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On paracompact spaces and projectively inductively closed functors

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Abstract. In this paper we introduce a notion of *projectively inductively closed* functor (p.i.c.-functor). We give sufficient conditions for a functor to be a p.i.c.-functor. In particular, any finitary normal functor is a p.i.c.-functor. We prove that every preserving weight p.i.c.-functor of a finite degree preserves the class of stratifiable spaces and the class of paracompact σ-spaces. The same is true (even if we omit a preservation of weight) for paracompact Σ-spaces and paracompact p-spaces.

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1. Introduction

By *Tych* we denote the category of all Tychonoff spaces and all their continuous functions. A Hausdorff compact space is called a compact space or just a *compactum*. By *Comp* we denote the full subcategory of *Tych*, whose objects are compacta.

Recall that a covariant functor $\mathcal{F}\colon Comp \to Comp$ is said to be *normal* [17] if it satisfies the following properties:

1. *preserves the empty set and singletons*, i.e., $\mathcal{F}(\varnothing) = \varnothing$ and $\mathcal{F}(\{1\}) = \{1\}$, where $\{k\}$ $(k \geq 0)$ denotes the set $\{0, 1, \ldots, k-1\}$ of nonnegative integers smaller than k. In this notation $0 = \{\varnothing\}$.
2. is *monomorphic*, i.e., for any (topological) embedding $f\colon A \to X$, the mapping $\mathcal{F}(f)\colon \mathcal{F}(A) \to \mathcal{F}(X)$ is also an embedding.
3. is *epimorphic*, i.e., for any surjective mapping $f\colon X \to Y$, $\mathcal{F}(f)\colon \mathcal{F}(X) \to \mathcal{F}(Y)$ is also surjective.
4. *continuous*, i.e., for any inverse spectrum $S = \{X_\alpha; \pi^\alpha_\beta : \alpha \in \mathcal{A}\}$ of compact spaces, the limit $f\colon \mathcal{F}(\lim S) \to \lim \mathcal{F}(S)$ of the mappings $\mathcal{F}(\pi_\alpha)$,

where $\pi_a\colon \lim S \to X_\alpha$ are the limiting projections of the spectrum S, is a homeomorphism.

(5) *preserves intersections*, i.e., for any family $\{A_\alpha\ \alpha \in \mathcal{A}\}$ of closed subsets of a compact space X, the mapping $\mathcal{F}(i)\colon \cap\{\mathcal{F}(A_\alpha):\ \alpha \in \mathcal{A}\} \to \mathcal{F}(X)$ defined by $\mathcal{F}(i)(x) = \mathcal{F}(i_\alpha)(x)$, where $i_\alpha\colon A_\alpha \to X$ is the identity embeddings for all $\alpha \in \mathcal{A}$, is an embedding.

(6) *preserves preimages*, i.e., for any mapping $f\colon X \to Y$ and an arbitrary closed set $A \subset Y$, we have $\mathcal{F}(f^{-1}(A)) = (\mathcal{F}(f))^{-1}(\mathcal{F}(A))$.

(7) *preserves weight*, i.e., $w(\mathcal{F}(X)) = w(X)$ for any infinite compactum X.

In what follows we shall use bigger than normal classes of functors. But any of them shall *preserve empty set, intersections and be monomorphic*. By exp we denote the well-known *hyperspace* functor of non-empty closed subsets. This functor takes every (nonempty) compact space X to the set of all its nonempty closed subsets endowed with the (finite) *Vietoris* topology (see [9]), and a continuous mapping $f\colon X \to Y$ to the mapping $\exp(f)\colon \exp(X) \to \exp(Y)$, defined by $\mathcal{F}(f)(A) = A$.

For a functor $\mathcal{F}$ and an element $a \in \mathcal{F}(X)$, the *support* of a is defined as intersection of all closed sets $A \subset X$ such that $a \in \mathcal{F}(A)$ (recall that we consider only monomorphic functors preserving intersections). This support we denote by $\mathrm{supp}_{\mathcal{F}(X)}(a)$. When it is clear what functor and space are meant, we denote the support of a merely by $\mathrm{supp}(a)$.

