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On some topological invariants for morphisms defined in homological spheres

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Abstract

In the paper one defines topological invariants of type degree for morphisms in the category $Top_{(2)}$ of topological pairs of spaces and continuous single valued maps, which admit homological n-spheres as target and arbitrary topological pairs of spaces as source. The different described degrees are acquired by means homological methods, and are a powerful tool in the root theory. Several existence theorems are obtained for equations with multivalued transformations.

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0. Introduction

The concept of topological degree *deg* is well know for maps of homological n-spheres and oriented n-dimensional manifolds (see for example, [4] and [2]). Recall this concept: if X and Y are both homological n-spheres and $f : X \to Y$ is a continuous single valued map by fixing some generated elements z_1 and z_2 of homology groups $H_n(X)$ and $H_n(Y)$ respectively, one obtains an equality $f_{n*}(z_1) = k \cdot z_2$. This number k is called the degree of f and is denoted by *deg* f. Topological degree theory plays a preponderant role in topology fixed points theory and non linear analysis. The different degrees can be considered as a generalization of the Winding number, Kronecker's characteristic and others

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topological invariants. Different generalizations of the topological degree has been studied for multivalued transformations (see for example [7] - [10]).

1. Homological invariants for some classes of morphisms in the categories Top and $Top_{(2)}$

1.1. **Notations and definitions.** *In the present section one introduces some basic topics which play an important role in the sequel.*

A pair (X, A) of topological spaces such that $A \subseteq X$ is called a pair of topological spaces, in this context, a topological space X is conceived as the pair $(X, \varnothing)$. Let (X, A) and (S, T) be some pairs of topological spaces and $f \in Mor_{Top}(X, S)$ such that $f(A) \subseteq T$, then f is named a continuous single valued map of pairs of topological spaces and denoted $f : (X, A) \to (S, T)$. The collections of pairs of topological spaces and continuous single valued maps of pairs of topological maps with the composition of maps define a category denoted $Top_{(2)}$, it admits as a full subcategory the category Top of topological spaces and continuous single valued maps. Two morphisms f, $g \in Mor_{Top_{(2)}}((X, A), (S, T))$ are called homotopic if and only if there exists a morphism

$$\Phi \in Mor_{Top_{(2)}}((X, A) \times [0, 1], (S, T))$$

such that $\Phi(x, 0) = f(x)$ and $\Phi(x, 1) = g(x)$ for every $x \in X$.

By H one denotes the covariant functor H of singular homology with coefficients in the abelian ring of integer $\mathbb{Z}$, defined from the category $Top_{(2)}$ in the category G_d of graded groups and homomorphisms of degree zero, where G the category of abelian groups and homomorphisms of groups is a full subcategory. Thus, for a given object $(X, A) \in Obj(Top_{(2)})$ and a morphism $f \in Mor_{Top_{(2)}}((X, A), (Y, B))$ the functor H assigns a graded group $\{H_i(X, A)\}_{i \geq 0}$ and a homomorphism of degree zero

$$\{H_i(f)\}_{i \geq 0} \in Mor_{G_d}(\{H_i(X, A)\}_{i \geq 0}, \{H_i(Y, B)\}_{i \geq 0}.$$

1.2. **Degree for a class of morphisms in the category Top.** An object Y in the category Top will be called a homological n-sphere if the topological space Y admits the same homological groups of the n-sphere.

Consider $f \in Mor_{Top}(X, Y)$ a morphism with source an arbitrary topological space X and target Y a homological n-sphere. Let $H_n(f) := f_{n*} \in Mor_G(H_n(X), H_n(Y))$ be the induced homomorphism of f and e be a generator of $H_n(Y)$.

Definition 1.1. The degree of a morphism $f \in Mor_{Top}(X, Y)$ is the integer denoted and defined as $dg(f, X, Y) =\mid k \mid$, where $k \in \mathbb{Z}$ verifies $a = k \cdot e$ and

$$\mathrm{Im} f_{n*} =< a > \subseteq H_n(S^n) =< e > .$$

Example 1.2. Consider the open subset:

$U = \{(x_1, x_2) \in \mathbb{R}^2 \mid (x_1 - 1)^2 + x_2^2 < 1\} \cup \{(x_1, x_2) \in \mathbb{R}^2 \mid (x_1 + 1)^2 + x_2^2 < 1\}$

of $\mathbb{R}^2$ and let $X = \partial U$ be the boundary of U.

If x_0 is a fixed element of $\mathbb{R}^2\backslash X$ one can define the morphism $f \in Mor_{Top}(X, \mathbb{R}^2\diagdown\{0\})$ given by the rule $f(x) = x - x_0$ for every $x \in X$. Then the next equalities are satisfied:

$$dg(f, X, \mathbb{R}^2\diagdown\{0\}) = \begin{cases} 1, & \text{if } x_0 \in U \\ 0, & \text{if } x_0 \notin \overline{U} \end{cases}$$

Let us give some properties of this topological invariant.

