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Attractors of reaction-diffusion equations in Banach spaces

José Valero

Abstract. In this paper we prove first some abstract theorems on existence of global attractors for differential inclusions generated by ω-dissipative operators. Then these results are applied to reaction-diffusion equations in which the Banach space L_p is used as phase space. Finally, new results concerning the fractal dimension of the global attractor in the space L_2 are obtained.

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1. Introduction

The theory of global attractors for partial differential equations have been applied to a wide range of equations of parabolic and reaction-diffusion types. It is very important in this theory to choose an appropriate phase space for the system. The existence, uniqueness and properties of solutions of partial differential equations depend strongly on the phase space.

The more common phase space used for reaction-diffusion equations is the space $L_2(\Omega)$ (see [3], [4], [5], [6], [7], [13], [17], [18], [21], [25], [26], [28], [29], [31], [32], [33]).

However, other spaces like $L_\infty(\Omega)$ [1], $H^\gamma(\Omega)$, $0 < \gamma \leq 1$ (see [2], [11], [12], [20] and their bibliography) or $C(\Omega)$ (see [15]-[16], [27]) have been also used succesfully.

As far as we know, the space $L_p(\Omega)$, where $2 < p < \infty$, has not been considered so far.

In this paper we shall consider the existence of global attractors for differential inclusions of the type

$$\begin{cases} x'(t) \in A(x(t)), \\ u(0) = u_0, \end{cases}$$

where A is a multivalued ω-dissipative operator.

This general framework allows us to apply the obtained results in order to prove the existence of a global attractor for the following reaction-diffusion equation

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u + f(u) \ni \omega u + h, \text{ on }]0,T[\times\Omega, \\ u = 0, \text{ on }]0,T[\times\Gamma, \\ u(0,x) = u_0(x) \in L_p(\Omega), \end{cases}$$

where $f : \mathbb{R} \to 2^{\mathbb{R}}$ is a multivalued maximal monotone map and $2 \leq p < \infty$. Therefore, we extend the results of attractors for reaction-diffusion equations to the case where the phase space $L_p(\Omega)$ is used. Moreover, the nonlinear function f, which is usually a continuous one, can be in our case multivalued. Hence, our results are valid also for differential equations with discontinuous nonlinearities. We obtain also a result (see Theorem 4.4), which is new in the case where $f \in C^1$ and $p = 2$.

In the case $p = 2$ we obtain some estimates of the fractal dimension of the global attractor. Such estimates are well known under different conditions on the function f (see [3]-[5], [7], [18], [25]). In all these papers the function f is at least Lipschitz on any bounded set of $\mathbb{R}$. We shall extend these result by considering a function $f(s)$ which is Lipschitz on a fixed bounded set $[-a,a]$ but can be even discontinuous for $s \notin [-a,a]$.

We shall recall now some definitions of the theory of dynamical systems (see [20], [22]-[23], [29] for more details). Let Y be a complete metric space with the metric denoted by $\rho(\cdot,\cdot)$, $V : \mathbb{R}_+ \times Y \to Y$ be a semigroup of operators, i.e., the following properties hold

$$V(t_1, V(t_2,x)) = V(t_1+t_2,x), \text{ for all } t_1,t_2 \in \mathbb{R}_+, x \in Y,$$
$$V(0,\cdot) = I.$$

The map $x \longmapsto V(t,x)$ is supposed to be continuous for each $t \in \mathbb{R}_+$.

Let us introduce the next notation:

$\mathcal{B}(Y)$ is the set of all bounded subsets of Y;

$\gamma_\tau^+(x) = \underset{t\geq\tau}{\cup} V(t,x)$, $\gamma_\tau^+(A) = \underset{x\in A}{\cup} \gamma_\tau^+(x)$, where $t,\tau \in \mathbb{R}_+$, $x \in Y$, $A \in \mathcal{B}(Y)$;

$\omega(B) = \underset{t\geq 0}{\cap} \overline{\gamma_t^+(B)}$ is the ω-limit set.

We denote by Z the set of stationary points of V, i.e.,

$$Z = \{u \in Y \mid V(t,u) = u, \quad \text{for all } t \in \mathbb{R}_+\}.$$

The continuous function $L : Y \to \mathbb{R}$ is called a Lyapunov function on Y for V if $L(V(t,x)) < L(x)$ for any $t > 0$, $x \in Y$, $x \notin Z$.

Let us recall the concept of distance between sets. Let $A, B \subset Y$. Then the distance from A to B is determined as follows:

$$d(A,B) = \sup_{x\in A} \left\{ \inf_{y\in B} \rho(x,y) \right\}.$$

Definition 1.1. *The compact set $\Re \subset Y$ is called a global attractor of the semigroup V if the following conditions hold:*

(1) *It attracts each set* $B \in \mathcal{B}(Y)$, *that is,*

$$d(V(t,B),\Re) \to 0, \text{ as } t \to +\infty.$$

(2) *It is invariant, i.e.,* $V(t,\Re) = \Re$, *for all* $t \geq 0$.

Definition 1.2. *The semigroup* V *is called "pointwise dissipative" if there exists a bounded set* B_1 *which attracts each* $x \in Y$.

Remark 1.3. We note that if for any $B \in \mathcal{B}(Y)$ there exists some $T = T(B) \in \mathbb{R}_+$ for which $\gamma_T^+(B) \in \mathcal{B}(Y)$, then pointwise dissipativeness is equivalent to the existence of a bounded set B_0 such that for each $x \in Y$ there exists $T(x)$ for which $V(T,x) \in B_0$. In such a case we can take $B_1 = \gamma^+_{T(\mathrm{B}_0)}(B_0)$.

Definition 1.4. *The semigroup* V *is said to be time-continuous if the map* $t \mapsto V(t,x)$ *is continuous for each* $x \in Y$.

Theorem 1.5. *(*[23, p.107-109] *and* [19, p.4-5]*)Let* $V(t,\cdot)$ *be compact for some* $t > 0$ *and let* V *be pointwise dissipative. Suppose that for any* $B \in \mathcal{B}(Y)$ *there exists* $T = T(B) \in \mathbb{R}_+$ *such that* $\gamma_T^+(B) \in \mathcal{B}(Y)$. *Then* V *has the global attractor* $\Re$. *If* V *is time-continuous and* Y *is connected, then* $\Re$ *is connected.*

This paper is organized as follows. In the second section we prove the existence of attractors for abstract differential inclusions in Banach spaces generated by ω-dissipative operators. In the third section we apply the abstract results of the preceding section to differential inclusions generated by subdifferential maps in Hilbert spaces. In the fourth section we prove the existence of global attractors for reaction-diffusion equations in $L_p(\Omega)$ spaces. Finally, in the fifth section we obtain estimates of the fractal dimension of the global attractor of reaction-diffusion equations in the Hilbert space $L_2(\Omega)$.

2. Existence of attractors of differential inclusions generated by ω-dissipative operators

Let X be a real Banach space with its dual denoted by X^*. We shall denote by $\|\cdot\|$ and $\|\cdot\|_*$ the norms in X and X^* respectively. $\langle \cdot,\cdot \rangle$ will denote pairing between the spaces X and X^*. Consider the next differential inclusion

$$\left\{ \begin{array}{l} \dfrac{du(t)}{dt} \in A(u(t)), \ 0 \leq t < \infty, \\ \qquad u(0) = u_0 \in X, \end{array} \right. \tag{2.1}$$

where $A : D(A) \subset X \to 2^X$ is a multivalued nonlinear operator and $u : \mathbb{R}_+ \to X$.

We remind that the dual operator $J : X \to 2^{X^*}$ (may be multivalued) is defined as follows:

$$J(x) = \left\{ j \in X^* \mid \|x\|^2 = \|j\|_*^2 = \langle x, j \rangle \right\}.$$

Consider the next conditions:

(A1) The operator A is ω-dissipative, i.e., for all $x_1, x_2 \in D(A)$, $y_1 \in A(x_1)$, $y_2 \in A(x_2)$, there exists $j(x_i, y_i) \in J(x_1 - x_2)$ such that

$$\langle y_1 - y_2,\, j\rangle \leq \omega \left\| x_1 - x_2 \right\|^2,$$

where $\omega \geq 0$.

When $\omega = 0$ the operator A is called dissipative. It is easy to check that condition (A1) is equivalent to the dissipativeness of the operator $A - \omega I$.

(A2) $\overline{D(A)} \subset \bigcap_{0<\lambda<\lambda_0} Im\,(I - \lambda A)$, where $\lambda_0 > 0$, $\lambda_0 \omega < 1$.

Notice that $\overline{D(A)}$ is a complete metric space endowed with the usual metric $\rho(x, y) = \|x - y\|$, for all $x, y \in \overline{D(A)}$.

If conditions A1 and A2 are satisfied, then there exists a semigroup of operators $V : \mathbb{R}_+ \times \overline{D(A)} \to \overline{D(A)}$ corresponding to the operator A (see [8, p.108] or [14, p.63]) such that V is determined by the next formula

$$V(t, x) = \lim_{n\to\infty} \left(I - \frac{t}{n} A \right)^{-n} x,\ x \in \overline{D(A)},\, t \in \mathbb{R}_+. \tag{2.2}$$

Moreover, for any fixed $t \in \mathbb{R}_+$ one has

$$\|V(t, x) - V(t, y)\| \leq \exp(\omega t)\, \|x - y\|, \text{ for all } x, y \in \overline{D(A)}. \tag{2.3}$$

It follows from inequality (2.3) that the map $x \longmapsto V(t, x)$ is continuous for each $t \in \mathbb{R}_+$.

A map $u : [0, T] \to X$ is called a strong solution of inclusion (2.1) on $[0, T]$ if:

(1) u is continuous on $[0, T]$ and $u(0) = u_0$;
(2) u is absolutely continuous on any compact subset of $(0, T)$;
(3) u is almost everywhere (in short, a.e.) differentiable on $(0, T)$ and satisfies inclusion (2.1) a.e. on $(0, T)$.

