseteq r_{2\epsilon}$ there exists $s, t \in S$ with $\overline{P}_{(s,t)} \not\subseteq r_{2\epsilon}$ so that for some $u, v \in S$ we have $P_s \not\subseteq Q_u$, $r_\epsilon \not\subseteq \overline{Q}_{(u,v)}$ and $r_\epsilon \not\subseteq \overline{Q}_{(v,t)}$. By M1, M2 and the definition of r_ϵ we obtain $\overline{\rho}(u, v) < \epsilon$, $\overline{\rho}(v, t) < \epsilon$, whence $\overline{\rho}(u, t) \le \overline{\rho}(u, v) + \overline{\rho}(v, t) < 2\epsilon$. This gives the contradiction $\overline{P}_{(s,t)} \subseteq r_{2\epsilon}$ so $r_\epsilon \circ r_\epsilon \subseteq r_{2\epsilon}$, and the dual result $R_{2\epsilon} \subseteq R_\epsilon \circ R_\epsilon$ is proved likewise. Finally (5) follows from $(r_\epsilon, R_\epsilon)^{\leftarrow} = (r_\epsilon, R_\epsilon)$. To prove this we need only show that $r_\epsilon^{\leftarrow} = R_\epsilon$ for any $\epsilon > 0$. Suppose that $R_\epsilon \not\subseteq r_\epsilon^{\leftarrow}$. Then we have $s, t \in S$ with $R_\epsilon \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s,t)} \not\subseteq r_\epsilon^{\leftarrow}$. Since $r_\epsilon^{\leftarrow}$ is a corelation, $\overline{P}_{(s,t)} \not\subseteq r_\epsilon^{\leftarrow}$ is equivalent to $r_\epsilon \not\subseteq \overline{Q}_{(t,s)}$ and so we have $s' \in S$ satisfying $\overline{P}_{(t,s')} \not\subseteq \overline{Q}_{(t,s)}$ for which we have $t' \in S$ with $P_t \not\subseteq Q_{t'}$ and $\overline{\rho}(t', s') < \epsilon$. By M1 we have $\overline{\rho}(t, s') < \epsilon$, whence $\underline{\rho}(s', t) < \epsilon$ by DM. Since $P_{s'} \not\subseteq Q_s$ we obtain $R_\epsilon \subseteq \overline{Q}_{(s,t)}$, which is a contradiction. This establishes $R_\epsilon \subseteq r_\epsilon^{\leftarrow}$, and the reverse inclusion is proved in the same way. This completes the proof that $\mathcal{U}_\rho$ is a direlational uniformity on $(S, \mathcal{S})$.

(ii) It is sufficient to show that M3 is equivalent to $\bigsqcap_{\epsilon>0} r_\epsilon \subseteq i$. Suppose that M3 holds but $\bigsqcap_{\epsilon>0} r_\epsilon \not\subseteq i$. Now we have $s, t \in S$ with $\bigsqcap_{\epsilon>0} r_\epsilon \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s,t)} \not\subseteq i$. Hence we have $t' \in S$ with $\overline{P}_{(s,t')} \not\subseteq \overline{Q}_{(s,t)}$ so that for some $s' \in S$ with $P_s \not\subseteq Q_{s'}$ we have $r_\epsilon \not\subseteq \overline{Q}_{(s',t')}\ \forall\, \epsilon > 0$. We deduce $\overline{\rho}(s', t') = 0$ and so $P_s \not\subseteq Q_t$ by M3. However now $i \not\subseteq \overline{Q}_{(s,t)}$, which contradicts $\overline{P}_{(s,t)} \not\subseteq i$. The proof that $\bigsqcap_{\epsilon>0} r_\epsilon \subseteq i$ implies M3 is left to the interested reader.

