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Algebraic and topological structures on rational tangles

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Abstract

In this paper we present the construction of a group Hopf algebra on the class of rational tangles. A locally finite partial order on this class is introduced and a topology is generated. An interval coalgebra structure associated with the locally finite partial order is specified. Irrational and real tangles are introduced and their relation with rational tangles are studied. The existence of the maximal real tangle is described in detail.

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1. Introduction

Rational tangles are not only beautiful mathematical objects but also have many applications in other fields such as biology and DNA synthesis [5]. The theory of tangles was invented in 1986 by Conway in his work [2]. He introduced the notion of rational tangles and with each rational tangle he associated a rational number by the continued fraction method. The associated rational

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number is based on the pattern of tangle twists. According to Conway's theorem [2, 3], two rational tangles are equivalent if and only if they represent the same rational number. Kauffman and Lambropoulou gave another proof of this theorem in [9]. The key value of tangles is in their hidden algebraic structures. The more we discover these structures, the better we can handle them mathematically and apply in life science.

Towards the better understanding of tangles as mathematical objects in this paper we concern ourselves to the algebraic structures of tangles.
The framework of this paper is as follows:

Section 2 is devoted to a review of the concept of rational tangles from [6, 7, 8, 9]. In this section we discuss basic definitions and explain the Conway's continued fraction method for assigning a rational number to a rational tangle. We also recall tangle operations. As we will see the class of rational tangles is not invariant under these operations. We modify this in the next section. In section 3 we are concerned in algebraic structures on rational tangles. We introduce a free product operation on rational tangles and show that the class of rational tangles is closed under this operation. The group Hopf algebra structure comes up in the next step. We present the group Hopf algebra structure of rational tangles explicitly. In section 4 we go towards a topological point of view in tangle study. We introduce a locally partial order to generate a topology on the class of rational tangles. There are tools to study the topology of partially ordered sets and we apply them [14]. The interval coalgebra structure is presented and its relation to the incidence algebra associated with the partial order is given. In section 5 we discuss the notion of irrational and real tangles and their infinite continued fractions. We see that any infinite chain of rational tangles has an upper bound in the set of real tangles and the existence of a maximal real tangle is explained.

2. Rational Tangles

In this section we present a review of the concept of rational tangles from [6, 7, 8, 9]. Basic definitions and the Conway's continued fraction method for assigning a rational number to a rational tangle is described in detail. Some tangle operations are also recalled.

Definition 2.1. Tangles consist of strings embedded in a 3-dimensional ball. Within the ball there are no free ends, the ends of the strings are restricted to move on the surface of the ball while the rest of the tangle remains inside the ball. Since a string has two ends, there is an even number of ends on the surface of the ball. Inside the ball there may exist closed loops that are linked with the tangle strings and the strings of the tangle may themselves be knotted and linked.

All throughout this paper we consider 2-tangles, i.e. 2-string tangles with four ends. We simply refer to them as tangles.

Definition 2.2. For $n \in \mathbb{Z}$, the horizontal tangles denoted by $[n]$, are made of n half twists of two horizontal strings and the vertical tangles denoted by

$\frac{1}{[n]}$ are made of n half twists of two vertical strings while the end points of the strings remain on the boundary of the ball. The directions of the half twists are specified by the $\pm$ signs, Fig. 1.

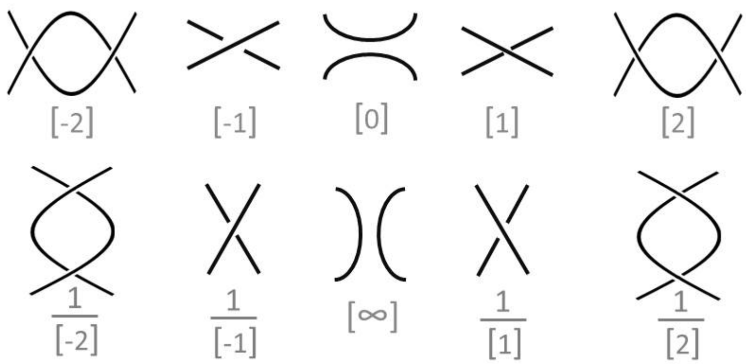

Figure 1- Horizontal and vertical tangles

Definition 2.3. The horizontal and vertical sums of two tangles T_1 and T_2 are denoted respectively by $T_1 + T_2$ and $T_1 \star T_2$. They are obtained by connecting the endpoints of T_1 to the endpoints of T_2 as are shown in Fig. 2.