Remark 2.1. If $u(\cdot), u(0) = u_0$, is a strong solution of (2.1) and (A1), (A2) hold, then $u(t) = V(t, u_0)$ [14, Theorem 3.1], where V is the semigroup defined by (2.2).

Remark 2.2. Suppose that the space X is reflexive, (A1), (A2) hold and the operator A is closed. Then the map $u(t) = V(t, u_0)$ is a strong solution of (2.1) for any $T > 0$,$u_0 \in D(A)$ [14, p.77].

We also must consider the next condition:

(A3) $Im\,(I - \lambda A) = X$,for all $\lambda > 0$.

A dissipative operator which satisfies A3 is called m-dissipative. In such a case A2 is always satisfied. Moreover, every m-dissipative operator is closed [8, p.75].

If $A = B + \omega I$ with B m-dissipative, then A is closed and (A1), (A2) are satisfied. Indeed,

$$Im\,(I - \lambda A) = Im\,(I - \lambda B - \lambda\omega I) = Im\,((1 - \lambda\omega)\, I - \lambda B) =$$

$$= Im\left(I - \frac{\lambda}{1-\lambda\omega}B\right) = X,$$

if $1-\lambda\omega > 0$. Hence,

$$\bigcap_{0<\lambda<\lambda_0} Im\,(I-\lambda A) = X, \; if \quad \lambda_0\omega < 1.$$

In that case we have the inclusion:

$$\frac{du\,(t)}{dt} \in B\,(u\,(t)) + \omega\, u\,(t)\,, \; u\,(0) = u_0. \tag{2.4}$$

Remark 2.3. It is easy to prove that if $u\,(t) = V\,(t,x)$ is a strong solution of (2.1) for each $x \in \overline{D\,(A)}$, then the set of stationary points Z can be characterized as follows:

$$Z = \{u \in D\,(A) \mid 0 \in A\,(u)\}\,.$$

It may be proved also that if $A = B + \omega I$, where B is m-dissipative, then this characterization is also true.

First we shall recall the next result, which states the existence of a compact attractor for m-dissipative operators.

Theorem 2.4. [30, p.609] *Let A be an m-dissipative operator. Suppose that the semigroup V generated by A is compact for some $t_0 > 0$ and that Z is nonempty and bounded. Assume also that the next condition is satisfied:*

(2.5) *If $x \in \overline{D\,(A)}$ and $\|V\,(t,x) - v\| = \|x - v\|$ for all $v \in Z$, for all $t \geq 0$, then $x \in Z$.*

Then Z is the global attractor of V. Moreover, the following conditions hold:

1. *Z is connected if $\overline{D\,(A)}$ is connected;*
2. *Every positive trajectory $\{x(t) = V(t,x_0),\ t \in \mathbb{R}_+,\ x_0 \in \overline{D(A)}\}$ converges to some element $z \in Z$, i.e., $\omega(x) = z \in Z$, for all $x \in \overline{D(A)}$.*

Remark 2.5. In fact, m-dissipativeness is too strong. The statement of the theorem remains valid if $(A1) - (A2)$ hold with $\omega = 0$.

Remark 2.6. The proof of Theorem 2.4 is based on the results from [23]. It is remarked in [19] that in the abstract result of that paper on the connectivity of the global attractor is necessary to supppose the semigroup V to be time-continuous. Since the semigroup V is time-continuous, the statement of point (1) remains valid.

To obtain suitable conditions on A and also to study the ω-dissipative case we need strong solutions of (2.1).

Further, we shall prove two results on existence of the global attractor for the semigroup V.

Lemma 2.7. *Suppose that A satisfies $(A1) - (A2)$ and that $u\,(t) = V\,(t,u_0)$ is a strong solution of (2.1) for any $u_0 \in \overline{D\,(A)}$. Suppose also that there exists $u \in Z$ such that the next condition holds:*

(2.6)

$$\text{for all } x \in D\,(A)\,,\, x \notin Z,\, y \in A\,(x)\,, \;\; \text{there exists } j \in J\,(x-u) \text{ for which } \langle y, j\rangle < 0.$$

Then the function $L(x) = \|x - u\|$ *is a Lyapunov function for* V *on* $\overline{D(A)}$.

Proof. Suppose that $x \notin Z$. Since the map $x(t) = V(t, x)$ is continuous on t, there exists an interval $[0, s)$ such that $x(t) \notin Z$ for any $t \in [0, s)$. Multiplying (2.1) by $j \in J(x(t) - u)$ and using (2.6), we have

$$\left\langle \frac{dx(t)}{dt}, j \right\rangle < 0 \text{ a.e. } t \in (0, s).$$

By Lemma 1.2 from [8, p.100] we have $\left\langle \frac{dx(t)}{dt}, j \right\rangle = \left(\frac{d}{dt}\|x(t) - u\|\right)\|x(t) - u\|$. Hence,

$$\left(\frac{d}{dt}\|x(t) - u\|\right)\|x(t) - u\| < 0 \text{ a.e. } t \in (0, s).$$

By integration we obtain

$$\|x(t) - u\| < \|x(0) - u\|, \text{ if } t \in (0, s).$$

It is easy to see that the last inequality holds for any $t > 0$. Indeed, let $x(s) \in Z$. Then $x(t) = x(s)$, for all $t \geq s$, and

$$\|x(s) - u\| = \lim_{\tau \to s} \|x(\tau) - u\| = \inf_{0 < \tau < s} \|x(\tau) - u\| < \|x(0) - u\|,$$

since $\|x(\tau) - u\|$ is a decreasing function on $[0, s)$. So, $L(x) = \|x - u\|$ is a Lyapunov function for V on $\overline{D(A)}$. $\square$

Theorem 2.8. *Let* A *satisfy conditions* $(A1) - (A2)$, V *be compact for some* $t_0 > 0$, *the map* $u(t) = V(t, u_0)$ *be a strong solution for all* $u_0 \in \overline{D(A)}$, Z *be nonempty and bounded and (2.6) hold for some* $u \in Z$. *Then* V *has the global attractor* $\Re$. *It is connected if* $\overline{D(A)}$ *is connected.*

Proof. It follows from Lemma 2.7 that $L(x) = \|x - u\|$ is a Lyapunov function for V on $\overline{D(A)}$ and also that for some $u \in Z$ one has

$$\|V(t, u_0) - u\| \leq \|u_0 - u\|, \text{ for all } u_0 \in B \in \mathcal{B}\left(\overline{D(A)}\right).$$

Then, it is obvious that $\gamma_0^+(B) \in \mathcal{B}\left(\overline{D(A)}\right)$.

Z is nonempty and bounded by assumption. It is well known from [23, Theorems 2.1 and 2.4] that if V is compact for some $t > 0$, there exists a Lyapunov function and $\gamma_0^+(x)$ is a bounded set, then $\omega(x)$ is nonempty, compact and attracts x. Moreover, $\omega(x) \subset Z$. It follows that Z attracts any $x \in \overline{D(A)}$, so that V is pointwise dissipative.

We conclude the proof by applying Theorem 1.5. $\square$

Corollary 2.9. *Under the conditions of Theorem 2.8,* $\Re = Z$ *if we suppose that one of the next conditions holds:*

(1) *(2.6) holds for any* $u \in Z$;
(2) A *is m-dissipative.*

Moreover, in the second case $\omega(x) = z \in Z$, *for all* $x \in \overline{D(A)}$.

Proof. Let us prove the first statement. Let $u \in \Re$. We must prove that in this case $u \in Z$.

Recall that the function $u(\cdot) : \mathbb{R} \to X$ is called a complete trajectory of V if $u(t+t_1) = V(t, u(t_1))$, for all $t_1 \in \mathbb{R}$, $t \in \mathbb{R}_+$. This function will be called a complete trajectory of the point x if $u(0) = x$.

From the definition, every global attractor is the union of all bounded complete trajectories of V (for a proof see [22, p.10]). Hence, the point u belongs to some bounded complete trajectory $\{u(t), t \in \mathbb{R}\}$. Without loss of generality we can put $u(0) = u$. From each sequence $\{u(t_n)\}$, $t_n \nearrow +\infty$ (that is, an increasing sequence of times converging to $+\infty$) or $t_n \searrow -\infty$ (that is, a decreasing sequence of times converging to $-\infty$), belonging to a complete bounded trajectory $\{u(t), t \in \mathbb{R}\}$, we can choose a subsequence converging to some stationary point (see [23, Theorem 2.4]). It follows that there exist two sequences $\{u(t_n)\}$, $t_n \nearrow +\infty$, $\{u(t_m)\}$, $t_m \searrow -\infty$, such that

$$\lim_{n\to\infty} u(t_n) = u_1 \in Z, \quad \lim_{m\to\infty} u(t_m) = u_2 \in Z.$$

Let us prove by contradiction that $u \in Z$. Suppose the opposite. Then by (2.6) and using Lemma 2.7 we have

$$\|u(t_n) - u_2\| < \|u - u_2\| < \|u(t_m) - u_2\|, \text{ for all } t_n > 0,\ t_m < 0.$$

Since $\lim_{t_m \to -\infty} \|u(t_m) - u_2\| = 0$, we obtain $u = u_1 = u_2 \in Z$. Thus, $\Re \subset Z$, from which $\Re = Z$.

Let us prove the second statement. Since $L(x) = \|x - u\|$ is a Lyapunov function for some $u \in Z$, (2.5) is immediately satisfied. Thus, the second statement is a consequence of Theorem 2.4. □

Remark 2.10. If $A = B + \omega I$, where B is m-dissipative (or satisfies $(A1)$ and $(A2)$ with $\omega = 0$), (2.6) can be written in the next way: there exists $u \in Z$ such that

(2.7)
$$\text{for all } x \in D(A),\ x \notin Z,\ y \in B(x), \text{ there exists } j \in J(x-u) \text{ for which } \langle y, j\rangle < -\omega \langle x, j\rangle.$$

In the preceding theorem we have used conditions providing the existence of a compact attractor. In the next one we shall consider another kind of conditions without using the set of stationary points Z.