One will begin by giving the relation between the Winding number $\deg f$ of a morphisms $f \in Mor_{Top}(S^n, S^n)$ defined on the sphere and its topological invariant $dg(f, S^n, S^n)$.

Proposition 1.3. *Let $f \in Mor_{Top}(S^n, S^n)$ then $dg(f, S^n, S^n) = \mid \deg f \mid$.*

Proof. Consider $H_n(S^n) =< e >$ then $\mathrm{Im} f_{n*} =< f_{n*}(e) >$, moreover $f_{n*}(e) = \deg f \cdot e$, hence $dg(f, S^n, S^n) = \mid \deg f \mid$. □

Definition 1.4. A morphism $f \in Mor_{Top_{(2)}}((X, A), (Y, B))$ is called h - sectional if f admits a right inverse homotopy.

It is not difficult to check the next properties of this degree:

Proposition 1.5. *The Topological degree satisfies the following assertions:*

1. *let $f \in Mor_{Top}(X, Y)$ be a constant map then $dg(f, X, Y) = 0$;*
2. *let $f \in Mor_{Top}(X, Y)$ be h sectional morphism then $dg(f, X, Y) = 1$;*
3. *Let Z be is a topological space and $(f, g) \in Mor_{Top}(X, Z) \times Mor_{Top}(Z, Y)$ then $dg(g \circ f, X, Y)$ is a multiple of $dg(g, Z, Y)$;*
4. *let X_0 be a subset of X an object in the category Top and $f_{X_0} \in Mor_{Top}(X_0, Z)$ be the restriction of a morphism $f \in Mor_{Top}(X, Y)$ then there exists a natural number $n \in \mathbb{N}$ such that $dg(f_{X_0}, X_0, Y) = n \cdot dg(f, X, Y)$.*

Proposition 1.6. *Let Y_1 and Y_2 be both some homological n-spheres in the category Top and $(f, g) \in Mor_{Top}(X, Y_1) \times Mor_{Top}(Y_1, Y_2)$ be a pair of morphisms then $dg(g \circ f, X, Y_2) = dg(f, X, Y_1) \cdot dg(g, Y_1, Y_2)$*

Proof. This is a consequence of the definition 1.1, of the topological degree. □

The topological degree is invariant for homotopic morphisms:

Proposition 1.7. *Let $(f, g) \in Mor_{Top}(X, Y) \times Mor_{Top}(X, Y)$ then if f and g are homotopic $dg(f, X, Y) = dg(g, X, Y)$.*

Let us consider some aspects of the degree $dg(f, X, Y)$ in that case where the target $Y = S^n$.

Proposition 1.8. *Let $f \in Mor_{Top}(X, S^n)$ such that $dg(f, X, S^n) \neq 0$ then f is an epimorphism in the category Top.*

Proof. Suppose that $f \in Mor_{Top}(X, S^n)$ is not an epimorphism so $f(X) \subset S^n$. Let $y \in S^n \diagdown f(X)$, then one can diagramed:

$$\begin{array}{ccc} X & \xrightarrow{f} & S^n \\ \widetilde{f} & \searrow & \uparrow i \\ & & S^n \diagdown \{y\} \end{array}$$

where $\widetilde{f}$ is the submap of f and i is the canonical injection. One can conclude by remarking that $H_n(S^n \diagdown \{y\})$ is a trivial group. $\square$

Proposition 1.9. *Let $f,\ g \in Mor_{Top}(X, S^n)$ then if $dg(f) \neq dg(g)$ the morphisms f and g admit at least a coincidence point in X.*

Proof. Indeed, if $f(x) \neq g(x)$ for every element $x \in X$ then for $(x,t) \in X \times [0,1]$ the vector field $v(x,t) = (1-t)\cdot f(x) + t \cdot (-g)(x) \in R^{n+1}$ is free of zero. This finding offers the opportunity to get the morphism $F \in Mor_{Top}(X \times [0,1], S^n)$ where, $F(x,t) = \frac{v(x,t}{\|v(x,t)\|}$ for all element (x,t) from the source $X \times [0,1]$. The morphism F defines a homotopy between f and $(-g)$. Hereafter, from propositions 1.7 and the definition 1.1, one takes $dg(f, X, S^n) = dg(-g, X, S^n) = dg(g, X, S^n)$. $\square$

1.3. **Degree for a class of morphisms in the category $Top_{(2)}$.** An object $(Y, B) \in Obj(Top_{(2)})$ is called a homological n-sphere if $H_0(Y,B) = H_n(Y,B)$ isomorphic to the abelian ring of integers $\mathbb{Z}$ and $H_i(Y,B) = \{0\}$ for all other indices. For more notions one this topics see [9]**.**

For instance, the pairs of spaces $(\mathbb{R}^n, \mathbb{R}^n \diagdown \{0\})$; (B^n, S^{n-1}) where B^n is the closed ball in $\mathbb{R}^n$ and $S^{n-1} = \partial B^n$, are some n-spheres in that category.