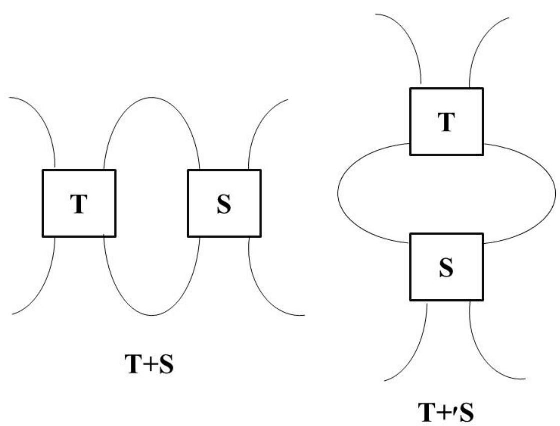

Figure 2- Horizontal and vertical sums

Rational tangles belong to a subclass of tangles introduced by Conway [2]. They are obtained from finite sums of horizontal and vertical tangles.

Definition 2.4. A tangle is rational if it can be obtained from the zero tangle [0] or the infinity tangle $[\infty]$ by twisting the strings while the endpoints of the strings remain moving on the surface of the ball.

Definition 2.5. Two rational tangles T_1 and T_2 are isotopic if T_2 is obtained from T_1 by moving the strings of T_1 continuously in such a way that no string penetrates either itself or another string. Isotopy of rational tangles is an equivalence relation and we use the notation $T_1 \sim T_2$ for equivalent rational tangles T_1 and T_2.

The set of rational tangles is not invariant under the horizontal and vertical sums, as we will see later. In order to modify this we apply the continued fractions representation of Conway [2] to encode the twists of rational tangles. In the following we introduce this representation in detail.

Definition 2.6. A rational tangle is said to be in standard form if it is created by consecutive horizontal sums of the tangles $[\pm 1]$ only on the right (left) and vertical sums of the tangles $[\pm 1]$ only at the bottom (top) starting from [0] or $[\infty]$.

So a rational tangle in standard form has an algebraic expression of the type:

$$[a_n] \star \frac{1}{a_{n-1}} + [a_{n-2}] \star \ldots \star \frac{1}{a_2} + [a_1].$$

For $a_2, a_3, \ldots, a_{n-1} \in \mathbb{Z} - 0$. We let a_1, a_n be [0] or $[\infty]$.

Remark 2.7. In the process of creating a rational tangle, we may start with horizontal or vertical tangles. We always assume that we start twisting from [0] tangle.

The shape of a rational tangle in standard form is encoded by associating to it a vector of integers $(a_1, a_2, \ldots, a_n)$, where the i-th term represents $|a_i|$ half twists in the directions as in Fig. 1. When i is odd it is horizontal and when i is even it is vertical. For a rational tangle this vector is unique, up to breaking the last term as

$$(a_1, a_2, \ldots, a_n) = (a_1, a_2, \ldots, a_n - 1, 1); a_n > 0$$
$$(a_1, a_2, \ldots, a_n) = (a_1, a_2, \ldots, a_n + 1, -1); a_n < 0.$$

For this reason the index n can be taken to be odd.

Remark 2.8. According to [9] every rational tangle is isotopic to a rational tangle in the standard form $T = (a_1, a_2, \ldots, a_n)$, where n is odd and all terms are positive or negative (except possibly the first term). Furthermore the standard form for a rational tangle is unique.

From now on all rational tangles are considered to be in standard form. John Conway [2, 3] associated a finite continued fraction to each rational tangle in the following way:

Definition 2.9. Let $T = (a_1, a_2, \ldots, a_n)$ be a rational tangle. Let

$$[a_1, a_2, \ldots, a_n] := a_1 + \frac{1}{a_2 + \ldots + \frac{1}{a_{n-1} + \frac{1}{a_n}}}$$

be the continued fraction associated with T. The sum of the continues fraction is called the fraction of the tangle.

For more details on continued fractions we refer to [4, 10].

Proposition 2.10. *Two rational tangles are isotopic if and only if they have the same fractions.*

For the proof see [2].