Theorem 2.11. *Let A satisfy $(A1)$-$(A2)$, V be compact for some $t_0 > 0$, $u(t) = V(t, u_0)$ be a strong solution for any $u_0 \in D(A)$ and the next condition hold: there exist $C > 0$, $\delta > 0$ such that*

$$\text{for all } u \in D(A),\ \|u\| > C,\ y \in A(u), \text{ there exists } j \in J(u) \text{ for which } < y, j > \le -\delta. \tag{2.8}$$

Then V has the global attractor $\Re$. It is connected if $\overline{D(A)}$ is connected.

Proof. First we shall prove that for all $x \in D(A)$ there exists $t(x)$, for which $V(t,x) \in B_0$, where

$$B_0 = \left\{ u \in \overline{D(A)} \mid \|u\| \le C + \epsilon \right\},$$

with $\epsilon > 0$.

Let $x \notin B_0$, so that $\|x\| > C + \epsilon$. Suppose that $u(t) = V(t, x) \notin B_0$, for all $t \geq 0$. Then, using (2.8) and arguing as in the proof of Lemma 2.7 we get

$$\|u(t)\|^2 \leq \|u(0)\|^2 - 2\delta t, \quad \text{for all } t \geq 0.$$

For t great enough $u(t) \in B_0$. The resulting contradiction proves that $u(t) \in B_0$ for some t.

Further, let us prove the inclusion $\gamma_0^+(B) \in \mathcal{B}\left(\overline{D(A)}\right)$, if $B \in \mathcal{B}\left(\overline{D(A)}\right)$, $B \subset D(A)$. For this purpose it suffices to show that if $\|u_0\| \leq M, u_0 \in D(A)$, $M > C$, then $\|u(t)\| = \|V(t, u_0)\| \leq M$, for all $t \geq 0$. Indeed, suppose that $\|u(t_1)\| > M$, for some $t_1 > 0$. Since the map $u(\cdot)$ is continuous on $[0, \infty)$ it is clear that there exists $t_2 < t_1$ such that $\|u(t_2)\| = M$ and $\|u(t)\| > M$ for $t_2 < t \leq t_1$. But in this case we obtain arguing as before that $\|u(t_1)\|^2 \leq \|u(t_2)\|^2 - 2\delta(t_2 - t_1)$, which is a contradiction.

It remains to consider the case where $x \in \overline{D(A)} \backslash D(A)$. Let us show first that $V(t, x) \in B_0$ for some $t(x) \geq 0$. Let it not be so, i.e., $V(t, x) \notin B_0$, for all $t \geq 0$. Since $x \in \overline{D(A)}$, we can find a sequence $\{x_k\} \subset D(A)$ such that $\lim_{k \to \infty} x_k = x$. Let t be fixed. Then using (2.3) we obtain that there exists k_0 such that $\|V(t, x_k)\| \geq C + \epsilon/2$, for all $k \geq k_0$. Since for any $M > C$ and $x \in D(A)$ we have that $\|V(\tau, x)\| \leq M$, for all $\tau \geq 0$, if $\|x\| \leq M$, we deduce that $\|V(\tau, x_k)\| \geq C + \epsilon/2$, for all $\tau \in [0, t]$, $k \geq k_0$. Then, using (2.8) again we get

$$\|V(t, x_k)\|^2 \leq \|V(\tau, x_k)\|^2 - 2\delta(t - \tau), \; t > \tau, \quad \text{for all } k \geq k_0.$$

When $k \to \infty$ we obtain

$$\|V(t, x)\|^2 \leq \|V(\tau, x)\|^2 - 2\delta(t - \tau), \; t > \tau.$$

It is easy to see that choosing t great enough we have that $V(t, x) \in B_0$. Hence, we obtain a contradiction.

It remains to prove that $\gamma_0^+(B) \in \mathcal{B}\left(\overline{D(A)}\right)$, for all $B \in \mathcal{B}\left(\overline{D(A)}\right)$. We shall prove that $V(t, B_M) \subset B_M$, for all $t \geq 0$, where

$$B_M = \left\{v \in \overline{D(A)} \mid \|v\| \leq M\right\}, \; M > C.$$

For any $B \subset B_M$, $B \subset D(A)$, we have already verified the inclusion $V(t, B) \subset B_M$, for all $t \geq 0$. Suppose that there exist $x \in \overline{D(A)} \backslash D(A)$, $x \in B_M$, and $t > 0$ such that $\|V(t, x)\| > M$. Choosing $\{x_k\} \subset D(A)$, $\|x - x_k\| \underset{k \to \infty}{\to} 0$, and using (2.3), we have $\|V(t, x) - V(t, x_k)\| \underset{k \to \infty}{\to} 0$. Then $\|V(t, x_k)\| > M$, for all $k \geq k_0$. The resulting contradiction shows that $V(t, B_M) \subset B_M$ and then $\gamma_0^+(B) \in \mathcal{B}\left(\overline{D(A)}\right)$, for all $B \in \mathcal{B}\left(\overline{D(A)}\right)$.

Therefore, $\gamma_0^+(B) \in \mathcal{B}\left(\overline{D(A)}\right)$, for all $B \in \mathcal{B}\left(\overline{D(A)}\right)$, and, in view of Remark 1.3, the semigroup V is pointwise dissipative . We conclude the proof by using Theorem 1.5. $\square$

Remark 2.12. In view of Remark 2.2 if X is reflexive and A is closed, then $u(t) = V(t, u_0)$ is a strong solution of (2.1) for any $u_0 \in D(A)$.

Remark 2.13. It is easy to see that if there exist $\epsilon > 0$, $M \geq 0$ such that

$$\text{for all } u \in D(A),\ y \in A(u),\ \text{there exists } j \in J(u) \text{ for which} \\ \langle y, j \rangle \leq -\epsilon \|u\|^2 + M, \tag{2.9}$$

then (2.8) holds.

3. Applications to inclusions generated by subdifferential maps

Let H be a real Hilbert space where is given the scalar product denoted by $(\cdot,\cdot)$. We identify H with its dual H^* and then the dual operator J is the identity map I. We recall that the multivalued operator $A : D(A) \subset H \to 2^H$ is called monotone if

$$(y_1 - y_2, x_1 - x_2) \geq 0, \text{ for all } x_1, x_2 \in D(A),\ y_1 \in A(x_1),\ y_2 \in A(x_2).$$

A monotone operator is called maximal monotone if there does not exist another monotone operator B such that $\mathrm{Graph}(A) \subset \mathrm{Graph}(B)$, where

$$\mathrm{Graph}(A) = \{(x,y) \in H \times H : y \in A(x)\}.$$

Remark 3.1. The operator A is maximal monotone if and only if $-A$ is m-dissipative [8, p.71].

Consider the problem

$$\begin{cases} \frac{du}{dt} \in -\partial\varphi(u(t)) + \omega u(t) + h, \\ u(0) = u_0, \end{cases} \tag{3.10}$$

where $\omega \geq 0$, $\partial\varphi : H \to H$ is the subdifferential of a convex proper lower semicontinuous function $\varphi : H \to]-\infty, +\infty]$ and $h \in H$. It is well known that $\overline{D(\varphi)} = \overline{D(\partial\varphi)}$ and also that $\partial\varphi$ is a maximal monotone operator [8, p.54]. Then $-\partial\varphi$ is m-dissipative in view of Remark 3.1 and $-\partial\varphi + h$ is also m-dissipative.

For any $u_0 \in \overline{D(\partial\varphi)}$ and $T > 0$ there exists a unique strong solution of (3.10), $u(\cdot) \in C([0,T], H)$ (see [9, p.82] or [21, p.1399]). In view of Remark 2.1 we have $u(t) = V(t, u_0)$.

Let us recall the next well-known criterion of compacity of the semigroup V (see [21, p.1398]). We shall give the proof for the sake of completeness.

Lemma 3.2. *The semigroup $V(t,\cdot)$ is compact (that is, the map $V(t,\cdot)$ is compact for any $t > 0$) if the following property is satisfied:*

(*L*) *The level sets B_C defined by*

$$B_C = \{u \in D(\varphi) \mid \|u\| \leq C,\ \varphi(u) \leq C\}$$

are compact in H for every $C \in \mathbb{R}_+$.

Moreover, for any $B \in \mathcal{B}\left(\overline{D(\varphi)}\right)$ and $t > 0$ there is $R > 0$ such that $V(t, B) \subset B_R$.