A. Ch. Chigogidze [7] extended an arbitrary intersection-preserving monomorphic functor $\mathcal{F}\colon \mathit{Comp} \to \mathit{Comp}$ to the category Tych by setting

$$\mathcal{F}_\beta(X) = \{a \in \mathcal{F}(\beta X):\ \mathrm{supp}(a) \subset X\}$$

for any Tychnoff space X. If $f\colon X \to Y$ is a continuous mapping of Tychonoff spaces and $\beta f\colon \beta X \to \beta Y$ is the (unique) extension of f over their Stone-Čech compactifications, then

$$\mathcal{F}(\beta f)(\mathcal{F}(\beta X)) \subset \mathcal{F}_\beta(X).$$

The last inclusion is a corollary of a trivial fact

$$f(\mathrm{supp}(a)) \supset \mathrm{supp}(\mathcal{F}(f)(a)). \tag{1.1}$$

Therefore, we can define the mapping

$$\mathcal{F}_\beta(f) = \mathcal{F}(\beta f)|X,$$

which makes $\mathcal{F}_\beta$ into a functor.

A. Ch. Chigogidze proved [7] that if a functor $\mathcal{F}$ has certain normality property, then $\mathcal{F}_\beta$ has the same property (modified when necessary). In what follows by a covariant functor $\mathcal{F}\colon \mathit{Tych} \to \mathit{Tych}$ we shall mean a functor of type $\mathcal{F}_\beta$. For such a functor $\mathcal{F}$ and any compact space X the space $\mathcal{F}(X)$ is a compact space.

For a set A by $|A|$ we denote the cardinality of A. For a subset A of a space X by $\overline{A}^X$ we denote the closure of A in X.

In this paper we introduce a notion of *projectively inductively closed* functor (p.i.c.-functor). We give sufficient conditions for a functor to be a p.i.c.-functor (Theorem 3.5). In particular, any finitary normal functor is a p.i.c.-functor (Corollary 3.6). We prove that every preserving weight p.i.c.-functor of a finite degree preserves the class of stratifiable spaces and the class of paracompact σ-spaces (Theorem 3.7). The same is true (even if we omit a preservation of weight) for paracompact Σ-spaces and paracompact p-spaces (Theorem 3.8). All spaces are assumed to be Tychonoff, and all mappings, continuous. Any additional information on general topology and covariant functors one can find, for example, in ([8], [9], [17]).

2. Preliminaries

In this section we recall some definitions and facts, which will be useful in establishing our main results (see Section 3).

Definition 2.1 ([2])**.** A *network* for a space X is a collection $\mathcal{N}$ of subsets of X such that whenever $x \in U$ with U open, there exists $F \in \mathcal{N}$ with $x \in F \subset U$.

An elementary corollary of this definition is that every base of a space X is a network of X.

A family $\mathcal{A}$ of subsets of X is said to be *σ-locally finite* if it is a union of countably many families $\mathcal{A}_n$ which are locally finite in X.

Definition 2.2 ([16])**.** A topological space X is called a *σ-space*, if it has a σ-locally finite network.

Remark 2.3. A rather simple observation of the definition 2.2 shows us that every closed subset of a σ-space is a σ-space.

Proposition 2.4 ([11])**.** *Every closed image of a σ-space is a σ-space.*

A well-known theorem of E. Michael [13] states that every closed image of a paracompact space is a paracompact space. So, from Proposition 2.4 we get

Theorem 2.5 ([11])**.** *Every closed image of a paracompact σ-space is a paracompact σ-space.*

Theorem 2.6 ([11])**.** *A countable product of paracompact σ-spaces is a paracompact σ-space.*

In 1969 K. Nagami [15] introduced more general class than class of σ-spaces.

Definition 2.7. A space X is a *Σ-space* if there exists a σ-discrete collection $\mathcal{N}$, and a cover c of X by closed countably compact sets such that, whenever $C \in c$ and $C \subset U$ with open U, then $C \subset F \subset U$ for some $F \in \mathcal{N}$.