3. Algebraic structures on rational tangles

In this section our concern is in algebraic structures on rational tangles. We introduce a free product operation on rational tangles and show that the class of rational tangles is closed under this operation. We present the Hopf algebraic structure of rational tangles. Details on Hopf algebra structure can be found in [1, 11, 12].

3.1. **Free product of rational tangles.** As we mentioned before the horizontal and vertical sums of two rational tangles are not necessarily a rational tangle. For example the horizontal sum of two rational tangles $T_1 = \frac{1}{[3]}, T_2 = \frac{1}{[2]}$ is not a rational tangle. In this section we define the free product of rational tangles and see that the class of rational tangles is closed under free product.

Definition 3.1. Let $T_1 = (a_1, a_2, \ldots, a_n)$, $T_2 = (b_1, b_2, \ldots, b_m)$ be two rational tangles in the standard form. Their free product which is denoted by $T_1 \circledast T_2$ is defined to be the standard form of the tangle $(a_1, a_2, \ldots, a_n + b_1, b_2, \ldots, b_m)$. We write

$$T_1 \circledast T_2 = standard(a_1, a_2, \ldots, a_n + b_1, b_2, \ldots, b_m).$$

Proposition 3.2. *The class of rational tangles is a non commutative group under the free product. This group is denoted by $\mathcal{RT}$.*

Proof. Obviously the class of rational tangles are closed under the free product. Moreover the free product is associative and the tangle [0] is the identity element.

Now to any rational tangle $T = (a_1, a_2, \ldots, a_n)$ we associate, as its inverse, the rational tangle $T^{-1} = (-a_n, -a_{n-1}, \ldots, -a_1)$. Obviously $T \circledast T^{-1} = T^{-1} \circledast T = [0]$. So the class of rational tangles is a group under the free product.

The free product is not commutative (up to isotopy). As we see for $T_1 = (1, 1, 2)$ and $T_2 = (1, 1, 3)$, we have

$$T_1 \circledast T_2 \nsim T_2 \circledast T_1$$

since they have different fractions and by the Conway's theorem they are not isotopic. □

Now we try to continue with more algebraic structures on rational tangles.

3.2. **Group Hopf Algebra Structure.**

Definition 3.3. Let G be a noncommutative group and $(\mathcal{A}, +, \bullet)$ be a (commutative or noncommutative) ring with the multiplicative unit element 1. Let $\Psi : \mathcal{A} \times G \to G$ be a mapping. The triple $(\mathcal{A}, G, \Psi)$ is called a noncommutative left pseudo-module over $\mathcal{A}$ if

- For all $g \in G : \Psi(1, g) = g$.
- For all $g \in G$ and $a, b \in \mathcal{A} : \Psi(a, \Psi(b, g)) = \Psi(a \bullet b, g)$.

We can define the noncommutative right pseudo-module over $\mathcal{A}$ in the same way.

Definition 3.4. For $T = (a_1, \ldots, a_n) \in \mathcal{RT}$, and $r \in \mathbb{Z}$, we define the left scalar multiplication by

$$rT := (ra_1, \ldots, ra_n).$$

Right scalar multiplication is also defined by $Tr := (a_1r, \ldots, a_nr)$. As we see $rT = Tr$.

Remark 3.5. For $r, s \in \mathbb{Z}$ and for $T \in \mathcal{RT}$ it is not always true that $(r + s)T = rT \circledast sT$. For if we take $r = -s$, then $-T = T^{-1}$. Which is not true since for $T = (a_1, a_2, \ldots, a_n)$,

$$-T = (-a_1, \ldots, -a_n); T^{-1} = (-a_n, \ldots, -a_1).$$

So the above scalar multiplication is not a module action of $\mathbb{Z}$ on $\mathcal{RT}$. But we have

Proposition 3.6. *Let $\Psi : \mathbb{Z} \times \mathcal{RT} \to \mathcal{RT}$ be defined by $\Psi(r, T) = rT$, for all $r \in \mathbb{Z}$ and $T \in \mathcal{RT}$. Then the triple $(\mathbb{Z}, \mathcal{RT}, \Psi)$ is a noncommutative left pseudo-module over $\mathbb{Z}$.*