Proof. We must prove that for every $B \in \mathcal{B}(\overline{D(\varphi)})$ and any $t > 0$ the set $V(t, B)$ is precompact in H. Let $B_N = \{u \in H \mid \|u\| < N\}$, $N > 0$. We take an arbitrary $u_0 \in B_N$, $u_0 \in D(\varphi)$, $u(t) = V(t, u_0)$. It follows from [9, p.82] that if $u_0 \in D(\varphi)$, then

$$t\left\|\frac{du}{dt}\right\|^2 + t\frac{d}{dt}\varphi(u) = t\left(h + \omega u, \frac{du}{dt}\right) \text{ a.e. on } (0, T),$$

and $\frac{du}{dt} \in L_2(0, T; H)$. Therefore, it follows that $\varphi(u(t))$ is absolutely continuous on $[0, T]$ [8, p.189]. Hence, integrating by parts we have

$$\int_0^T t\left\|\frac{du}{dt}\right\|^2 dt + T\varphi(u(T)) = \int_0^T t(h + \omega u, \frac{du}{dt})dt + \int_0^T \varphi(u(t))dt,$$

and then

$$T\varphi(u(T)) \leq \frac{1}{2}\int_0^T t\left\|\frac{du}{dt}\right\|^2 dt + T\varphi(u(T))$$

$$\leq \frac{1}{2}\int_0^T t\left(\|h\| + \omega\|u\|\right)^2 dt + \int_0^T \varphi(u(t))dt. \tag{3.11}$$

On the other hand, without loss of generality we can assume that $\min\{\varphi(u) : u \in H\} = \varphi(x_0) = 0$. Indeed, let $x_0 \in D(\partial\varphi)$, $y_0 \in \partial\varphi(x_0)$. If we introduce the new function $\widetilde{\varphi}(u) = \varphi(u) - \varphi(x_0) - (y_0, u - x_0)$, the inclusion

$$\frac{du}{dt} + \partial\varphi(u) \ni h + \omega u$$

is equivalent to

$$\frac{du}{dt} + \partial\widetilde{\varphi}(u) \ni h - y_0 + \omega u = \widetilde{h} + \omega u$$

and $\min\{\widetilde{\varphi}(u) : u \in H\} = \widetilde{\varphi}(x_0) = 0$. It is clear that $\widetilde{\varphi}$ satisfies (L).

Hence, since $h + \omega u(t) - \frac{du(t)}{dt} \in \partial\varphi(u(t))$ a.e. on $(0, T)$, we have

$$\varphi(u(t)) \leq (h + \omega u(t) - \frac{du(t)}{dt}, u(t) - x_0).$$

Integrating over $(0, T)$ we get

$$\int_0^T \varphi(u(t))dt$$

$$\leq \frac{1}{2}\|u(0) - x_0\|^2 - \frac{1}{2}\|u(T) - x_0\|^2 + \int_0^T (\|h\| + \omega\|u(t)\|)\|u(t) - x_0\| dt$$

$$\leq \frac{1}{2}\|u(0) - x_0\|^2 + \int_0^T (\|h\| + \omega\|u(t)\|)\|u(t) - x_0\| dt.$$

Let $v_0 \in \overline{D(\varphi)}$ be fixed. Obviously, there exists a constant C such that $\|v(t)\| \leq C$, for all $t \in [0, T]$, where $v(t) = V(t, v_0)$. It follows from inequality (2.3) that

$$\|u(t) - v(t)\| \leq \exp(\omega T)\|u_0 - v_0\|, \text{ for } 0 \leq t \leq T.$$

Therefore, there are constants D_1, D_2 (depending on T and N, but not on $u_0 \in B_N$) for which

$$\|u(t)\| \le D_1, \text{ for all } t \in [0,T], \tag{3.12}$$

$$\int_0^T \varphi(u(t))dt \le D_2 < \infty. \tag{3.13}$$

Using (3.12)-(3.13) in relation (3.11) we obtain that for any $T > 0$ there exists $K > 0$ such that $\varphi(u(T)) \le K$.

Consider now that $u_0 \in \overline{D(\varphi)}$, $u_0 \in B_N$. We take a sequence $\{u_0^n\} \subset D(\varphi)$ such that $u_0^n \to u_0$, as $n \to \infty$. It is clear by inequality (2.3) that $u^n(T) \to u(T)$. Since φ is lower semicontinuous, we get

$$\varphi(u(T)) \le \liminf_{n\to\infty} \varphi(u^n(T)) \le K.$$

Then

$$V(T, B_N) \subset B_{R,}$$

where $R = \max\{K, D_1\}$. It follows from (L) that $V(T, B_N)$ is precompact in H. □

Thus, as a consequence of Theorem 2.11 we obtain the following result, which is a slight improvement of Corollary 2.1 from [21].

Corollary 3.3. *Let φ satisfy (L) and let the following condition hold: there exist $\delta, C > 0$ such that for all $u \in D(\partial\varphi)$, $\|u\| > C$, for all $y \in -\partial\varphi(u)$ one has*

$$(y, u) \le -\delta - \omega\|u\|^2 - (h, u). \tag{3.14}$$

Then V has the global attractor $\Re$, which satisfies the next property of smoothness: there exists $C \in \mathbb{R}_+$ such that $\Re \subset B_C$.

Proof. It is straightforward to prove by using (3.14) that (2.8) holds. Hence, by Lemma 3.2 all conditions of Theorem 2.11 hold. Finally, since the attractor $\Re$ is invariant, we have $V(t, \Re) = \Re$, for all $t \ge 0$. Taking $t > 0$ we obtain by Lemma 3.2 that $\Re \subset B_C$ for some $C > 0$, as claimed. □

Further, as a consequence of Theorem 2.8 and Lemma 3.2 we obtain:

Corollary 3.4. *Let (L) be satisfied and let the following conditions hold:*

(1) *There exists $u \in Z$ such that for all $x \in D(\partial\varphi)$, $x \notin Z$, for all $y \in \partial\varphi(x)$, one has*

$$(y, x-u) > \omega(x, x-u) + (h, x-u). \tag{3.15}$$

(2) *The set of stationary points Z, i.e., the set of solutions of the inclusion*

$$\partial\varphi(u) \ni \omega u + h, \; u \in H, \tag{3.16}$$

is bounded.

Then V has the global attractor $\Re$.

Remark 3.5. It is possible to transform (3.15) as follows. By the definition of the subdifferential map we have

$$(y, x-u) \geq \varphi(x) - \varphi(u), \text{ for all } y \in \partial\varphi(x).$$

Hence, (3.15) will be satisfied if

$$\varphi(x) - \varphi(u) > \omega(x, x-u) + (h, x-u), \text{ for all } x \notin Z.$$

As a consequence of Corollary 2.9 we have:

Corollary 3.6. *Let* $\omega = 0$, $h = 0$, *and let* (L) *be satisfied. If* φ *is coercive, i.e.,* $\varphi(u) \to +\infty$, *as* $\|u\| \to +\infty$, *then* $\Re = Z \neq \varnothing$ *is the global attractor of* V.

Proof. Since φ is coercive, we have [8, p.52]

$$Z = \{x \mid 0 \in \partial\varphi(x)\} = \left\{x \in D(\varphi) \mid \varphi(x) = \min_{y \in D(\varphi)} \varphi(y)\right\} \neq \varnothing. \tag{3.17}$$

On the other hand, using again that φ is coercive it is easy to check that Z is bounded. Moreover, in view of Remark 3.5 and since ω and h are equal to zero, in order to prove (3.15) it suffices to satisfy the next condition: $\varphi(x) - \varphi(u) > 0$, for all $x \notin Z$, and some fixed $u \in Z$. This condition is satisfied for any $u \in Z$ in view of (3.17). It remains to apply Lemma 3.2 and Corollary 2.9. □

Remark 3.7. From the coerciveness of φ it follows that Z is non-empty and bounded. We can also use these two conditions instead of coerciveness. On the other hand, in Corollaries 3.4 and 3.6, $\Re \subset B_C$ for some $C > 0$.

4. Applications to reaction-diffusion equations

In the sequel Ω wil be an open bounded subset of $\mathbb{R}^n$ with sufficiently smooth boundary Γ. Consider the boundary value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + f(u) \ni \omega u + h, \text{ on }]0,T[\times \Omega, \\ u = 0, \text{ on }]0,T[\times \Gamma, \\ u(0,x) = u_0(x) \text{ on } \Omega, \end{cases} \tag{4.18}$$

where $\omega \geq 0$, $f : D(f) \subset \mathbb{R} \to 2^{\mathbb{R}}$ is a maximal monotone multivalued map such that $D(f) = \mathbb{R}$, $h \in L_p(\Omega)$ and $2 \leq p < \infty$.

We shall use the Banach space $X = L_p(\Omega)$ as phase space. Define the operator $B_p : D(B_p) \to 2^X$,

$$B_p(u) = \{\xi \in L_p(\Omega) : \xi(x) \in \Delta u(x) - f(u(x)), \text{ a.e. on } \Omega\},$$

$$D(B_p) = \Big\{u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) : \\ \exists y \in L_p(\Omega) \text{ such that } y(x) \in f(u(x)) \text{ a.e. on } \Omega\Big\}.$$

The operator B_p is m-dissipative (see [8, p.87]). Hence, the operator $A_p(u) = B_p(u) + \omega u + h$, $D(A_p) = D(B_p)$, satisfies $A1$–$A2$.

Recall that in $L_p(\Omega)$ the dual operator $F : L_p(\Omega) \to L_q(\Omega)$, $\frac{1}{p} + \frac{1}{q} = 1$, is single-valued and defined by $F(u) = \frac{|u|^{p-2}u}{\|u\|_{L_p}^{p-2}}$.

Let us denote the semigroup generated by (4.18) in the complete metric space $\overline{D(B_p)} \subset L_p(\Omega)$ by V_p.

Proposition 4.1. *Let f satisfy the next condition: there exist $M \geq 0$, $\varepsilon > 0$ such that for all $s \in D(f)$, $y \in f(s)$ one has*

$$y\, s - \omega |s|^2 \geq \varepsilon |s|^2 - M. \tag{4.19}$$

Then condition (2.8) holds.