Definition 1.10. The degree of a morphism $f \in Mor_{Top_{(2)}}((X, A), (Y, B))$ where (Y, B) is a homological n-sphere with $H_n(Y, B) =< \eta >$ is denoted and defined by $dgr(f, (X, A), (Y, B)) =\mid k \mid$, where $\mathrm{Im} f_{n*} =< b >$ and $b = k \cdot \eta$.

The next properties are obvious.

Proposition 1.11. *The following assertions are satisfied:*

(1) *if a morphism $f \in Mor_{Top_{(2)}}((X, A), (Y, B))$ is a constant map then $dgr(f, (X, A), (Y, B)) = 0$;*
(2) *if $(f, g) \in Mor_{Top_{(2)}}((X, A), (X', A')) \times Mor_{Top_{(2)}}((X', A'), (Y, B))$ then there exits an integer $k \in \mathbb{N}$ such that :*

$$dgr(g \circ f, (X, A), (Y, B)) = k \cdot dgr(g, (X', A'), (Y, B));$$

(3) *let $(X_0, A_0) \subseteq (X, A)$ and $f_0 \in Mor_{Top_{(2)}}((X_0, A_0), (Y, B))$ be the submap of the morphism $f \in Mor_{Top_{(2)}}((X, A), (Y, B))$ then there exists a natural number $k \in \mathbb{N}$ such that :*

$$dgr(f_0, (X_0, A_0), (Y, B)) = k \cdot dgr(f, (X, A), (Y, B));$$

(4) *let* (Y_1, B_1) *and* (Y_2, B_2) *be some* n*-spheres in the category* $Top_{(2)}$ *and* $(f, g) \in Mor_{Top_{(2)}}((X, A), (Y_1, B_1)) \times Mor_{Top_{(2)}}((Y_1, B_1), (Y_2, B_2))$ *be a pair of morphisms then :*

$$dgr(g \circ f, (X, A), (Y_2, B_2)) = dgr(f, (X, A), (Y_1, B_1)) \cdot dgr(g, (Y_1, B_1), (Y_2, B_2))$$

(5) *if* $f, g \in Mor_{Top_{(2)}}((X, A), (Y, B))$ *are some homotopic morphisms then*

$$dgr(f, (X, A), (Y, B)) = dgr(g, (X, A), (Y, B)).$$

This homological invariant satisfies some more specific properties. Let us describe some of them.

Proposition 1.12. *Let* $Z \subset Int(A) \subseteq A \subseteq \overline{A} \subset X$ *and*

$$f \in Mor_{Top_{(2)}}((X, A), (Y, B))$$

then

$$dgr(\widetilde{f}, (X \smallsetminus Z, A \smallsetminus Z), (Y, B)) = dgr(f, (X, A), (Y, B))$$

where $\widetilde{f} \in Mor_{Top_{(2)}}((X \smallsetminus Z, A \smallsetminus Z), (Y, B))$ *is the submap of the morphism* f *on the pair* $(X \smallsetminus Z, A \smallsetminus Z)$.

Proof. This is a consequence of the following commutative diagram:

$$\begin{array}{cc} H_n(X, A) & \searrow f_{n*} \\ i_{n*} \uparrow & H_n(Y, B) \\ H_n(X \smallsetminus Z, A \smallsetminus Z) & \nearrow \widetilde{f}_{n*} \end{array}$$

where $i \in Mor_{Top_{(2)}}((X \smallsetminus Z, A \smallsetminus Z), (X, A))$ is the natural injection. From excision theorem one infers that i_{n*} is an isomorphism and concludes the proof. $\square$

Proposition 1.13. *Let* $f \in Mor_{Top_{(2)}}((X, A), (Y, B))$ *be a morphism such that* $dgr(f, (X, A), (Y, B)) \neq 0$ *then there exist* $x \in X \smallsetminus A$ *and* $y \in Y \smallsetminus B$ *such that* $f(x) = y$.

Proof. Indeed, if $f(x) \notin Y \smallsetminus B$ for every element $x \in X \smallsetminus A$ on can get the following commutative diagram:

$$\begin{array}{ccc} H_n(X, A) & \overset{f_{n*}}{\rightarrow} & H_n(Y, B) \\ \widetilde{f}_{n*} \downarrow & \nearrow i_{n*} & \\ H_n(B, B) & & \end{array}$$

where $i \in Mor_{Top_{(2)}}((B, B), (Y, B))$ is the natural injection and $\widetilde{f} = f$. One concludes by observing that $H_n(B, B)$ is a trivial group. $\square$

Corollary 1.14. *Let* $f \in Mor_{Top}(X, \mathbb{R}^n)$ *be a morphism in the category* Top *and* A *be a closed subset of* X *such that* $f(a) \neq 0$ *for every* $x \in A$*, then if the degree of the morphism* $f \in Mor_{Top_{(2)}}((X, A), (\mathbb{R}^n, \mathbb{R}^n \smallsetminus \{0\}))$ *is not zero, there exists* $x_0 \in X \smallsetminus A$ *such that* $f(x_0) = 0$.