(iii) By Lemma 4.3 the set $G \in \mathcal{S}$ is open for the uniform ditopology if and only if $G \not\subseteq Q_s \implies \exists\, \epsilon > 0$ with $r_\epsilon[s] \subseteq G$. Since $P_s \subseteq r_\epsilon(P_s) = r_\epsilon[s]$, if we can show that $r_\epsilon[s]$ is uniformly open it will follow by ([6], Theorem 4.2) that the family $r_\epsilon[s]$, $s \in S^\flat$, $\epsilon > 0$, is a base for $\tau_{\mathcal{U}_\rho}$. However, if we take $r_\epsilon[s] \not\subseteq Q_t$, we then have $t' \in S$ with $\overline{P}_{(s,t')} \not\subseteq \overline{Q}_{(s,t)}$, so that $\overline{\rho}(s', t') < \epsilon$ for some $s' \in S$ with $P_s \not\subseteq Q_{s'}$. Since $P_{t'} \not\subseteq Q_t$ we have $\overline{\rho}(t', t) = 0$ and so $\overline{\rho}(s', t) < \epsilon$, whence we may choose $\delta > 0$ with $\overline{\rho}(s', t) + \delta < \epsilon$ and it is now easy to show that $r_\delta[t] \subseteq r_\epsilon[s]$. This establishes that $r_\epsilon[s]$ is uniformly open, as required. Finally it is straightforward to verify that

$$r_\epsilon[s] = N_\epsilon^\rho(s),$$

so the family $N_\epsilon^\rho(s)$, $s \in S^\flat$, $\epsilon > 0$, is a base for both $\tau_{\mathcal{U}_\rho}$ and τ_ρ, whence these topologies coincide. Likewise, the cotopologies coincide. $\square$

Corollary 6.5. *A pseudo metric ditopology is completely biregular. It is T_0, and hence bi–$T_{3\frac{1}{2}}$ [7] and in particular bi–T_2 [7], if and only if ρ is a dimetric.*

Proof. Immediate from Theorem 6.4, Theorem 4.14 and Theorem 4.16. □

For the dimetric of Example 6.2 (3) we obtain the discrete direlational uniformity with base $\{(i, I)\}$. The metric di-uniformity of the usual dimetric on $(\mathbb{I}, \mathcal{I})$ is the usual di-uniformity, while the dicovering uniformity corresponding to the usual dimetric on $(L, \mathcal{L})$ has base $\{((0, s+\epsilon], (0, s-\epsilon]) \mid 0 < s < 1\}$ (cf. Example 3.8).

Definition 6.6. A direlational uniformity $\mathcal{U}$ on $(S, \mathcal{S})$ is called (*pseudo*) *metrizable* if there exists a (pseudo) dimetric ρ with $\mathcal{U} = \mathcal{U}_\rho$.

Theorem 6.7. *A direlational uniformity $\mathcal{U}$ is pseudo metrizable if and only if it has a countable base. It is metrizable if and only if it is also separated.*

Proof. If $\mathcal{U}$ is pseudo metrizable there is a pseudo dimetric ρ with $\mathcal{U} = \mathcal{U}_\rho$. But now, for example, $(r^\rho_{1/n}, R^\rho_{1/n})$, $n \geq 1$, is a countable base of $\mathcal{U}_\rho$, and hence of $\mathcal{U}$.

Conversely, let $\mathcal{U}$ have the countable base (b_n, B_n), $n \geq 1$. Take $(d_1, D_1) \in \mathcal{U}$ symmetric with $(d_1, D_1) \sqsubseteq (b_1, B_1)$, and by induction for $n > 1$ choose a symmetric $(d_n, D_n) \in \mathcal{U}$ so that $(d_n, D_n)^3 \sqsubseteq (d_{n-1}, D_{n-1}) \sqcap (b_n, B_n)$. Then $(d_{n+1}, D_{n+1})^3 \sqsubseteq (d_n, D_n)$ for all $n = 1, 2, \dots$ and $(d_n, D_n) \sqsubseteq (d_n, D_n)^3 \sqsubseteq (b_n, B_n)$, so $\{(d_n, D_n) \mid n = 1, 2, \dots\}$ is also a base of $\mathcal{U}$. Now let q, q^* be as defined in Lemma 4.12 and Lemma 4.13 for the sequence (d_n, D_n). Clearly $\rho = (q, q^*)$ is a pseudo dimetric, so we may consider the direlational uniformity $\mathcal{U}_\rho$. However, Lemma 4.12 (1) and Remark 4.13 (1) immediately give

$$(d_{n+1}, D_{n+1}) \sqsubseteq (r^\rho_{2^{-n}}, R^\rho_{2^{-n}}) \sqsubseteq (d_n, D_n),$$

whence $\mathcal{U} = \mathcal{U}_\rho$.