Clearly, from Definitions 2.1 and 2.7 we have.

Proposition 2.8. *Every perfect preimage of a σ-space is a Σ-space. In particular, every σ-space is a Σ-space.*

Proposition 2.9. *Every closed subspace Y of a Σ-space X is a Σ-space.*

Indeed, evidently, that the families $\mathcal{N}|Y$ and $c|Y$, where $\mathcal{N}$ and c are from Definition 2.7, satisfy Definition 2.7 for Y.

K. Nagami [15] has shown that the class of Σ-spaces is strictly larger than the class of perfect preimages of σ-spaces. On the other hand, the class of perfect preimages of σ-spaces is much larger than the class of σ-spaces. For example, every compact σ-space is metrizable.

Paracompact Σ-spaces behave nicely with respect to countable products and perfect images.

Proposition 2.10 ([15]). *The countable product of paracompact Σ-spaces is a paracompact Σ-space.*

Proposition 2.11 ([15]). *Every perfect image of a paracompact Σ-space is a paracompact Σ-space.*

The class of paracompact p-spaces in sense of A. V. Arhangel'-skii is a proper subclass of paracompact Σ-spaces.

Definition 2.12 ([3]). A space X is called a *p-space* if there exists a countable family u_n such that:

1) u_n consists of open subsets of βX;
2) $X \subset \cup u_n$ for each n;
3) $\cap_n st(x, u_n) \subset X$ for every $x \in X$.

Here for a family v of subsets of a space Y by $st(y, v)$ we denote the set $\cup\{V \in v : \ y \in V\}$.

Theorem 2.13 ([3]). *The class of paracompact p-spaces coincides with the class of perfect preimages of metrizable spaces.*

Corollary 2.14. *Every paracompact p-space is a perfect preimage of a paracompact σ-space and, consequently, is a paracompact Σ-space.*

Theorem 2.13 also yields

Corollary 2.15 ([3]). *Every countable product of paracompact p-spaces is a paracompact p-space.*

Proposition 2.16 ([3]). *Every closed subspace of a paracompact p-space is a paracompact p-space.*

Theorem 2.17 ([10]). *Every perfect image of a paracompact p-space is a paracompact p-space.*

Let us recall some more notions and facts.

Definition 2.18 ([6]). A space X is *stratifiable* if there is a function G which assigns to each $n \in \omega$ and closed set $H \subset X$, an open set $G(n, H)$ containing H such that

(1) if $H \subset K$, then $G(n, H) \subset G(n, K)$;
(2) $H = \cap_n \overline{G(n, H)}$.

The class of stratifiable spaces was defined in 1961 by J.Ceder [6]. But he called these spaces by M_3-spaces. The latter form was proposed by C.R. Borges [5] in 1966.

In the definition of a stratifiable space we can also use the following additional condition:

(3) $$G(n+1,H) \subset G(n,H).$$

Indeed, we define new stratification G' by

$$G'(n,H) = \cap_{i \le n} G(i,H).$$

The following dual characterization of stratifiable spaces is sometimes useful:

X is stratifiable if and only if for each open $U \subset X$ and $n \in \omega$ one can assign an open set U_n such that $\overline{U}_n \subset U$, $U = \cup_n U_n$ and $U \subset V$ implies $U_n \subset V_n$.

To get this characterization from a function G satisfying definition 2.18, let $U_n = X \backslash \overline{G(n, X \backslash U)}$. On the other hand, to get G from U_n's let $G(n,X) = X \backslash \overline{(X \backslash H)}_n$.

From this characterization of stratifiable spaces and Michael's theorem [14] characterizing a paracompact space by σ-cushioned coverings, we get

Theorem 2.19 ([6]). *Stratifiable spaces are paracompact.*

Corollary 2.20 ([5]). *Stratifiable spaces are perfectly normal.*

Indeed, every stratifiable space is normal in view of Theorem 2.19. On the other hand, each closed subset of X is a G_δ-set by Definition 2.18.