Proof. For $u \in D(A_p)$ and $y \in f(u)$, a.e. on Ω, $y \in L_p(\Omega)$, we obtain by using (4.19) and integrating over Ω that

$$\int_\Omega (y(x) - \omega u(x)) \frac{|u(x)|^{p-2} u(x)}{\|u\|_{L_p}^{p-2}} dx \geq \varepsilon \int_\Omega \frac{|u(x)|^p}{\|u\|_{L_p}^{p-2}} dx - M \int_\Omega \frac{|u|^{p-2}}{\|u\|_{L_p}^{p-2}} dx.$$

Then, using integration by parts, for any $\xi \in A_p(u)$ we get

$$\langle \xi, F(u) \rangle =$$

$$= \int_\Omega \Delta u \frac{|u(x)|^{p-2} u(x)}{\|u\|_{L_p}^{p-2}} dx - \int_\Omega (y(x) - \omega u(x) + h(x)) \frac{|u(x)|^{p-2} u(x)}{\|u\|_{L_p}^{p-2}} dx$$

$$\leq -(p-1) \int_\Omega |\nabla u(x)|^2 \frac{|u(x)|^{p-2}}{\|u\|_{L_p}^{p-2}} dx - \varepsilon \int_\Omega \frac{|u(x)|^p}{\|u\|_{L_p}^{p-2}} dx$$

$$+ M \int_\Omega \frac{|u|^{p-2}}{\|u\|_{L_p}^{p-2}} dx + \int_\Omega |h(x)| \frac{|u(x)|^{p-1}}{\|u\|_{L_p}^{p-2}} dx.$$

Further, Hölder inequality implies

$$\langle \xi, F(u) \rangle \leq -\varepsilon \|u\|_{L_p}^2 + M |\Omega|^{\frac{2}{p}} \frac{\|u\|_{L_p}^{p-2}}{\|u\|_{L_p}^{p-2}} + \|h\|_{L_p} \frac{\|u\|_{L_p}^{p-1}}{\|u\|_{L_p}^{p-2}}$$

$$= -\varepsilon \|u\|_{L_p}^2 + M |\Omega|^{\frac{2}{p}} + \|h\|_{L_p} \|u\|_{L_p} \leq -\frac{\varepsilon}{2} \|u\|_{L_p}^2 + M |\Omega|^{\frac{2}{p}} + \frac{1}{2\varepsilon} \|h\|_{L_p}^2,$$

where $|\Omega|$ denotes the Lebesgue measure of Ω. Fix $\delta > 0$. Then there exists $C > 0$ such that if $\|u\|_{L_p} > C$ we obtain

$$\langle \xi, F(u) \rangle \leq -\delta.$$

Thus, condition (2.8) holds. □

Remark 4.2. In the case $p = 2$ condition (4.19) can be weakened by putting $\varepsilon - \lambda_1$ instead of ε, where λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$.

Theorem 4.3. *If condition (4.19) is satisfied and*

$$2 \leq p \leq \frac{2n}{n-2}, \text{ if } n \geq 3, \tag{4.20}$$

then the semigroup generated by (4.18) has the global attractor $\Re$, which is compact in $L_p(\Omega)$ and bounded in $H_0^1(\Omega)$. If $p = 2$, then $\Re$ is connected.

Proof. We have to prove first that the semigroup is compact. Consider first the case where $p = 2$. Denote in this case the semigroup by V_2.

Note that there exists a proper convex lower semicontinuous map $j : \mathbb{R} \to]-\infty, +\infty]$ such that $f = \partial j$, where ∂j is the subdifferential of j [8, p.60].

Let us determine the function $\varphi : H \to]-\infty, +\infty]$, $H = L_2(\Omega)$, by

$$\varphi(u) = \begin{cases} \frac{1}{2}\int_\Omega |\nabla u(x)|^2 \, dx + \int_\Omega j(u(x))\, dx, & \text{if } u \in D(\varphi), \\ +\infty, & \text{otherwise}, \end{cases}$$

where $D(\varphi) = \{u \in H_0^1(\Omega),\ j(u) \in L_1(\Omega)\}$. It is well known [8, p.88] that $y \in \partial\varphi(u)$ if and only if $u \in D(\partial\varphi)$, $y(x) \in -\Delta u(x) + f(u(x))$ a.e. on Ω, and $y(\cdot) \in L_2(\Omega)$, where

$$D(\partial\varphi) = \{u \in H^2(\Omega) \cap H_0^1(\Omega) \mid \\ \text{there exists } v \in L_2(\Omega) \text{ such that } v(x) \in f(x) \text{ a.e. on } \Omega\},$$

$$\overline{D(\varphi)} = \overline{D(\partial\varphi)} = L_2(\Omega).$$

Then $B_2 = \partial\varphi$ and we have problem (3.10).

We must prove that $\varphi(u)$ satisfies (L). Indeed, since the function $j(u)$ is bounded from below by an affine function [8, p.51], for any $u \in B_C$ we have

$$\int_\Omega (\mu + vu(x))\, dx + \frac{1}{2}\int_\Omega |\nabla u(x)|^2\, dx \le \varphi(u) \le C,$$

where $\mu, v \in \mathbb{R}$. Since the norms $\|u\|_{H_0^1(\Omega)}$ and $\|\nabla u\|_{L_2(\Omega)}$ are equivalent, the preceding inequality implies that B_C is bounded in $H_0^1(\Omega)$. Finally, we use the fact that the inclusion $H_0^1(\Omega) \subset L_2(\Omega)$ is compact [10, p.169].

Now Lemma 3.2 implies the compacity of $V_2(t, \cdot)$ for any $t > 0$. Moreover, since for any bounded set $B \subset L_2(\Omega)$ and $t > 0$ there exists C such that $V_2(t, B) \subset B_C$, the set $V_2(t, B)$ is bounded in $H_0^1(\Omega)$.

Further, let $p > 2$. It is easy to check using the unicity of solutions that $V_p(t, u_0) = V_2(t, u_0)$, for any $u_0 \in \overline{D(B_p)}$, $t \ge 0$. Hence, for any bounded set $B \subset \overline{D(B_p)}$ (in the topology of $L_p(\Omega)$) and $t > 0$ we have that $V_p(t, B) = V_2(t, B)$ is bounded in $H_0^1(\Omega)$. Since in view of condition (4.20) the injection $H_0^1(\Omega) \subset L_p(\Omega)$ is compact (see [10, p.169]), $V_p(t, B)$ is a precompact set.

We note that for any $u_0 \in D(A_p)$ there exists a unique strong solution $u(\cdot)$ of inclusion (4.18) (see [8, p.146]), so that since in view of Remark 2.1, $u(t) = V_p(t, u_0)$, the existence of the global attractor follows from Proposition 4.1 and Theorem 2.11.

Further, since the global attractor is bounded in $L_p(\Omega)$, we get that $V_p(t, \Re)$ is bounded in $H_0^1(\Omega)$. Therefore, we obtain that $\Re$ is bounded in $H_0^1(\Omega)$ by the equality $V_p(t, \Re) = \Re$.

Finally, if $p = 2$ we have that $\overline{D(B_2)} = \overline{D(\partial\varphi)} = L_2(\Omega)$. Then since $L_2(\Omega)$ is connected, the global attractor is connected. $\square$

Theorem 4.4. *Let $p = 2$, $h \equiv 0$, $0 \in f(0)$ and let f satisfy the next condition*

$$ys \ge (-\lambda_1 + \omega)\, s^2, \text{ for all } s \in \mathbb{R},\ y \in f(s), \tag{4.21}$$

where λ_1 *is the first eigenvalue of* $-\Delta$ *in* $H_0^1(\Omega)$. *Moreover, there exists* $C > 0$ *such that if* $|s| \geq C$, *then*

$$ys > (-\lambda_1 + \omega)\, s^2, \text{ for all } y \in f(s). \tag{4.22}$$

Then the semigroup generated by (4.18) has the global connected attractor $\Re$, *which is compact in* $L_2(\Omega)$ *and bounded in* $H_0^1(\Omega)$.

Proof. We have seen in the proof of the previous theorem that $B_2 = \partial\varphi$ and (L) holds. We must verify that (3.15)-(3.16) hold. It is clear that the function $v(x) \equiv 0$ is a stationary point of V_2. Let us check (3.15) for this point. It follows from (4.21) that for all $u \in D(\partial\varphi)$, $y \in f(u)$ a.e. on Ω, $y \in L_2(\Omega)$, one has

$$y(x)u(x) \geq (-\lambda_1 + \omega)\, u^2(x), \text{ a.e. on } \Omega,$$

and so

$$(y, u) = \int_\Omega y(x)\, u(x) dx \geq (-\lambda_1 + \omega)\, \|u\|_{L_2}^2 .$$

Therefore,

$$(-\Delta u + y, u) \geq \omega\, \|u\|_{L_2}^2, \text{ for all } u \in D(\partial\varphi),\ y \in L_2(\Omega), y \in f(u) \text{ a.e. on } \Omega.$$

It remains to show that this inequality is strict when $u \notin Z$. Suppose that $(-\Delta u + y, u) = \omega\, \|u\|_{L_2}^2$, $y(x) \in f(u(x))$ a.e. on Ω, $y \in L_2(\Omega)$. This implies that $y(x)u(x) = (-\lambda_1 + \omega)\, u^2(x)$ a.e. on Ω, and $u \in span\{e_1, e_2, ..., e_m\}$, i.e., it belongs to the space generated by the eigenfunctions corresponding to λ_1. Indeed, if $y(x)u(x) > (-\lambda_1 + \omega)\, u^2(x)$ on a set $\Omega_1 \subset \Omega$ such that $|\Omega_1| \neq 0$, then

$$(y, u) > (-\lambda_1 + \omega)\, \|u\|_{L_2}^2 .$$

Hence $(-\Delta u + y, u) > \omega\, \|u\|_{L_2}^2$, which is a contradiction. On the other hand, by using the equality $y(x)u(x) = (-\lambda_1 + \omega)\, u^2(x)$, we have

$$(-\Delta u, u) + (y - \omega, u) = (-\Delta u, u) - \lambda_1\, \|u\|_{L_2}^2 = 0.$$

The last equality can hold only if $u \in span\{e_1, e_2, ..., e_m\}$. Let us suppose that it is not the case. Then

$$(-\Delta u, u) = \left(\sum_{i=1}^{\infty} \lambda_i \gamma_i e_i, \sum_{i=1}^{\infty} \gamma_i e_i \right) = \sum_{i=1}^{\infty} \lambda_i \gamma_i^2 > \lambda_1 \sum_{i=1}^{\infty} \gamma_i^2 = \lambda_1\, \|u\|_{L_2}^2 ,$$

and thus we obtain a contradiction. We must check that $u \in Z$. We take the partition $\Omega = \Omega_1 \cup \Omega_2$, where $u(x) = 0$ a.e. on Ω_1, $u(x) \neq 0$ a.e. on Ω_2, and define the function

$$\xi(x) = \begin{cases} 0, & \text{on } \Omega_1, \\ y(x), & \text{on } \Omega_2. \end{cases}$$

Since $0 \in f(0)$, we have $\xi(x) \in f(u(x))$ a.e. on Ω. On the other hand $\xi(x) = (-\lambda_1 + \omega)\, u(x)$ a.e. on Ω. Since $-\Delta e_k = \lambda_1 e_k$ on Ω, for all $k = 1, ..., m$ [10, p.192], we get

$$-\Delta u(x) + \xi(x) = \omega u(x) \text{ a.e. on } \Omega.$$

Thus, $u \in Z$ and condition (3.15) holds.