Proof. Obviously $\psi(1, T) = T$, for all $T \in \mathcal{RT}$. Also for $r, s \in \mathbb{Z}$ and $T = (a_1, a_2, \ldots, a_n) \in \mathcal{RT}$, we have

$$\Psi(r, \Psi(s, T)) = r(sT) = (r(sa_1), \ldots, r(sa_n))$$
$$= ((rs)a_1, \ldots, (rs)a_n) = (rs)(a_1, \ldots, a_n) = \Psi(rs, T).$$

□

Remark 3.7. We can also make RT into a noncommutative right pseudo-module by defining

$$\Phi : \mathcal{RT} \times \mathbb{Z} \to \mathcal{RT}; \Phi(T, r) = Tr$$

for all $r \in \mathbb{Z}$ and $T \in \mathcal{RT}$. Also for all $r, s \in \mathbb{Z}$ and all $T \in \mathcal{RT}$, we have $(rT)s = r(Ts)$. We call $\mathcal{RT}$ a noncommutative bi-pseudo-module over $\mathbb{Z}$.

Definition 3.8. The pseudo-tensor product of two pseudo bi-modules $\mathcal{M}$ and $\mathcal{N}$ over the ring of integers $\mathbb{Z}$ is the quotient of the free abelian group with basis the symbols $m \otimes n$; for $m \in M$ and $n \in N$; by the subgroup generated by

$$-(m_1 + m_2) \otimes n + m_1 \otimes n + m_2 \otimes n$$
$$-m \otimes (n_1 + n_2) + m \otimes n_1 + m \otimes n_2$$

$$(m, r) \otimes n - m \otimes (r, n)$$

where $m, m_1, m_2 \in \mathcal{M}, n, n_1, n_2 \in \mathcal{N}, r \in \mathbb{Z}$.

Remark 3.9. We can construct the pseudo-tensor product $\mathcal{RT} \otimes \mathcal{RT}$ over the ring of integers $\mathbb{Z}$.

We define the following mappings:

$$\begin{aligned} \Delta :&\mathcal{RT} \longrightarrow \mathcal{RT} \otimes \mathcal{RT}; \Delta(T) = T \otimes T \\ \Sigma :&\mathcal{RT} \longrightarrow \mathbb{Z}; \Sigma(T) = 1 \\ \mathcal{S} :&\mathcal{RT} \longrightarrow \mathcal{RT}; \mathcal{S}(T) = T^{-1} \\ \mu :&\mathcal{RT} \times \mathcal{RT} \longrightarrow \mathcal{RT}; \mu(T, S) = T \otimes S \end{aligned}$$

Proposition 3.10. *The group of rational tangles $\mathcal{RT}$ together with the co-multiplication Δ, counit Σ and antipodal $\mathcal{S}$ as defined above is a group Hopf algebra.*

Proof. For all $T \in \mathcal{RT}$, we have

$$(id \otimes \Delta) \circ \Delta(T) = T \otimes (T \otimes T) = (T \otimes T) \otimes T = (\Delta \otimes id) \circ \Delta(T)$$

$$(id \otimes \Sigma) \circ \Delta(T) = T \otimes 1 \sim 1 \otimes T = (\Sigma \otimes id) \circ \Delta(T)$$

$$\mu \circ (\mathcal{S} \otimes id) \circ \Delta(T) = \mathcal{S}(T) \circledast T = T^{-1} \circledast T = [0] = \Sigma(T) 1_{\mathcal{RT}}$$

$$\mu \circ (id \otimes \mathcal{S}) \circ \Delta(T) = T \circledast \mathcal{S}(T) = T \circledast T^{-1} = [0] = \Sigma(T) 1_{\mathcal{RT}}$$

Where $1_{\mathcal{RT}} = [0]$ is the identity element of the group $\mathcal{RT}$.

So the axioms of a group Hopf algebra satisfies. □

4. Topology generated by partial order on rational tangles

In this section we go through topological point of view in tangle study. We introduce a locally finite partial order generating a topology on the class of rational tangles. The interval coalgebra structure is presented and its relation to the incidence algebra associated with the partial order is given. More details on the coalgebra structure can be found in [1, 11, 12].

Definition 4.1. Let $T = (a_1, a_2, \ldots, a_n)$ be a rational tangle. For $1 \le k \le n$, the rational tangle $C_k = (a_1, a_2, \ldots, a_k)$ is called the k-th convergent of T.