Theorem 2.21 ([12]). *Every stratifiable space is a σ-space.*

As a corollary of Theorems 2.19 and 2.21 we get

Theorem 2.22. *Every stratifiable space is a paracompact σ-space.*

Theorem 2.23 ([5]). *Every subspace of a stratifiable space is stratifiable.*

Theorem 2.24 ([5]). *A countable product of stratifiable spaces is stratifiable.*

Theorem 2.25 ([5]). *Stratifiable spaces are preserved by closed mappings.*

From Theorem 2.25 we get

Corollary 2.26. *An image of a metrizable space under closed mapping is stratifiable. In particular, every metrizable space is stratifiable.*

Going back to functors $\mathcal{F}\colon Comp \to Comp$, we, evidently, have

(2.1) $$a \in \mathcal{F}(\mathrm{supp}(a)).$$

If a functor $\mathcal{F}$ preserves preimages, then $\mathcal{F}$ *preserves supports* [17], i.e.

(2.2) $$f(\mathrm{supp}(a) = \mathrm{supp}(\mathcal{F}(f)(a)).$$

The property (2.2) can be conversed.

Proposition 2.27 ([17]). *Any monomorphic preserving intersections functor preserves supports if and only if it preserves preimages.*

Definition of the functor $\mathcal{F}$ and property (2.2) imply that

$$f(\operatorname{supp}_{\mathcal{F}(X)}(a)) = \operatorname{supp}_{\mathcal{F}_\beta(Y)} \mathcal{F}_\beta(f)(a) \tag{2.3}$$

for any preimage preserving functor $\mathcal{F}\colon Comp \to Comp$, continuous mapping $f\colon X \to Y$, and $a \in \mathcal{F}_\beta(X)$.

Now we recall one construction given by V. N. Basmanov [4]. Let $\mathcal{F}\colon Comp \to Comp$ be a functor. By $C(X,Y)$ we denote the space of all continuous mappings from X to Y with compact-open topology.

In particular, $C(\{k\},Y)$ is naturally homeomorphic to the k-th power Y^k of the space Y; the homeomorphism takes each mapping $\xi\colon \{k\} \to Y$ to the point $(\xi(0),\dots,\xi(k-1)) \in Y^k$.

For a functor $\mathcal{F}$, compact space X, and a positive integer k, V.N. Basmanov [4] defined the mapping

$$\pi_{\mathcal{F},X,k}\colon C(\{k\},X) \times \mathcal{F}(\{k\}) \to \mathcal{F}(X)$$

by

$$\pi_{\mathcal{F},X,k}(\xi,a) = \mathcal{F}(\xi)(a)$$

for any $\xi \in C(\{k\},X)$ and $a \in \mathcal{F}(\{k\})$.

When it is clear what functor $\mathcal{F}$ and what space X are meant, we omit the subscripts $\mathcal{F}$ and X and write $\pi_{X,k}$ or π_k instead of $\pi_{\mathcal{F},X,k}$.

According to Shcepin's theorem ([17], Theorem 3.1). the mapping

$$\mathcal{F}\colon C(Z,Y) \to \mathcal{F}(\mathcal{F}(Z),\mathcal{F}(Y))$$

is continuous for any *continuous* functor $\mathcal{F}$ and compact spaces Z and Y. This implies the following assertion.

Proposition 2.28 ([4])**.** *If $\mathcal{F}$ is a continuous functor, X is a compact space, and k is a positive integer, then the mapping $\pi_{\mathcal{F},X,k}$ is continuous.*

Let $\mathcal{F}_k$ be a subfunctor of a functor $\mathcal{F}$ defined as follows. For a compact space X, $\mathcal{F}_k(X)$ is the image of the mapping $\pi_{\mathcal{F},X,k}$ and for a mapping $f\colon X \to Y$, $\mathcal{F}_k(f)$ is the restriction of $\mathcal{F}(f)$ to $\mathcal{F}_k(X)$. Denote by $\overline{f}\colon C(\{k\},X) \to (C\{k\},Y)$ the mapping which takes ξ to composition $f \circ \xi$. It is easy to see that

$$\pi_{Y,k} \circ \overline{f} \times \mathrm{id}_{\mathcal{F}(\{k\})} = \mathcal{F}(f) \circ \pi_{X,k}. \tag{2.4}$$

Therefore, $\mathcal{F}(f)(\mathcal{F}_k(X)) \subset \mathcal{F}_k(Y)$. Hence, $\mathcal{F}_k$ is a functor.