Next, we must prove that Z is bounded in $L_2(\Omega)$. If $u \in Z$, then for some $y \in L_2(\Omega)$, $y(x) \in f(u(x))$ a.e. on Ω, we have

$$(-\Delta u + y, u) = \omega \|u\|_{L_2}^2 ,$$

and then, as we have already proved,

$$y(x)u(x) = (-\lambda_1 + \omega) u^2(x) \text{ a.e. on } \Omega.$$

It follows from this equality that $|u(x)| < C$ a.e. on Ω, because by assumption $y(x)u(x) > -(\lambda_1 + \omega) u^2(x)$ if $|u(x)| \geq C$. Thus, Z is bounded in $L_\infty(\Omega)$ and consequently in $L_2(\Omega)$.

The properties of the attractor may be proved in the same way as in Theorem 4.3. □

Remark 4.5. We note, as a particular case, that if $ys > -\lambda_1 s^2$, for all $s \in \mathbb{R}\setminus\{0\}$, $y \in f(s)$, then $u \equiv 0$ is the unique stationary point. It follows from Corollary 2.9 that $\Re = Z = \{0\}$.

5. Dimension of the global attractor of reaction-diffusion equations in the case $p = 2$

We are now interested in the estimation of the fractal dimension of the global attractor of (4.18) in the Hilbert space $L_2(\Omega)$. Such estimates are well known in the case of a differentiable function f (see [3]-[5], [25]). For non-differentiable maps a similar result was obtained in [7] supposing that f is Lipschitz and in [18] in the case where $f \in W_{loc}^{1,\infty}(\mathbb{R})$. In all these papers the function f is at least Lipschitz on any bounded set of $\mathbb{R}$.

We shall extend these results by considering a function $f(s)$ which is Lipschitz on a fixed bounded set $[-a, a]$ but can be even discontinuous for $s \notin [-a, a]$.

Recall that the fractal dimension of a compact set $\mathcal{A}$ is defined by

$$d_f(\mathcal{A}) = \inf\{d > 0 \mid \mu_f(\mathcal{A}, d) = 0\},$$

where

$$\mu_f(\mathcal{A}, d) = \overline{\lim_{\epsilon \to 0}} \, \epsilon^d n_\epsilon,$$

and n_ϵ is the minimum number of balls of radius $r \leq \epsilon$ which is necessary to cover $\mathcal{A}$.

First we have to obtain an estimate of the elements of the global attractor in the norm of the space $L_\infty(\Omega)$. For this goal we need to impose a dissipative condition which is stronger than (4.19).

Proposition 5.1. *Let us assume that there exist $\varepsilon > 0, r > 2, M \geq 0$ such that*

$$ys - \omega s^2 \geq \varepsilon |s|^r - M, \text{ for any } s \in \mathbb{R}, y \in f(s). \tag{5.23}$$

Let $k \geq 0$, $u_0 \in D(A_{k+r})$ and $h \in L_\infty(\Omega)$. Then $u(t) = V_2(t, u_0)$ satisfies:

$$\|u(t)\|_{L_{k+2}} \leq |\Omega|^{\frac{1}{k+2}} \left(2^{\frac{1}{k+r}} \left(\left(\tfrac{4M}{\varepsilon}\right)^{\frac{1}{r}} + \left(\tfrac{4\|h\|_{L_\infty}}{\varepsilon}\right)^{\frac{1}{r-1}} \right) + \left(\tfrac{\varepsilon}{2}(r-2)t\right)^{-\frac{1}{r-2}} \right), \tag{5.24}$$

for all $t > 0$.

Proof. Let $k \geq 0$ be arbitrary. We note that since $u_0 \in D(A_{k+r})$, we have that $V_2(t,u_0) = V_{k+r}(t,u_0)$, so that $u(\cdot) \in C([0,T], L_{k+r}(\Omega))$ for any $T > 0$. Denote $v(t) = \|u(t)\|_{L_{k+2}}^{k+2}$. Due to the regularity of u_0 we have that $u(\cdot)$ is a strong solution of (4.18), $u \in W^{1,\infty}(0,T; L_{k+r}(\Omega))$, for any $T > 0$, and $u(t) \in W_0^{1,k+r}(\Omega) \cap W^{2,k+r}(\Omega)$, for any $t \in [0,T]$ (see [8, p.146]). It follows from [8, Lemma 1.2, p.100] that $\frac{1}{k+2}\frac{d}{dt}\|u\|_{L_{k+2}}^{k+2} = \int_\Omega \frac{du}{dt}|u|^k u dx$. Multiplying (4.18) by $|u|^k u$ and using the Green formula and (5.23) we obtain

$$\begin{aligned}&\tfrac{1}{k+2}\tfrac{d}{dt}\|u\|_{L_{k+2}}^{k+2} + (k+1)\textstyle\int_\Omega |\nabla u|^2 |u|^k dx + \varepsilon \int_\Omega |u|^{k+r} dx \\ &\leq \quad M \textstyle\int_\Omega |u|^k dx + \int_\Omega h|u|^k u dx.\end{aligned}$$

Now the Hölder inequalities

$$\|u\|_{L_k}^k \leq \|u\|_{L_{k+2}}^k |\Omega|^{\frac{2}{k+2}}, \ \|u\|_{L_{k+2}}^{k+r} \leq \|u\|_{L_{k+r}}^{k+r} |\Omega|^{\frac{r-2}{k+2}}$$

imply

$$\frac{1}{k+2}\frac{d}{dt}\|u\|_{L_{k+2}}^{k+2} + \varepsilon|\Omega|^{\frac{2-r}{k+2}}\|u\|_{L_{k+2}}^{k+r} \leq M\|u\|_{L_{k+2}}^k |\Omega|^{\frac{2}{k+2}} + \int_\Omega h|u|^k u dx.$$

Further Young inequality with exponent $q = \frac{k+r}{k}$ and coefficients

$$a = \frac{\varepsilon}{4}|\Omega|^{-\frac{r}{k+2}}, C_a = \left(\frac{\varepsilon}{4}|\Omega|^{-\frac{r}{k+2}}\right)^{-\frac{k}{r}}$$

gives

(5.25)
$$\frac{1}{k+2}\frac{d}{dt}\|u\|_{L_{k+2}}^{k+2} + \frac{3\varepsilon}{4}|\Omega|^{\frac{2-r}{k+2}}\|u\|_{L_{k+2}}^{k+r} \leq |\Omega|\left(\frac{\varepsilon}{4}\right)^{-\frac{k}{r}} M^{\frac{k+r}{r}} + \int_\Omega h|u|^k u dx.$$

Using the Hölder inequalities

$$\int_\Omega h|u|^k u dx \leq \|h\|_{L_\infty}\|u\|_{L_{k+1}}^{k+1}, \ \|u\|_{L_{k+1}}^{k+1} \leq \|u\|_{L_{k+2}}^{k+1}|\Omega|^{\frac{1}{k+2}}$$

we have

$$\begin{aligned}&\tfrac{1}{k+2}\tfrac{d}{dt}\|u(t)\|_{L_{k+2}}^{k+2} + \tfrac{3\varepsilon}{4}|\Omega|^{\frac{2-r}{k+2}}\|u\|_{L_{k+2}}^{k+r} \\ &\leq \quad |\Omega|\left(\tfrac{\varepsilon}{4}\right)^{-\frac{k}{r}} M^{\frac{k+r}{r}} + \textstyle\int_\Omega h|u|^k u dx \\ &\leq \quad |\Omega|\left(\tfrac{\varepsilon}{4}\right)^{-\frac{k}{r}} M^{\frac{k+r}{r}} + \|h\|_{L_\infty}\|u\|_{L_{k+1}}^{k+1} \\ &\leq \quad |\Omega|\left(\tfrac{\varepsilon}{4}\right)^{-\frac{k}{r}} M^{\frac{k+r}{r}} + \|h\|_{L_\infty}\|u\|_{L_{k+2}}^{k+1}|\Omega|^{\frac{1}{k+2}}.\end{aligned}$$

Now applying the Young inequality with exponent $q = \frac{k+r}{k+1}$ and coefficients $a = \frac{\varepsilon}{4}|\Omega|^{\frac{1-r}{k+2}}$, $C_a = \left(\frac{\varepsilon}{4}|\Omega|^{\frac{1-r}{k+2}}\right)^{\frac{k+1}{1-r}}$ to the last term of the inequality we get

$$\begin{aligned}&\tfrac{1}{k+2}\tfrac{d}{dt}\|u\|_{L_{k+2}}^{k+2} + \tfrac{\varepsilon}{2}\|u\|_{L_{k+2}}^{k+r}|\Omega|^{\frac{2-r}{k+2}} \\ &\leq \quad |\Omega|\left(\left(\tfrac{\varepsilon}{4}\right)^{-\frac{k}{r}} M^{\frac{k+r}{r}} + \left(\tfrac{\varepsilon}{4}\right)^{-\frac{k+1}{r-1}}\|h\|_{L_\infty}^{\frac{k+r}{r-1}}\right).\end{aligned}$$

Hence,

$$\frac{d}{dt}v(t) + \gamma(v(t))^q \leq \delta, \tag{5.26}$$

where $\delta = (k+2)\,|\Omega|\left(\left(\frac{\varepsilon}{4}\right)^{-\frac{k}{r}} M^{\frac{k+r}{r}} + \left(\frac{\varepsilon}{4}\right)^{-\frac{k+1}{r-1}} \|h\|_{L_\infty}^{\frac{k+r}{r-1}}\right)$, $\gamma = \frac{\varepsilon}{2}(k+2)\,|\Omega|^{\frac{2-r}{k+2}}$. $q = \frac{k+r}{k+2}$.