The final statement is immediate from Theorem 4.16 and Corollary 6.5. □

Definition 6.8. A ditopology on $(S, \mathcal{S})$ is called *(pseudo) metrizable* if it is the metric ditopology of some (pseudo) dimetric on $(S, \mathcal{S})$.

Theorem 6.9. *The ditopology (τ, κ) on $(S, \mathcal{S})$ is pseudo metrizable if and only if there exists a family $\mathcal{C}_n$, $n = 1, 2, \dots$ of anchored dicovers of $(S, \mathcal{S})$ satisfying the conditions*

1. $\mathcal{C}_{n+1} \prec_{(\star)} \mathcal{C}_n$ *for all* $n \geq 1$.
2. $G \in \tau \iff (G \not\subseteq Q_s \implies \exists\, n,\ \mathrm{St}(\mathcal{C}_n, P_s) \subseteq G)$.
3. $F \in \kappa \iff (P_s \not\subseteq F \implies \exists\, n,\ F \subseteq \mathrm{CSt}(\mathcal{C}_n, Q_s))$.

Proof. Suppose first that $(\tau, \kappa) = (\tau_\rho, \kappa_\rho)$ for some pseudo dimetric ρ, and consider the direlational uniformity $\mathcal{U}_\rho$ on $(S, \mathcal{S})$. Note that $(r^\rho_{4^{-n}}, R^\rho_{4^{-n}})$, $n \geq 1$, is a base of $\mathcal{U}_\rho$, whence $\mathcal{C}_n = \gamma(r^\rho_{4^{-n}}, R^\rho_{4^{-n}})$, $n \geq 1$, is a base of anchored dicovers for $\upsilon_\rho = \Gamma(\mathcal{U}_\rho)$. By the hypothesis and Theorem 6.4 (iii), (τ, κ) is the uniform ditopology of $\mathcal{U}_\rho$, while $\mathcal{U}_\rho = \Delta(\upsilon_\rho)$ by Theorem 3.7 (3). Clearly (2) and (3) now follow from Proposition 4.4, so it remains to show (1). However noting that $(r^\rho_\epsilon, R^\rho_\epsilon)$ is symmetric we may apply Proposition 2.4 to $(r^\rho_\epsilon, R^\rho_\epsilon)^2 \sqsubseteq (r^\rho_{2\epsilon}, R^\rho_{2\epsilon})$ for $\epsilon = 4^{-(n+1)}$ and $\epsilon = 2 \times 4^{-(n+1)}$ to give

$$\mathcal{C}_{n+1} \prec_{(\Delta)} \gamma(r^\rho_{2\times 4^{-(n+1)}},\ R^\rho_{2\times 4^{-(n+1)}}) \prec_{(\Delta)} \mathcal{C}_n,$$

whence $\mathcal{C}_{n+1} \prec_{(\star)} \mathcal{C}_n$ by Lemma 2.2 (3 ii), since the dicovers are anchored.

Conversely, suppose that there exists a sequence of anchored dicovers $\mathcal{C}_n$ satisfying (1)–(3). Then these form a base for a dicovering uniformity υ. Moreover, by Proposition 4.4, conditions (2) and (3) imply that the uniform topology of υ, and hence of $\mathcal{U} = \Delta(\upsilon)$, is (τ, κ). Clearly $\delta(\mathcal{C}_n)$, $n = 1, 2, \ldots$ is a countable base of $\mathcal{U}$, so by Theorem 6.7 there is a pseudo dimetric ρ for which the uniform ditopology of $\mathcal{U}_\rho = \mathcal{U}$ is the metric ditopology of ρ. Hence $(\tau, \kappa) = (\tau_\rho, \kappa_\rho)$, so (τ, κ) is pseudo metrizable. □

Clearly conditions (2) and (3) may also be given in terms of the dineighbourhood system and Theorem 6.9 is then seen as a ditopological analogue of the Alexandroff-Urysohn metrization theorem [12].

We end by showing that arbitrary di-uniformities may be defined using pseudo dimetrics.