Remark 4.2. The following relations are true for the convergents [4, 10]:

$$a_2, \ldots, a_n > 0 \Longrightarrow C_1 < C_3 < C_5 < \ldots < C_6 < C_4 < C_2$$

$$a_2, \ldots, a_n < 0 \Longrightarrow C_1 > C_3 > C_5 > \ldots > C_6 > C_4 > C_2$$

Definition 4.3. Let $T_1 = (a_1, a_2, \ldots, a_n), T_2 = (b_1, b_2, \ldots, b_m)$ be two rational tangles. We write $T_1 \le T_2$ if $n \le m$ and T_1 is the n-th convergent of T_2.

Proposition 4.4. *The relation $\le$ on the class of rational tangles is a locally finite partial order.*

Proof. Since each rational tangle is the convergent of itself, so the relation is reflexive. Now let $T_1 = (a_1, a_2, \ldots, a_n), T_2 = (b_1, b_2, \ldots, b_m)$ be two rational tangles with $T_1 \leq T_2, T_2 \leq T_1$. Then $n \leq m, m \leq n$ and both T_1 and T_2 are convergents of each other. So $T_1 \sim T_2$.

Now let $T_1 = (a_1, a_2, \ldots, a_n), T_2 = (b_1, b_2, \ldots, b_m), T_3 = (c_1, c_2, \ldots, c_l)$ be rational tangles with $T_1 \leq T_2, T_2 \leq T_3$. Then $n \leq m, m \leq l$. Also T_1 is the n-th convergent of T_2 and T_2 is the m-th convergent of T_3. So $n = l$ and T_1 is the n-th convergent of T_3.

Now for each pair of rational tangles $T_1 \leq T_2$ we define the interval $[T_1, T_2]$ by

$$[T_1, T_2] = \{T : T \in \mathcal{RT}, T_1 \leq T \leq T_2\}.$$

Then $[T_1, T_2]$ has a finite number of elements (up to isotopy) and so the partial order is locally finite. □

For each rational tangle $T \in \mathcal{RT}$, we define $\Lambda(T) = \{S : S \in \mathcal{RT}, S \leq T\}$.

Proposition 4.5. *The family* $\mathcal{B} = \{\Lambda(T)\}_{T \in \mathcal{RT}}$ *form a basis for a topology on* $\mathcal{RT}$.

Proof. Since for $T \in \mathcal{RT}, T \leq T$, we have $T \in \Lambda(T)$. Also for the rational tangles $T_1, T_2, T \in \mathcal{RT}$, if

$T \in (T_1) \cap \Lambda(T_2)$, then

$$T \in \Lambda(T_1) \Longrightarrow T \leq T_1 \Longrightarrow \Lambda(T) \subseteq \Lambda(T_1)$$

$$T \in \Lambda(T_2) \Longrightarrow T \leq T_2 \Longrightarrow \Lambda(T) \subseteq \Lambda(T_2)$$

So $\Lambda(T) \subseteq \Lambda(T_1) \cap \Lambda(T_2)$. So the family $\mathcal{B}$ satisfies the axioms of a basis and generates a topology on $\mathcal{RT}$. □

Definition 4.6. A subset $U \in \mathcal{RT}$ is open if it is a union of the sets $\Lambda(T_i) \in \mathcal{B}$, where the index i belongs to some index set.

So far we have seen that $\mathcal{RT}$ is a locally finite partial order set (a locally finite poset). The incidence algebra associated with this poset is defined by the following procedure:

Let $\mathcal{I} = \{[T, S] : T, S \in \mathcal{RT}, T \leq S\}$. Let $C(\mathcal{I})$ be the set of all functions from $\mathcal{I}$ to $\mathbb{Z}$. Consider the convolution

$$f \star g([T, S]) = \sum_{T \leq X \leq S} f([T, X]) g([X, S])$$

$C(\mathcal{I})$ is called the incidence algebra associated with the partial order. Details on incidence algebra can be found in [13]. For all $f, g \in C(\mathcal{I})$ we define

$$\Delta : \mathcal{I} \longrightarrow \mathcal{I} \otimes \mathcal{I}; \Delta([T, S]) = \sum_{T \leq X \leq S} [T, X] \otimes [X, S]$$

$$\Sigma : \mathcal{I} \longrightarrow \mathbb{Z}; \Sigma([T, S]) = 0.$$

Proposition 4.7. *The set* $\mathcal{I}$ *together with the comultiplication* Λ *and counit* Σ *is a coalgebra. It is called the interval coalgebra.*