Further we shall use the following version of the Gronwall lemma [29, Chapter III, p.163]:

Lemma 5.2. *Let $y(t) \geq 0$ be absolutely continuous on $(0,\infty)$. Suppose that there exist $q > 1$, $\gamma > 0$, $\delta \geq 0$ such that*

$$\frac{d}{dt}y(t) + \gamma y^q(t) \leq \delta. \tag{5.27}$$

Then

$$y(t) \leq \left(\frac{\delta}{\gamma}\right)^{\frac{1}{q}} + (\gamma(q-1)t)^{-\frac{1}{q-1}}, \text{ for all } t > 0. \tag{5.28}$$

Thus, Lemma 5.2 implies

$$\begin{aligned}
\|u(t)\|_{L_{k+2}} \leq & \left(\frac{|\Omega|^{\frac{k+r}{k+2}}}{2}\left(\left(\frac{\varepsilon}{4}\right)^{-\frac{k+r}{r}} M^{\frac{k+r}{r}} + \left(\frac{\varepsilon}{4}\right)^{-\frac{k+r}{r-1}} \|h\|_{L_\infty}^{\frac{k+r}{r-1}}\right)\right)^{\frac{1}{k+r}} \\
& + \left(\frac{\varepsilon}{2}|\Omega|^{\frac{2-r}{k+2}}(r-2)\,t\right)^{-\frac{1}{r-2}} \\
& \leq |\Omega|^{\frac{1}{k+2}}\left(\frac{1}{2^{\frac{1}{k+r}}}\left(\left(\frac{4M}{\varepsilon}\right)^{\frac{1}{r}} + \left(\frac{4\|h\|_{L_\infty}}{\varepsilon}\right)^{\frac{1}{r-1}}\right) + \left(\frac{\varepsilon}{2}(r-2)\,t\right)^{-\frac{1}{r-2}}\right),
\end{aligned}$$

for all $t > 0$. $\square$

Corollary 5.3. *Let $u_0 \in C_0^\infty(\Omega)$, $h \in L_\infty(\Omega)$ and let (5.23) hold. Then for any $\eta > 0$ we have*

$$\|u\|_{L_\infty(\eta,\infty;L_\infty(\Omega))} \leq \left(\frac{4M}{\varepsilon}\right)^{\frac{1}{r}} + \left(\frac{4\|h\|_{L_\infty}}{\varepsilon}\right)^{\frac{1}{r-1}} + \frac{1}{\left((r-2)\,\eta\frac{\varepsilon}{2}\right)^{\frac{1}{r-2}}}. \tag{5.29}$$

Proof. First we shall show that $u_0 \in D(A_p)$ for any $p \geq 2$. We take an arbitrary single-valued function $g(s)$ such that $g(s) \in f(s)$, for all $s \in \mathbb{R}$. Since $f(s)$ is maximal monotone, $g(s)$ is non-decreasing. Any non-decreasing function is measurable, so that the composition $l(x) = g(u_0(x))$ is a measurable selection of $f(u_0(x))$. Let us check that $l(x) \in L_\infty(\Omega)$. Since $u_0 \in C_0^\infty(\Omega)$, there exists $b > 0$ such that $|u_0(x)| \leq b$, for all $x \in \Omega$. Hence,

$$g(-b) \leq g(u_0(x)) \leq g(b), \quad \text{for all } x \in \Omega,$$

so that $l(x) \in L_\infty(\Omega)$. It follows that $l(x) \in L_p(\Omega)$, for any $p \geq 2$, and that $l(x) \in f(u_0(x))$, a.e. on Ω. Therefore, $u_0 \in D(A_p)$, for all $p \geq 2$.

Proposition 5.1 implies that (5.24) is satisfied for any $k \geq 0$. Passing to the limit as $k \to \infty$ we obtain

$$\|u(t)\|_{L_\infty} \leq \left(\frac{4M}{\varepsilon}\right)^{\frac{1}{r}} + \left(\frac{4\|h\|_{L_\infty}}{\varepsilon}\right)^{\frac{1}{r-1}} + \frac{1}{\left((r-2)\,t\frac{\varepsilon}{2}\right)^{\frac{1}{r-2}}}, \text{ for all } t > 0,$$

and then for any $\eta > 0$ we get

$$\|u\|_{L_\infty(\eta,T;L_\infty(\Omega))} \le \left(\frac{4M}{\varepsilon}\right)^{\frac{1}{r}} + \left(\frac{4\|h\|_{L_\infty}}{\varepsilon}\right)^{\frac{1}{r-1}} + \frac{1}{\left((r-2)\,\eta\frac{\varepsilon}{2}\right)^{\frac{1}{r-2}}}.$$

□

Corollary 5.4. *Let $u_0 \in L_2(\Omega)$, $h \in L_\infty(\Omega)$ and let (5.23) hold. Then inequality (5.29) holds.*

Proof. Let $u_0^n \in C_0^\infty(\Omega)$ be a sequence such that $u_0^n \to u_0$ in $L_2(\Omega)$. Then inequality (2.3) implies that $u^n(t) = V_2(t, u_0^n)$ converges to $u(t) = V_2(t, u_0)$ in $C([0,T], L_2(\Omega))$. The sequence u^n is bounded in $L_\infty(\eta, T; L_\infty(\Omega))$ by (5.29). Hence, there exists a subsequence converging to u weakly star in $L_\infty(\eta, T; L_\infty(\Omega))$. Therefore, (5.29) holds. □

Now we can obtain an estimate of the elements of the global attractor $\Re$ (which exists in view of Theorem 4.3) in the norm of the space $L_\infty(\Omega)$.

Theorem 5.5. *Let $h \in L_\infty(\Omega)$ and let (5.23) hold. Then for any $y \in \Re$ the following estimate holds:*

$$\|y\|_{L_\infty(\Omega)} \le \left(\frac{4M}{\varepsilon}\right)^{\frac{1}{r}} + \left(\frac{4\|h\|_{L_\infty}}{\varepsilon}\right)^{\frac{1}{r-1}}. \tag{5.30}$$

Proof. Let $\delta > 0$, $y \in \Re$ be arbitrary. We choose η such that

$$\left((r-2)\,\eta\frac{\varepsilon}{2}\right)^{-\frac{1}{r-2}} < \delta.$$

Since $\Re$ is invariant, there exists $u_0 \in \Re$ for which $y = u(2\eta) = V_2(2\eta, u_0)$. Corollary 5.4 implies

$$\|u\|_{L_\infty(\eta,3\eta;L_\infty(\Omega))} \le \left(\frac{4M}{\varepsilon}\right)^{\frac{1}{r}} + \left(\frac{4\|h\|_{L_\infty}}{\varepsilon}\right)^{\frac{1}{r-1}} + \delta.$$

Since $u \in C(\eta, 3\eta; L_2(\Omega))$ and it is bounded in $L_\infty(\eta, 3\eta; L_\infty(\Omega))$, we have $u \in C_w(\eta, 3\eta; L_q(\Omega))$, for any $2 \le q < \infty$, where C_w denotes the weak topology (see [24, p.275]). Hence,

$$\|y\|_{L_q(\Omega)} = \|u(2\eta)\|_{L_q(\Omega)} \le \left(\left(\frac{4M}{\varepsilon}\right)^{\frac{1}{r}} + \left(\frac{4\|h\|_{L_\infty}}{\varepsilon}\right)^{\frac{1}{r-1}} + \delta\right)|\Omega|^{\frac{1}{q}},$$

so that

$$\|y\|_{L_\infty(\Omega)} = \|u(2\eta)\|_{L_\infty(\Omega)} \le \left(\frac{4M}{\varepsilon}\right)^{\frac{1}{r}} + \left(\frac{4\|h\|_{L_\infty}}{\varepsilon}\right)^{\frac{1}{r-1}} + \delta.$$

Since δ is arbitrary, (5.30) holds. □

Theorem 5.6. *Let $h \in L_\infty(\Omega)$ and let (5.23) hold. Suppose that there exists $a > 0$ such that*

$$\left(\frac{4M}{\varepsilon}\right)^{\frac{1}{r}} + \left(\frac{4\|h\|_{L_\infty}}{\varepsilon}\right)^{\frac{1}{r-1}} \le a \tag{5.31}$$

and in the interval $[-a,a]$ the function $f(s)$ is Lipschitz (with Lipschitz constant ξ).