A functor $\mathcal{F}$ is called a *functor of degree n*, if $\mathcal{F}_n(X) = \mathcal{F}(X)$ for any compact space X, but $\mathcal{F}_{n-1}(X) \neq \mathcal{F}(X)$ for some X. The next assertion (Proposition 2.29) is Shcepin's definition of the functor $\mathcal{F}_k$. But using Basmanov's definition we should prove it. One can find the proof in [28].

Proposition 2.29. *For any continuous functor $\mathcal{F}$ and a compact space X, we have*

$$\mathcal{F}_k(X) = \{a \in \mathcal{F}(X) : \ |\operatorname{supp}(a)| \le k\}.$$

Corollary 2.30. *For any compact space X, we have*

$$\exp_k(X) = \{a \in \exp(X) : \ |a| \le k\}.$$

The definition of a support and the property (2.1) imply.

Proposition 2.31. *For a functor $\mathcal{F}$, a compact space X, and a closed subset A of X,*

$$\mathcal{F}(A) = \{a \in \mathcal{F}(X) : \operatorname{supp}(a) \subset A\}.$$

For a Tychonoff space X, a functor $\mathcal{F}\colon Comp \to Comp$, and a positive integer k, we put

$$\mathcal{F}_k(X) = \pi_{\mathcal{F},\beta X,k}(C(\{k\}), X) \times \mathcal{F}(\{k\}))$$

and denote the restriction of $\pi_{\mathcal{F},\beta X,k}$ to $C(\{k\}) \times \mathcal{F}(\{k\})$ by $\pi_{\mathcal{F},X,k}$. If $f\colon X \to Y$ is a continuous mapping, then

$$\mathcal{F}(\beta f)(\mathcal{F}_k(X)) \subset \mathcal{F}_k(Y),$$

in view of the equality (2.4) for the mapping βf. Therefore, setting

$$\mathcal{F}_k(f) = \mathcal{F}_k(\beta f)|\mathcal{F}(X),$$

we obtained a mapping

$$\mathcal{F}_k(f)\colon \mathcal{F}_k(X) \to \mathcal{F}_k(Y).$$

Thus, we have defined the covariant functor

$$\mathcal{F}_k\colon Tych \to Tych,$$

that extends the functor $\mathcal{F}_k\colon Comp \to Comp$ to the category $Tych$. Proposition 2.29 implies the following assertion.

Proposition 2.32 ([18])**.** *If $\mathcal{F}\colon Comp \to Comp$ is a continuous functor, then $\mathcal{F}_k\colon Tych \to Tych$ is a subfunctor of the functor $\mathcal{F}_\beta$, and*

$$\mathcal{F}_k(X) = \mathcal{F}_\beta(X) \cap \mathcal{F}_k(\beta X). \tag{2.5}$$

Proposition 2.33 ([18])**.** *For a compact space X, a continuous functor $\mathcal{F}$ and a positive integer k, the set $\mathcal{F}_k(X)$ is closed in $\mathcal{F}(X)$.*

Propositions 2.32 and 2.33 imply

Proposition 2.34 ([18])**.** *For a Tychonoff space X, a continuous functor $\mathcal{F}$, and a positive integer k, the set $\mathcal{F}_k(X)$ is closed in $\mathcal{F}_\beta(X)$.*

Corollary 2.35. *For a Tychonoff space X, a continuous functor $\mathcal{F}$, and a positive integer k, the set $\mathcal{F}_k(X)$ is closed in $\mathcal{F}_{k+1}(X)$.*

3. Projectively inductively closed functors

We start recalling that a functor $\mathcal{F}$ is said to be *finitely open* [18], if the set $\mathcal{F}_k(\{k+1\})$ is open in $\mathcal{F}(\{k+1\})$ for any positive integer k. The dual for this definition states that $\mathcal{F}(\{k+1\}) \backslash \mathcal{F}_k(\{k+1\})$ is closed in $\mathcal{F}(\{k+1\})$.