Corollary 1.15. *Let $f \in Mor_{Top}(B^n, \mathbb{R}^n)$ such that $f(x) \neq 0$ for every element $x \in \partial B^n = S^{n-1}$ then if the degree of the morphism*

$$f \in Mor_{Top_{(2)}}((B^n, S^{n-1}), (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}))$$

is not zero there exists at least an element x in the interior of the ball such that $f(x) = 0$.

2. Homological invariant for a class of multivalued transformations

Let (X, A), $(S, T) \in Obj(Top_{(2)})$ a correspondence $F : (X, A) \to (S, T)$ which assigns for each element $x \in X$ a subset $F(x) \subseteq S$, and $F(A) = \underset{a\in A}{\cup} F(a) \subseteq T$ is named a multivalued transformation, the graph of F denoted Γ_F is the pair $(\Gamma_F^X, \Gamma_F^A) \in Obj(Top_{(2)})$, where $\Gamma_F^X = \{(x, s) \in X \times S \mid s \in F(x)\}$ and $\Gamma_F^A = \{(a, t) \in A \times T \mid t \in F(a)\}$.

A representation of a multivalued transformation $F : (X, A) \to (S, T)$ is a quintuple $Q = [(X, A), (S, T), (M, N), p, q]$ where

$$(p, q) \in Mor_{Top_{(2)}}((M, N), (X, A)) \times Mor_{Top_{(2)}}((M, N), (S, T))$$

and $q(p^{-1}(x)) = F(x)$ for every element $x \in X$. In the case when $p := t_F \in Mor_{Top_{(2)}}((\Gamma_F^X, \Gamma_F^A), (X, A))$ and $q := r_F \in Mor_{Top(2)}((\Gamma_F^X, \Gamma_F^A), (S, T))$ are the natural projections the quintuple $\widetilde{Q} = [(X, A), (S, T), (\Gamma_F^X, \Gamma_F^A), t_F, r_F]$ is named the canonical representation of F.

2.1. Degree for multivalued transformations defined in homological n-spheres.

Let Y_1 and Y_2 be both some homological n-spheres and $F : Y_1 \to Y_2$ be a multivalued transformation with a representation $Q = [Y_1, Y_2, X, p, q]$.

Definition 2.1. The degree of a multivalued transformation $F : Y_1 \to Y_2$ relative to the representation Q is denoted and defined by $\mathfrak{D}g(F, Q) = dg(p, X, Y_1) \cdot dg(q, X, Y_2)$.

The degree $\mathfrak{D}g(F, \widetilde{Q})$ of F relative to the canonical representation $\widetilde{Q}$ will be called the degree of the multivalued transformation F and will denoted by $\mathfrak{D}g(F)$.

Let us give some properties of this homological invariant.

Proposition 2.2. *Let $Q = [Y_1, Y_2, X, p, q]$ be a representation of F, then $\mathfrak{D}g(F, Q)$ is a multiple of $\mathfrak{D}g(F)$.*

Proof. One can consider the following commutative diagram:

$$\begin{tikzcd} & \Gamma_F \arrow[dl, "t_F"'] \arrow[dr, "r_F"] & \\ Y_1 & X \arrow[l, "p"] \arrow[r, "q"'] \arrow[u, "\lambda"'] & Y_2 \end{tikzcd}$$

where $\lambda(x) = (p(x), q(x))$ for every element $x \in X$. One concludes by using assertion 3 of proposition 1.5. $\square$

Proposition 2.3. *Let $F, G : Y_1 \to Y_2$ be two multivalued transformations such that $G(x) \subseteq F(x)$ for every element $x \in Y_1$ then $\mathfrak{D}g(G) = k \cdot \mathfrak{D}g(F)$ for some integer $k \in \mathbb{N}$.*

Proof. Under the hypothesis, one obtains that $\Gamma_G \subseteq \Gamma_F$. One concludes the proof with the following commutative diagram:

$$\begin{tikzcd} & \Gamma_F \arrow[dl, "t_F"'] \arrow[dr, "r_F"] & \\ Y_1 & & Y_2 \\ & \Gamma_G \arrow[ul, "t_G"] \arrow[ur, "r_G"'] \arrow[uu, "i"'] & \end{tikzcd}$$

and by referring to the assertion 3 of proposition 1.5. $\square$

What happened if one gets a morphism $f \in Mor_{Top}(Y_1, Y_2)$ and considers it as a multivalued morphism in the following sense $F(x) = \{f(x)\}$ for every element $x \in Y_1$. In such situation, one has which follows:

Proposition 2.4. *Let $f \in Mor_{Top}(Y_1, Y_2)$ and $F : Y_1 \to Y_2$ be the multivalued transformation given by the rule $F(x) = \{f(x)\} := f(x)$ for every element $x \in Y_1$, then $\mathfrak{D}g(F) = dg(f, Y_1, Y_2)$.*