Then there exists K depending on Ω and n for which

$$d_f(\Re) \leq K(\omega+\xi)^{\frac{n}{2}}. \tag{5.32}$$

Proof. Let $\{\lambda_N\}_{N=1}^{\infty}$ be the eigenvalues of $-\Delta$ in $H_0^1(\Omega)$, P^N be the orthoprojector to the subspace generated by the eigenfunctions corresponding to the first N eigenvalues and $Q^N = I - P^N$. It follows from [7, Theorem 4] that if we find $t > 0$, $l \in [1, +\infty)$, $\delta \in (0, \frac{1}{\sqrt{2}})$ such that

$$\|V_2(t,u_0) - V_2(t,v_0)\| \leq l\,\|u_0 - v_0\|, \tag{5.33}$$

$$\left\|Q^N V_2(t,u_0) - Q^N V_2(t,v_0)\right\| \leq \delta\,\|u_0 - v_0\|, \tag{5.34}$$

for all $u_0, v_0 \in \Re$, then for any $\eta > 0$ such that $\left(\sqrt{2}6l\right)^N \left(\sqrt{2}\delta\right)^{\eta} = \sigma < 1$ the next estimate holds

$$d_f(\mathcal{A}) \leq N + \eta. \tag{5.35}$$

In view of inequality (2.3) condition (5.33) holds with $l(t) = \exp(\omega t)$.

Further we note that from Theorem 5.5 and the Lipschitz condition of f on $[-a,a]$ it follows that

$$\|f(u) - f(v)\|_{L_2} \leq \xi\,\|u - v\|_{L_2}, \text{ for any } u, v \in \Re. \tag{5.36}$$

We take two arbitrary initial conditions $u_0, v_0 \in \Re$. Let now $w(t) = u(t) - v(t)$, $w^N(t) = Q^N w(t)$, where $u(t) = V_2(t,u_0)$, $v(t) = V_2(t,v_0)$. From (4.18) we can easily obtain

$$\frac{1}{2}\frac{d}{dt}\left\|w^N(t)\right\|_{L_2}^2 + \left\|\nabla w^N\right\|_{L_2}^2 + \left(f(u) - f(v), w^N\right) = \omega\left(w, w^N\right).$$

Using the inequality $\left\|\nabla w^N\right\|_{L_2}^2 \geq \lambda_{N+1}\left\|w^N(t)\right\|_{L_2}^2$, (2.3) and (5.36) we obtain

$$\frac{d}{dt}\left\|w^N(t)\right\|_{L_2}^2 + 2\lambda_{N+1}\left\|w^N(t)\right\|_{L_2}^2 \leq 2(\omega+\xi)\exp(2\omega t)\,\|w(0)\|_{L_2}^2.$$

Multiplying both sides by $\exp(2\lambda_{N+1}t)$ and integrating over $(0,t)$ we get

$$\begin{aligned}
&\left\|w^N(t)\right\|_{L_2}^2 \\
&\leq \|w(0)\|_{L_2}^2\left(\exp(-2\lambda_{N+1}t) + \tfrac{\omega+\xi}{\omega+\lambda_{N+1}}\left(\exp(2\omega t) - \exp(-2\lambda_{N+1}t)\right)\right) \\
&\leq \|w(0)\|_{L_2}^2\left(\exp(-2\lambda_{N+1}t) + \tfrac{\omega+\xi}{\omega+\lambda_{N+1}}\exp(2\omega t)\right) \\
&= \delta^2(t,N)\,\|w(0)\|_{L_2}^2.
\end{aligned}$$

Repeating exactly the same proof of Theorem 7 from [7] we can obtain that for $t = \frac{\log\left(\sqrt{2}12\right)}{\lambda_{N+1}-\omega}$ there exists a constant D (depending on Ω) such that if $N = \left[(D(\omega+\xi))^{\frac{n}{2}}\right]$, where $[x]$ denotes the integer part of x, then $\delta(t,N) < \frac{1}{\sqrt{2}}$ and $(12\delta(t,N)\,l(t))^N < 1$, so that $\left(\sqrt{2}6l\right)^N\left(\sqrt{2}\delta\right)^{\eta} < 1$ for $\eta = N$. It follows from (5.35) that

$$d_f(\mathcal{A}) \leq 2N \leq K(\omega+\xi)^{\frac{n}{2}},$$

with $K = 2D^{\frac{n}{2}}$. $\square$

Remark 5.7. We note that if f is locally Lipschitz, then it is Lipschitz on any interval $[-b, b]$. We allow the function $f(s)$ to be not Lipschitz (and even discontinuous) for $s \notin [-a, a]$.

References

[1] F. Andreu, J.M. Mazón, F. Simondon, J. Toledo, *Attractor for a degenerate nonlinear diffusion problem with nonlinear boundary condition,* J. Dynamics Differential Equations **10** (1998), 347-377.

[2] J.M.Arrieta, A.N.Carvalho, A.Rodríguez-Bernal, *Attractors of parabolic problems with nonlinear boundary conditions. Uniform bounds*, Commun. Partial Differential Equations **25** (2000), 1-37.

[3] A.V. Babin , M.I. Vishik *Attractors of evolution partial differential equations and estimation of their dimension* (in russian), Uspekhi Mat. Nauk **38** (1983), 133-187 (English translation in Russian Math. Surveys **38** (1983), 151-213).

[4] A.V. Babin, M.I. Vishik, *Attracteurs maximaux dans les équations aux dérivées partielles*, in "Nonlinear partial differential equations and their applications, Collegue de France Seminar", Vol. 7, Research Notes in Math. $n^0$122, Pitman, 1985, 11-34.

[5] A.V. Babin, M.I. Vishik, *Attractors of Evolution Equations*, (Nord Holland, Amsterdam, 1992).

[6] F. Balibrea, J. Valero, *Estimates of dimension of attractors of reaction-diffusion equations in the nondifferentiable case*, C.R.Acad.Sci, Paris, Série I **325** (1997), 759-764.

[7] F. Balibrea, J. Valero, *On dimension of attractors of differential inclusions and reaction-diffusion equations*, Discrete Cont. Dynam. Systems **5** (1999), 515-528.

[8] V. Barbu, *Nonlinear semigroups and differential equations in Banach spaces*, (Editura Academiei, Bucuresti, 1976).

[9] H. Brezis, *Problèmes unilatéraux*, J. Math. Pures Appl. **51** (1972), 1-168.

[10] H. Brezis, *Análisis Funcional*, (Alianza Editorial, Madrid, 1984) (Translated from *Analyse fonctionalle*, (Masson Editeur, Paris, 1983)).

[11] A.N. Carvalho, S.M. Oliva, A.L. Pereira, A. Rodriguez-Bernal, *Attractors for parabolic problems with nonlinear boundary conditions*, J. Math. Anal. Appl, **207** (1997), 409-461.

[12] A.N. Carvalho, J.G. Ruas-Filho, *Global attractors for parabolic problems in fractional power spaces,* SIAM J.Math.Anal. **26** (1995), 415-427.

[13] V.V. Chepyzhov, M.I. Vishik, *Attractors of non-autonomous dynamical systems and their dimension*, J. Math. Pures Appl. **73** (1994), 279-333.

[14] M.G. Crandall, A. Pazy, *Nonlinear evolution equations in Banach spaces*, Israel J. Math. **11** (1972), 57-94.

[15] L. Dung, *Dissipativity and global attractors for a class of quasilinear parabolic systems,* Commun. Partial Differential Equations **22** (1997), 413-433.

[16] L. Dung, *Global attractors and steady state solutions for a class of reaction-diffusion systems,* J. Differential Equations **147** (1998), 1-29.

[17] A.Eden, B. Michaux, J.M. Rakotoson, *Doubly nonlinear parabolic-type equations as dynamical systems,* J. Dynamics Differential Equations **3** (1991), 87-131.

[18] A. Eden, J.M. Rakotoson, *Exponential attractors for a doubly nonlinear equation*, J. Math. Anal. Appl. **185** (1994), 321-339.

[19] M. Gobbino, M. Sardella, *On the connectedness of attractors for dynamical systems*, J. Differential Equations **133** (1997), 1-14.

[20] J.K. Hale, *Asymptotic Behavior of Dissipative Systems*, Math. Surv. Mono. 25 (AMS, Providence, 1988).

[21] A. Haraux, *Attractors of asymptotically compact processes and applications to nonlinear partial differential equations*, Comm. Partial Differential Equations **13** (1988), 1383-1414.

[22] O.A. Ladyzhenskaya, *Attractors for Semigroups and Evolution Equations*, (Cambridge University Press, Cambridge, New York, 1991).

[23] O.A. Ladyzhenskaya, *Some comments to my papers on the theory of attractors for abstract semigroups*, Zap. Nauchn. Sem. LOMI **182** (1990), 102-112 (English translation in J. Soviet Math. **62** (1992), 1789-1794).

[24] J.L. Lions, E. Magenes, *Non-homogeneus boundary value problems and applications*, (Springer-Verlag, Berlin, 1972).

[25] M. Marion, *Attractors for reaction-diffusion equations: existence and estimate of their dimension*, Appl. Anal. **25** (1987), 101-147.

[26] M. Marion, *Finite-dimensional attractors associated with partly dissipative reaction-diffusion systems*, SIAM J. Math. Anal. **20** (1989), 816-844.

[27] S. Merino, *On the existence of the compact global attractor for semilinear reaction diffusion systems on* $\mathbb{R}^n$, J. Differential Equations **132** (1996), 87-106.

[28] V.S. Melnik, J. Valero, *On attractors of multivalued semi-flows and differential inclusions,* Set-Valued Analysis **6** (1998), 83-111.

[29] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics,* (Springer-Verlag, New York, 1988).

[30] J. Valero, *On evolutionary inclusion attractors in Banach spaces*, Ukrain. Mat. Zh. **47** (1995), 602-610.

[31] J. Valero, *On locally compact attractors of dynamical systems*, J. Math. Anal. Appl. **237** (1999), 43-54.

[32] J. Valero, *Attractors of parabolic equations without uniqueness*, J. Dynamics Differential Equations **13** (2001).

[33] B. Wang, *Attractors for reaction-diffusion equations in unbounded domains*, Physica D **128** (1999), 41-52.

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J.Valero
Universidad Cardenal Herrera CEU
Campus de Elche
C/Comissari, n 3
03203 Elche(Alicante), Spain
E-mail address: valer.el@ceu.es