Remark 3.1. As an example of a finitely open functor one can take any *finitary* functor $\mathcal{F}$, i.e., a functor $\mathcal{F}$ such that $\mathcal{F}(\{k\})$ is finite for any positive integer k. In particular, the hyperspace functor exp is a finitary and, consequently, a finitely open functor.

Lemma 3.2. *For any continuous, preserving preimages functor $\mathcal{F}_\beta$, the mapping $\pi_{\mathcal{F}_\beta,X,1}$ is a homeomorphism.*

Proof. At first we show that $\pi_{\mathcal{F}_\beta,X,1}$ is a bijective mapping. In view of (2.3) for any $\xi \in C(\{1\}, X)$ and $a \in \mathcal{F}(\{1\})$ we have $\mathcal{F}(\xi)(a) = \xi(0)$, since we consider functors preserving empty set. Since the set $\{1\}$ consists of one point 0, every mapping $\xi\colon \{1\} \to X$ is a monomorphism. But we consider only monomorphic functors. Hence, the mapping $\mathcal{F}(\xi)$ is a monomorphism. On the other hand,

$$\pi_{\mathcal{F}_\beta,X,1}(\xi, a) = \mathcal{F}(\xi)(a) = \xi(0).$$

Consequently, π_1 is an injective mapping. Furthere, the mapping

$$\pi_{\mathcal{F}_\beta,X,k}\colon C(\{k\}, X) \times \mathcal{F}(\{k\}) \to \mathcal{F}_k(X)$$

is epimorphic for any positive integer k, in particular, for $k = 1$ by definition of $\mathcal{F}(X)$. Thus, π_1 is a bijective mapping.

Hence, π_1 is a homeomorphism for a compact space X (π_1 is continuous in view of Proposition 2.28). If X is a Tychonoff space, then by definition, the mapping $\pi_{\mathcal{F}_\beta,X,k}$ is a restriction of $\pi_{\mathcal{F},\beta X,k}$ to $C(\{k\}, X) \times \mathcal{F}(\{k\})$. Therefore, the mapping $\pi_{\mathcal{F},X,1}$ is a homeomorphism as a restriction of the homeomorphism $\pi_{\mathcal{F},\beta X,1}$ to a subset. The proof is complete. □

Definition 3.3. An epimorphism $f\colon X \to Y$ is called *inductively closed* if there exists a closed subset A of X such that $f(A) = Y$ and $f|A$ is a closed mapping.

Definition 3.4. A functor $\mathcal{F}_\beta$ is said to be *projectively inductively closed* (p.i.c.) if the mapping $\pi_{\mathcal{F}_\beta,X,k}$ is inductively closed for any Tychonoff space X and positive integer k.

The next theorem gives us sufficient conditions for a functor $\mathcal{F}_\beta$ to be projectively inductively closed (a p.i.c.-functor).

Theorem 3.5. *Every continuous, monomorphic, finitely open functor $\mathcal{F}_\beta\colon Tych \to Tych$, that preserves empty set, intersections, and preimages is a p.i.c.-functor.*

Proof. It is necessary to check, that for any Tychonoff space X and positive integer k, the mapping

$$\pi_{\mathcal{F}_\beta,X,k}\colon X^k \times \mathcal{F}(\{k\}) \to \mathcal{F}_k(X)$$

is inductively closed. We shall prove it by induction on k. If $k = 1$, the mapping $\pi_{\mathcal{F},X,1}$ is inductively closed, since it is a homeomorphism by Lemma 3.2.