Definition 6.10. Let $\mathcal{U}$ be a direlational uniformity on $(S, \mathcal{S})$. Then a pseudo dimetric ρ on $(S, \mathcal{S})$ is called *uniform for* $\mathcal{U}$ if $(r^\rho_\epsilon, R^\rho_\epsilon) \in \mathcal{U}\ \forall \epsilon > 0$.

For pseudo metrics ρ_1, ρ_2 on $(S, \mathcal{S})$, $\rho_1 \vee \rho_2 = (\overline{\rho}_1 \vee \overline{\rho}_2,\ \underline{\rho}_1 \vee \underline{\rho}_2)$ is a pseudo metric on $(S, \mathcal{S})$, and clearly $(r^{\rho_1}_\epsilon, R^{\rho_1}_\epsilon) \sqcap (r^{\rho_2}_\epsilon, R^{\rho_2}_\epsilon) = (r^{\rho_1 \vee \rho_2}_\epsilon, R^{\rho_1 \vee \rho_2}_\epsilon)$. Hence the family $\mathcal{G}$ of pseudo dimetrics on $(S, \mathcal{S})$ uniform for $\mathcal{U}$ has the property $\rho_1, \rho_2 \in \mathcal{G} \implies \rho_1 \vee \rho_2 \in \mathcal{G}$. This leads to the following:

Theorem 6.11. *Let* $\mathcal{G}$ *be a non-empty family of pseudo dimetrics on* $(S, \mathcal{S})$ *which is closed under finite suprema. Then*

$$\mathcal{U}_\mathcal{G} = \{(r, R) \mid \exists\, \rho \in \mathcal{G},\ \epsilon > 0 \text{ with } (r^\rho_\epsilon, R^\rho_\epsilon) \sqsubseteq (r, R)\}$$

is a direlational uniformity on $(S, \mathcal{S})$*. Moreover, if* $\mathcal{U}$ *is a direlational uniformity on* $(S, \mathcal{S})$ *and* $\mathcal{G}$ *the set of pseudo dimetrics uniform for* $\mathcal{U}$ *then* $\mathcal{U}_\mathcal{G} = \mathcal{U}$.

Proof. Straightforward. □

Acknowledgements. The authors would like to thank the referees for their helpful comments

References

[1] L. M. Brown, *Relations and functions on textures*, preprint.

[2] L. M. Brown and M. Diker, *Ditopological texture spaces and intuitionistic sets*, Fuzzy Sets and Systems **98** (1998), 217–224.

[3] L. M. Brown and M. Diker, *Paracompactness and full normality in ditopological texture spaces*, J. Math. Anal. Appl. **227** (1998), 144–165.
[4] L. M. Brown and R. Ertürk, *Fuzzy sets as texture spaces, I. Representation theorems*, Fuzzy Sets and Systems **110 (2)** (2000), 227–236.
[5] L. M. Brown and R. Ertürk, *Fuzzy sets as texture spaces, II. Subtextures and quotient textures*, Fuzzy Sets and Systems **110 (2)** (2000), 237–245.
[6] L. M. Brown, R. Ertürk and Ş. Dost, *Ditopological texture spaces and fuzzy topology, I. General principles*, submitted.
[7] L. M. Brown, R. Ertürk and Ş. Dost, *Ditopological texture spaces and fuzzy topology, II. Separation axioms*, preprint, 2002.
[8] M. Diker, *Connectedness in ditopological texture spaces*, Fuzzy Sets and Systems **108** (1999), 223–230.
[9] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove and D. S. Scott, *A compendium of continuous lattices* (Springer–Verlag, Berlin, 1980).
[10] J. L. Kelley, *General Topology* (D. Van Nostrand, Princeton, 1955).
[11] G. Sambin, *Some points in formal topology*, (Proc. Dagstuhl Seminar *Topology in Computer Science*, June 2000), Theoretical Computer Science, to appear.
[12] J. W. Tukey, *Convergence and uniformity in topology*, (Ann. of Math. Studies **2**, 1940).
[13] A. Weil, *Sur les espaces à structure uniforme et sur la topologie générale* (Actualités Sci. Ind. **551**, Paris, 1937).

Received July 2002 Revised December 2002

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