Proof. It is a consequence of the following commutative diagram:

$$\begin{tikzcd} \Gamma_f \arrow[r, "r_f = r_F"] \arrow[d, "t_f = t_F"'] & Y_2 \\ Y_1 \arrow[ur, "f"'] & \end{tikzcd}$$

where the morphism $t_f \in Mor_{Top}(\Gamma_f, Y_1)$ realizes a homeomorphism. $\square$

Corollary 2.5. *Let $G : Y_1 \to Y_2$ be a multivalued mapping which admits a selector $f \in Mor_{Top}(Y_1, Y_2)$ then $dg(f, Y_1, Y_2) = k \cdot \mathfrak{D}g(G)$ for some natural number $k \in \mathbb{N}$.*

Proof. Indeed $F(x) := \{f(x)\} \subseteq G(x)$ for every $x \in Y_1$ and thus one can conclude by referring to the propositions 2.3 and 2.4. $\square$

Proposition 2.6. *Let Y_1, Y_2 and Y_3 be some homological n-spheres, $F : Y_1 \to Y_2$ be a multivalued transformation and $f \in Mor_{Top}(Y_2, Y_3)$ then $dg(f, Y_1, Y_2) \cdot \mathfrak{D}g(F) = k \cdot \mathfrak{D}g(f \circ F)$ for some $k \in \mathbb{N}$.*

Proof. Of course, the quintuple $Q = [Y_1, Y_2, \Gamma_F, t_F, f \circ r_F]$ is a representation of the multivalued morphism $f \circ F : Y_1 \to Y_2$ therefore, from proposition 1.6 one obtains $\mathfrak{D}g(f \circ F, Q) = dg(f, Y_1, Y_2) \cdot \mathfrak{D}g(F)$ one concludes thanks to proposition 2.2. $\square$

Proposition 2.7. *Let Y be a homological n-sphere and $F : Y \to S^n$ be a multivalued transformation such that $\mathfrak{D}g(F)$ is different from zero then $F(Y) = S^n$.*

Proof. Of course, in this case $dg(r_F, \Gamma_F, S^n) \neq 0$ one concludes by referring to the proposition 1.8. $\square$

Definition 2.8. Two multivalued transformations $F_0, F_1 : Y_1 \to Y_2$ defined on some homological n-spheres Y_1 and Y_2 are called homotopic if there exists a quintuple $[Y_1, Y_2, X \times [0,1], \Phi, \Psi]$ such that $Q_0 = [Y_1, Y_2, X, \Phi_0, \Psi_0]$ and $Q_1 = [Y_1, Y_2, X, \Phi_1, \Psi_1]$ realize some representations of F_0 and F_1 respectively, where $\Phi_t : X \to S^n$ and $\Psi_t : X \to S^n$ are defined by the rules $\Phi_t(x) = \Phi(x,t)$, $\Psi_t(x) = \Psi(x,t)$ for every element $(x,t) \in X \times \{0,1\}$.

Proposition 2.9. *Let $F_0, F_1 : Y_1 \to Y_2$ be some multivalued transformations defined on some homological n-spheres Y_1 and Y_2 then if F_0 and F_1 are homotopic there exist some natural numbers k_0, $k_1 \in \mathbb{N}$ such that $k_0 \cdot \mathfrak{D}g(F_0) = k_1 \cdot \mathfrak{D}g(F_1)$.*

Proof. For this purpose one refers to propositions 1.7 and 2.2. □

2.2. **Degree for multivalued transformations with images in homological n-spheres in the category $Top_{(2)}$.** *In this section one displays a homological invariant for multivalued transformations acting between homological n-spheres of the category $Top_{(2)}$*

Let (Y_0, B_0) and (Y_1, B_1) be some homological n-spheres and $F : (Y_0, B_0) \to (Y_1, B_1)$ be a multivalued transformation that admits a quintuple

$$Q = [(Y_0, B_0), (Y_1, B_1), (M, N), p, q]$$

as a representation.

Definition 2.10. The degree of a multivalued transformation $F : (Y_0, B_0) \to (Y_1, B_1)$ relative to the representation Q is denoted and defined by

$$\mathfrak{D}gr(F, Q) = dgr(p, (M, N), (Y_0, B_0)) \cdot dgr(q, (M, N), (Y_1, B_1)).$$

The degree of F relative to the canonical representation

$$\widetilde{Q} = [(Y_0, B_0), (Y_1, B_1), (\Gamma_F^{Y_0}, \Gamma_F^{B_0}), t_F, r_F]$$

will be denoted by $\mathfrak{D}gr(F) := \mathfrak{D}gr(F, \widetilde{Q})$.

In the sequel, one describes some properties of this homological invariant.