Assume that our assertion is proved for all integers $k \le l$. Let us prove it for $k = l + 1$. Fix some point $x_0 \in X^l$. Consider the embedding $i\colon X^l \to X^{l+1}$ defined as

$$i(x_1, \dots, x_l) = (x_1, \dots, x_l x_0).$$

Define a mapping $j\colon \mathcal{F}(\{l\}) \to \mathcal{F}(\{l + 1\})$ by the equality $j(a) = \mathcal{F}(h)(a)$, where $h\colon \{l\} \to \{l + 1\}$ is an identical embedding, i.e., $h(m) = m$ for any

$m \le l-1$. Since $\mathcal{F}$ is a monomorphic functor, the mapping j is an embedding. Hence, we defined the embedding

$$e = i \times j \colon X^l \times \mathcal{F}(\{l\}) \to X^{l+1} \times \mathcal{F}(\{l+1\}).$$

It follows from definitions that

$$\pi_{\mathcal{F}_\beta,X,l+1} \circ e = \pi_{\mathcal{F}_\beta,X,l}. \tag{3.1}$$

From property (3.1) we get, that on the set $e(X^l \times \mathcal{F}(\{l\}))$ the next equality holds:

$$\pi_{\mathcal{F},X,l+1} = \pi_{\mathcal{F}_\beta,X,l} \circ e^{-1}. \tag{3.2}$$

Since the mapping $\pi_{\mathcal{F},X,l}$ is inductively closed by an inductive assumption, there exists a closed subset A of $X^l \times \mathcal{F}(\{l\})$ such that $\pi_{\mathcal{F},X,l}(A) = \mathcal{F}_l(X)$, and the mapping $\pi_{\mathcal{F},X,l}|A$ is closed. Since the mapping e^{-1} is a homeomorphism on the set $e(A)$, equality (3.2) and Corollary 2.35 imply that

$$\pi_{\mathcal{F},X,l+1}|A \text{ is a closed mapping.} \tag{3.3}$$

Moreover, it is clear, that

$$\pi_{\mathcal{F},X,l+1}(A) = \mathcal{F}_l(X). \tag{3.4}$$

Now we put

$$\Phi = \mathcal{F}(\{l+1\}) \backslash \mathcal{F}_l(\{l+1\}).$$

The set Φ is compact, because the functor $\mathcal{F}$ is finitely open. Now we define the sets

$$Z_0 = X^{l+1} \times \Phi \tag{3.5}$$

and

$$Z_1 = (\beta X)^{l+1} \times \Phi. \tag{3.6}$$

By $f_i,\ i < 2$, we denote restrictions of the mapping $\pi_{\beta X,l+1}$ to the sets Z_i. Let us show that

$$Z_0 = f_1^{-1}(f_1(Z_0)). \tag{3.7}$$

To verify this equality, we remark that the functor $\mathcal{F}_\beta$ preserves monomorphisms, intersections, and preimages. Hence, it preserves supports (look at (2.2)). Therefore,

$$\operatorname{supp}(\pi_{l+1}(\xi, a)) = \xi(\operatorname{supp}(a)) \tag{3.8}$$

for any $\xi \in C(\{l+1\}, \beta X)$ and $a \in \mathcal{F}(\{l+1\})$. But if $(\xi, a) \in Z_1$, then $\operatorname{supp}(a) = \{l+1\}$. Consequently,

$$\operatorname{supp}(f_1(\xi, a)) = \xi(\{l+1\}). \tag{3.9}$$

Hence,

$$Z_0 = \{(\xi, a) \in Z_1 :\ \xi(\{l+1\}) \subset X\}. \tag{3.10}$$

Thus, if $f_1(\xi_0, a_0) = f_1(\xi_1, a_1)$ and $(\xi_0, a_0) \in Z_0$, then $(\xi_1, a_1) \in Z_0$. Hence, equality (3.7) is verified.

Compactness of the set Z_1 and the equality (3.7) imply that the mapping $f_0\colon Z_0 \to f_0(Z_0)$ is closed.