Proof. We check the axioms of the coalgebra. Obviously for all $[T, S] \in \mathcal{I}$,

$$(id \otimes \Delta) \circ \Delta([T, S]) = (\Delta \otimes id) \circ \Delta([T, S])$$
$$(id \otimes \Sigma) \circ \Delta([T, S]) = (\Sigma \otimes id) \circ \Delta([T, S])$$

□

Remark 4.8. The incidence algebra convolution is related to the comultiplication Δ by

$$f \star g([T, S]) = \mu_{\mathbb{Z}} \circ (f \otimes g) \circ \Delta([T, S])$$

where $\mu_{\mathbb{Z}}$ is the multiplication in $\mathbb{Z}$.

5. Irrational tangles

To each infinite standard continued fraction, there corresponds a 2-tangle. They are called irrational tangles. In this section we study their properties and their relation to rational tangles.

Definition 5.1. Let $\{a_n\}_{n\geq 1}$ be a sequence of integers all positive except the first term. The number

$$(a_1, a_2, \ldots) = \lim_{n\to\infty} (a_1, \ldots, a_n)$$

is called an infinite standard continued fraction.

Proposition 5.2 ([9])**.** *The above limit exists and is finite. Furthermore to each irrational number, there corresponds a unique infinite standard continued fraction.*

Definition 5.3. To each infinite standard continued fraction $(a_1, a_2, \ldots)$ there corresponds an irrational tangle obtained from consecutive a_i half twists, starting with a horizontal $|a_1|$ half twist in direction according to the sign of a_1. The number $C_n = (a_1, \ldots, a_n)$ is called the n-th convergent of the irrational tangle. The set of all rational and irrational tangles is called real tangles.

Definition 5.4. For two irrational tangles T and S we write $T \leq S$ if every convergent of T is a convergent of S.

Definition 5.5. Two irrational tangles $T = (a_1, a_2, \ldots)$ and $S = (b_1, b_2, \ldots)$ are said to be equivalent if

$$\lim_{n\to\infty} (a_1, \ldots, a_n) = \lim_{m\to\infty} (b_1, \ldots, b_m).$$

In this case we write $T \sim S$.

Proposition 5.6. *For any two irrational tangles T and S we have*

$$T \leq S \Longleftrightarrow T \sim S \Longleftrightarrow S \leq T.$$

Moreover for any rational tangle $T = (a_1, \ldots, a_n)$ there exists an irrational tangle S with $T \leq S$.

Proof. The first part is obvious from the definition. For the second part let $S = \overline{(a_1, a_2, \ldots, a_n)}$, where the bar sign is for the repetition of the sequence. □

From the above proposition we have

Proposition 5.7. *Any infinite chain $T_1 \leq T_2 \leq T_3 \leq \ldots$ of rational tangles has an upper bound in the set of real tangles.*

Proof. There is a sequence $\{n_i\}_{i\geq 1}$ of positive integers and a sequence $\{a_n\}_{n\geq 1}$ of integers all positive except the first term such that,

$$T_1 = (a_1, \ldots, a_{n_1}), T_2 = (a_1, \ldots, a_{n_1}, a_{n_1+1}, \ldots, a_{n_2})$$

$$, \ldots, T_k = (a_1, a_2, \ldots, a_{n_1}, a_{n_1+1}, \ldots, a_{n_2}, \ldots, a_{n_k}), \ldots$$

for all $k \geq 1$. Since all a_i s (except possibly a_1) are positive, then the irrational tangle

$$S = (a_1, a_2, \ldots) = \lim_{k\to\infty} (a_1, \ldots, a_{n_k})$$

satisfies $T_i \leq S$, for all $i \geq 1$. □

Corollary 5.8. *Any chain $T_1 \leq T_2 \leq T_3 \leq \ldots.$ of real tangles has an upper bound in the set of real tangles.*

Proof. If all T_i s are rational, then from the above proposition, the upper bound exists. Now if T_i is irrational for some $i \geq 1$, then for all $j \geq i$ we have $T_i \sim T_j$. So the irrational tangle T_i is the upper bound. □

Corollary 5.9. *There exists a maximal real tangle.*

Proof. By the Zorn's lemma and the above discussion, the maximal real tangle exists. □

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