Proposition 2.11. *Let $Q = [(Y_0, B_0), (Y_1, B_1), (X, X'), p, q]$ be a representation of a multivalued transformation $F : (Y_0, B_0) \to (Y_1, B_1)$ then there exists a natural number $k \in \mathbb{N}$ such that $\mathfrak{D}gr(F, Q) = k \cdot \mathfrak{D}gr(F)$.*

Proof. Of course, one can consider the next commutative diagram:

$$\begin{array}{ccccc} (Y_0, B_0) & \xleftarrow{p} & (X, X') & \xrightarrow{q} & (Y_1, B_1) \\ & t_F \nwarrow & \downarrow \lambda & \nearrow r_F & \\ & & (\Gamma_F^{Y_0}, \Gamma_F^{B_0}) & & \end{array}$$

where $\lambda(x) = (p(x), q(x))$ for every element $x \in X$. One concludes by using the assertion 2 of the proposition 1.11. □

Corollary 2.12. *Let $F : (Y_0, B_0) \to (Y_1, B_1)$ be a multivalued transformation $Q = [Y_0, Y_1, X, p, q]$ be a representation of $F : Y_0 \to Y_1$ then the quintuple $Q = [(Y_0, B_0), (Y_1, B_1), (X, p^{-1}(B_0)), p, q]$ is a representation of $F : (Y_0, B_0) \to (Y_1, B_1)$ and there exists a natural number $k \in \mathbb{N}$ such that $\mathfrak{D}gr(F, Q) = k \cdot \mathfrak{D}gr(F)$.*

Proof. This is a consequence of the definition 2.10 and the proposition 2.11. □

Example 2.13. Let $B_1(0)$ be the unit ball of the complex plane $\mathbb{C}$ and $S_1(0) = \partial B_1(0)$ be the boundary of $B_1(0)$ and let $F : (B_1(0), S_1(0)) \to (\mathbb{C}, \mathbb{C} \setminus \{0\})$ be the multivalued transformation defined by the rule $F(z) = \sqrt[n]{z}$. The quintuple

$$Q = [(B_1(0), S_1(0)), (\mathbb{C}, \mathbb{C} \setminus \{0\}), (B_1(0), S_1(0)), p, q]$$

where $p(w) = w^n$ and $q(w) = w$ for every element $w \in B_1(0)$, is a representation of the multivalued mapping F.

Moreover, $dgr(p, (B_1(0), S_1(0)), (B_1(0), S_1(0))) = dg(p) = n$ and $dgr(q, (B_1(0), S_1(0)), (\mathbb{C}, \mathbb{C} \setminus \{0\})) = dg(q) = 1$ so $Dgr(F, Q) = n$.

On the other hand, the single valued map $\lambda : (B_1(0), S_1(0)) \to (\Gamma_F^{B_1(0)}, \Gamma_F^{S_1(0)})$ where $\lambda(w) = (p(w), q(w))$ is an isomorphism in the category $Top_{(2)}$ this implies that the induced homomorphism in homology is an isomorphism in the category $\mathcal{G}$ of groups and homomorphisms of groups and thus $Dgr(F) = Dgr(F, Q) = n$.

Proposition 2.14. *Let F, $G : (Y_0, B_0) \to (Y_1, B_1)$ be both some multivalued transformations such that $G(x) \subseteq F(x)$ for every element $x \in Y_0$ then $\mathfrak{D}gr(G) = k \cdot \mathfrak{D}gr(F)$ for some natural number $k \in \mathbb{N}$.*

Proof. Under the hypothesis, one obtains that $(\Gamma_G^{Y_0}, \Gamma_G^{B_0}) \subseteq (\Gamma_F^{Y_0}, \Gamma_F^{B_0})$. Therefore one infers the assertion from, the following commutative diagram:

$$\begin{array}{ccccc} & {}^{t_F}\swarrow & (\Gamma_F^{\mathbf{Y}_0}, \Gamma_F^{\mathbf{B}_0}) & \searrow^{r_F} & \\ (Y_0, B_0) & & \uparrow i & & (Y_1, B_1) \\ & {}_{t_G}\nwarrow & (\Gamma_G^{\mathbf{Y}_0}, \Gamma_G^{\mathbf{B}_0}) & \nearrow_{r_G} & \end{array}$$

and by referring to the assertion 2 of the proposition 1.11 □

Proposition 2.15. *Let $f \in Mor_{Top_{(2)}}((Y_0, B_0), (Y_1, B_1))$ and $F : (Y_0, B_0) \to (Y_1, B_1)$ be the multivalued transformation given by the rule $F(x) = \{f(x)\} := f(x)$ for every element $x \in Y_0$, then $\mathfrak{D}gr(F) = dgr(f, (Y_0, B_0), (Y_1, B_1))$.*

Proof. For this purpose one can consider the next diagram:

$$(Y_0, B_0) \overset{t_f = t_F}{\longleftarrow} (\Gamma_f^{\mathbf{Y}_0}, \Gamma_f^{\mathbf{B}_0}) \overset{r_f = r_F}{\rightarrow} (Y_1, B_1)$$

□

Corollary 2.16. *Let $G : (Y_0, B_0) \to (Y_1, B_1)$ be a multivalued transformation which admits a selector $f \in Mor_{Top(2)}((Y_0, B_0), (Y_1, B_1))$ then:*

$$dgr(f, (Y_0, B_0), (Y_1, B_1)) = k \cdot \mathfrak{D}gr(G)$$

for some natural number $k \in \mathbb{N}$.