Now we shall verify that

$$f_0(Z_0) = f_1(Z_1) \cap \mathcal{F}_{l+1}(X). \tag{3.11}$$

It is sufficient to check the inclusion $\supset$. Let $f_1(\xi, a) \in f_1(Z_1) \cap \mathcal{F}_{l+1}(X)$. Then $X \supset \operatorname{supp}(f_1(\xi, a)) = \xi(\{l+1\})$ by (3.9). Consequently, $(\xi, a) \in Z_0$ according to (3.10). Thus, the equality (3.11) is checked.

This equality and compactness of $f_1(Z_1)$ imply that $f_0(Z_0)$ is closed in $\mathcal{F}_{l+1}(X)$. Hence, the mapping $f_0\colon Z_0 \to \mathcal{F}_{l+1}(X)$ is closed. Let $B = A_0 \cup Z_0$. Then the mapping

$$\pi_{\mathcal{F}_\beta, X, l+1}|B\colon B \to \mathcal{F}_{l+1}(X)$$

is closed as a union of closed mappings on two closed subsets A and Z_0.

To complete the proof, it suffices to check that

$$f_0(Z_0) \supset \mathcal{F}_{l+1}(X) \backslash \mathcal{F}_l(X). \tag{3.12}$$

But this inclusion is a corollary of (3.11) and an evident inclusion

$$f_1(Z_1) \supset \mathcal{F}_{l+1}(X) \backslash \mathcal{F}_l(X).$$

The proof is complete. □

Corollary 3.6. *Every finitary normal functor, in particular the functor* $\exp_n$*, is a p.i.c.-functor.*

Theorem 3.7. *Let $\mathcal{F}_\beta$ be a weight preserving p.i.c.-functor of a finite degree m. Then $\mathcal{F}_\beta$ preserves the class of stratifiable spaces and the class of paracompact σ-spaces.*

Proof. First we consider the case of stratifiable spaces. By Theorem 2.24 the space $X^m \times \mathcal{F}_\beta(\{m\})$ is stratifiable as a finite product of stratifiable spaces ($\mathcal{F}_\beta(\{m\})$ is stratifiable being a metrizable compact space, because $\mathcal{F}_\beta$ is a weight preserving functor). Since $\mathcal{F}_\beta$ is a p.i.c.-functor, there exists a closed subset A of $X^m \times \mathcal{F}_\beta(\{m\})$ such that $\pi_{\mathcal{F}_\beta, X, m}(A) = \mathcal{F}_\beta(X)$ and the mapping $\pi_{\mathcal{F}_\beta, X, m}|A \to \mathcal{F}_\beta(X)$ is closed. But every subspace of a stratifiable space is stratifiable in view of Theorem 2.23. Hence, A is a stratifiable space. Then the space $\mathcal{F}_\beta(X)$ is stratifiable like an image of a stratifiable space under closed mapping (look at Theorem 2.25). □

The proof of the assertion for paracompact σ-spaces repeats the previous proof. The necessary changings are the following: instead of Theorems 2.24, 2.23, and 2.25, we use Theorem 2.6, Remark 2.3, and Theorem 2.5 respectively.

By the same procedure we get

Theorem 3.8. *Let $\mathcal{F}_\beta$ be a p.i.c.-functor of a finite degree. Then $\mathcal{F}_\beta$ preserves the class of paracompact Σ-spaces and the class of paracompact p-spaces.*

Proof of this theorem repeats the proof of Theorem 3.7 for stratifiable spaces. The necessary changings are: 1) we don't need that $\mathcal{F}_\beta(m)$ is metrizable; 2) instead of Theorems 2.24, 2.23, and 2.25, we use Propositions 2.10, 2.9, and 2.11 in the case of paracompact Σ-spaces; 3) in the case of paracompact p-spaces we use respectively Corollary 2.15, Proposition 2.16, and Theorem 2.17.

Corollary 3.6, Theorems 3.7, and 3.8 yield

Corollary 3.9. *Every normal finitary functor of a finite degree, in particular the functor* $\exp_m$*, preserves the class of stratifiable spaces, the class of paracompact* σ*-spaces, and the class of paracompact* Σ*-spaces.*

Remark 3.10. As for paracompact p-spaces, they are preserved by any normal functor $\mathcal{F}_\beta$ (look at [1]).

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