Proof. This is a consequence of the propositions 2.14 and 2.15. □

Proposition 2.17. *Let* (Y_0, B_0), (Y_1, B_1) *and* (Y_2, B_2) *be some homological* n*-spheres,* $F : (Y_0, B_0) \to (Y_1, B_1)$ *be a a multivalued transformation and* $f \in Mor_{Top_{(2)}}((Y_1, B_1), (Y_2, B_2))$ *then*

$$dgr(f, (Y_1, B_1), (Y_2, B_2)) \cdot \mathfrak{D}gr(F) = k \cdot \mathfrak{D}gr(f \circ F)$$

for some natural number $k \in \mathbb{N}$.

Proof. Of course, the quintuple $Q = [(Y_0, B_0), (Y_2, B_2), (\Gamma_F^{Y_0}, \Gamma_F^{B_0}), t_F, f \circ r_F]$ is a representation of the multivalued transformation $f \circ F : (Y_0, B_0) \to (Y_2, B_2)$ therefore, from the assertion 4 of proposition 1.11 one obtains the next equality:

$$\mathfrak{D}gr(f \circ F, Q) = dgr(f, (Y_1, B_1), (Y_2, B_2)) \cdot \mathfrak{D}gr(F),$$

one concludes due to proposition 2.11 □

Definition 2.18. Let $F_0, F_1 : (X, A) \to (S, T)$ be some multivalued transformations, F_0 and F_1 are called homotopic if there exists a quintuple

$$Q = [(X, A), (S, T), (M, N) \times [0, 1], \Phi, \Psi]$$

such that the following quintuples:

$$Q_0 = [(X, A), (S, T), (M, N), \Phi_0, \Psi_0]$$

and

$$Q_1 = [(X, A), (S, T), (M, N), \Phi_1, \Psi_1]$$

are some representations of F_0 and F_1 respectively and where $\Phi_t : (M, N) \to (X, A)$ and $\Psi_t : (M, N) \to (X, A)$ are defined by the rules $\Phi_t(m) = \Phi(m, t)$, $\Psi_t(m) = \Psi(m, t)$ for every element $(m, t) \in M \times \{0, 1\}$.

Proposition 2.19. *Let* $F_0, F_1 : (Y_0, B_0) \to (Y_1, B_1)$ *be some multivalued transformations then if* F_0 *and* F_1 *are homotopic there exist some representations* Q_0 *and* Q_1 *of* F_0 *and* F_1 *respectively such that* $\mathfrak{D}gr(F_0, Q_0) = \mathfrak{D}gr(F_1, Q_1)$.

Proof. It is a consequence of assertion 5 of proposition 1.11. □

Proposition 2.20. *Let* $F : (Y_0, B_0) \to (\mathbb{R}^n, \mathbb{R}^n \diagdown \{0\})$ *be a multivalued transformation such that* $\mathfrak{D}gr(F) \neq 0$ *then there exists an element* $y \in Y_0 \diagdown B_0$ *such that* $0 \in F(y)$.

Proof. Indeed, $\mathfrak{D}gr(F) \neq 0$ so $dgr(r_F, (\Gamma_F^{Y_0}, \Gamma_F^{B_0}), (\mathbb{R}^n, \mathbb{R}^n \diagdown \{0\})) \neq 0$ after which one can conclude due to the proposition 1.13. □

Let S be the boundary of a closed ball B of $\mathbb{R}^n$ and $F : B \to \mathbb{R}^n$ be a multivalued transformation. The multivalued vector field induced by F noted by Φ is the multivalued transformation given by the rule $\Phi(x) = x - F(x)$ for every element $x \in B$. It is obvious that if $Q = [B, \mathbb{R}^n, \Gamma_F, p, q]$ is the canonical representation of F and the multivalued vector field induced by F is such that $\Phi : (B, S) \to (\mathbb{R}^n, \mathbb{R}^n \diagdown \{0\})$ then the quintuple $\widehat{Q} =$

$[(B,S),(\mathbb{R}^n,\mathbb{R}^n\diagdown\{0\}),(\Gamma_F^B,\Gamma_F^S),p,p-q)]$ is a representation of the multivalued vector field Φ.

In the sequel the degrees:

$$dgr(p,(\Gamma_F^B,\Gamma_F^S),(\mathbb{R}^n,\mathbb{R}^n\diagdown\{0\}))$$

and

$$dgr(p-q,(\Gamma_F^B,\Gamma_F^S),(\mathbb{R}^n,\mathbb{R}^n\diagdown\{0\}))$$

will be denoted by $dgr(p)$ and $dgr(p-q)$ respectively.

Proposition 2.21. *Let $F : B \to \mathbb{R}^n$ be a multivalued transformation which is free of fixed point on the boundary S of a closed ball B then if the topological degree $\mathfrak{D}gr(\Phi)$ of the multivalued vector field induced by F is not zero i.e. $Dgr(\Phi)\neq 0$, the multivalued transformation $F : B \to \mathbb{R}^n$ admits a fixed point in the interior of the ball B.*

Proof. This is a consequence of proposition 2.20. □

Proposition 2.22. *Let $F : B \to \mathbb{R}^n$ be a multivalued transformation free of fixed point on the boundary S of a ball B and $F(S)\subseteq B$ then the following equivalence is satisfied:*

$$\mathfrak{D}gr(\Phi,\widehat{Q})\neq 0 \textit{ if and only if } dgr(p)\neq 0.$$

Proof. Consider the morphisms $p,\ p-q \in Mor_{Top_{(2)}}((\Gamma_F^B,\Gamma_F^S),(\mathbb{R}^n,\mathbb{R}^n\backslash\{0\}))$ and let $\Psi \in Mor_{Top}(\Gamma_F^B\times[0,1]),\mathbb{R}^n)$ be a morphism given by the rule:

$$\Psi((x,y),\lambda)=p(x,y)-\lambda\cdot q(x,y)$$

for every element $((x,y),\lambda)\in\Gamma_F^B\times[0,1]$.

It follows that the morphism:

$$\Psi\in Mor_{Top_{(2)}}((\Gamma_F^B,\Gamma_F^S)\times[0,1]),(\mathbb{R}^n,\mathbb{R}^n\backslash\{0\}))$$

is a homotopy between the morphisms:

$$p,p-q\in Mor_{Top_{(2)}}((\Gamma_F^B,\Gamma_F^S),(\mathbb{R}^n,\mathbb{R}^n\backslash\{0\}))$$

therefore from assertions 5 of proposition 1.11 one deduces what follows :

$$dgr(p)=dgr(p-q)$$

and thus one obtains the next equality:

$$\mathfrak{D}gr(\Phi,\widehat{Q})=(dgr(p))^2\,.$$

Hence, $\mathfrak{D}gr(\Phi,\widehat{Q})\neq 0$ if and only if $dgr(p)\neq 0$. □

The last proposition 2.22, permits to obtain a generalization of the theorems due to Eilenberg-Montgomery [3] and Kakutani [8].

Theorem 2.23. *Let $F : B \to \mathbb{R}^n$ be a multivalued transformation which satisfies the following conditions:*

(1) $dgr(p) \neq 0$,
(2) $F(S) \subseteq B$.

Then the multivalued transformation F admits in the ball a fixed point.

Proof. Of course, if F has a fixed point on S then the conclusion of the theorem is satisfied. Otherwise, if F is free of fixed point on S then $\mathfrak{D}gr(\Phi) \neq 0$. One concludes the proof from proposition 2.21. $\square$

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References

[1] Yu. G. Borisovich, B. D. Gel'man, A. D. Myshkis and V. V. Obukhovskii, *Topological methods in the fixed point theory of multivalued maps,* Uspehi Mat. Nauk **35**, no. 1 (1980), 59–126 (in Russian); English translation: Russian Math. Surveys **35** (1980), 65–143.

[2] A. Dold , *Lectures on Algebraic Topology,* Springer-Verlag, Berlin-Heidelberg -New York, 1972.

[3] S. Eilenberg and D. Montgomery, *Fixed point theorems for multivalued transformations,* Amer. J. Math **68** (1946), 214–222.

[4] S. Eilenberg and N. Steenrod, *Foundations of Algebraic Topology,* Princeton university Press, Princeton, New Jersey, 1952.

[5] L. Górniewicz, *Homological methods in fixed point theory of multi-valued maps,* Dissertationes Math. Warsaw **129** (1976), 1–66.

[6] L. Górniewicz, *Topological Fixed Point Theory of Multivalued Mappings,* Kluwer Acad. Publ., Dordrecht-Boston-London, 1999.

[7] A. Granas, *Sur la notion du degre topologique pour une certaine classe de transformations multivalentes dans les spaces de Banach,* Bull. Acad. polon. sci. Ser. sci. math., astron. et phys. **7**, no. 4 (1959), 191–194.

[8] S. Kakutani, *A generalization of Brouwer's fixed point theorem,* Duke Mathematical Journal **8** (1941), 457–459.

[9] R. M. Switzer, *Algebraic Topology- Homotopy and Homology,* Springer Verlag, Berlin-Heidelberg-New York, 1975.

[10] N. M Benkafadar and B. D. Gel'man, *Generalized local degree for multi-valued mappings,* International Journal of Math. Game Theory and Algebra **10**, no. 5 (2000), 413–434.

