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Homeomorphisms on compact metric spaces with finite derived length

V. Kannan [a] and Sharan Gopal [b]

[a] School of Mathematics and Statistics, University of Hyderabad, Hyderabad, India. (vksm@uohyd.ernet.in)
[b] Department of Mathematics, BITS-Pilani, Hyderabad campus, Hyderabad, India. (sharanraghu@gmail.com)

Abstract

The sets of periodic points of self homeomorphisms on an ordinal of finite derived length are characterised, thus characterising the same for homeomorphisms on compact metric spaces with finite derived length. A partition of ordinal is introduced to study this problem which is also used to solve two more problems: one about an equivalence relation and the other about a group action, both on an ordinal of finite derived length.

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1. Introduction

Periodicity is one of the important properties that are well studied in Dynamical systems. The study of sets of periodic points for a class of dynamical systems has been an interesting one in the literature (See [1], [3]). Some recent results characterise these sets for toral automorphisms and solenoids (See [12], [11]). In this paper, we do this for all self-homeomorphisms on compact metric spaces with finite derived length. It can be proved that a compact metric space with finite derived length is countable and a countable compact metric space is homeomorphic to a countable ordinal (See [6]). So the sets of periodic points for self homeomorphisms on an ordinal with finite derived length are characterised here.

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Another main result is on the group actions. Given a topological space X, the study of action of group of homeomorphisms on it is well studied in the literature (See [5]). Here we consider the actions of a particular kind of subgroups of this group of homeomorphisms on metric spaces. It is proved that the separable metric spaces with finite derived length have finitely many orbits under the action of all these subgroups.

An account of some preliminaries and notations that will be used in the paper is given in this and the next paragraph. By a dynamical system (X, f), we mean a topological space X together with a continuous self map f on X. The set $\{f^l(x) : l \in \mathbb{N}_0\}$ is called the orbit of $x \in X$. A subset $A \subset X$ is said to be forward $f-$invariant if $f(A) \subset A$ and backward $f-$invariant if $f^{-1}(A) \subset A$. A is said to be $f-$invariant (or just invariant, when the context is clear) if A is both forward $f-$invariant and backward $f-$invariant.

If A and B are two ordered sets, then we define a relation, saying that $A \sim B$ if there is an order preserving bijection from A to B. This is an equivalence relation and each equivalence class (in fact a representative of this class) is called an ordinal number. Hereafter, by an ordinal number, we actually refer to a representative of the corresponding class. This makes it easy to convey the idea of order among the ordinals. If α and β are two ordinal numbers such that there is an order preserving bijection from α to a subset of β, then define $\alpha \leq \beta$. This relation "$\leq$" is an order on the set of ordinal numbers according to which, this is well ordered. Note that any non-negative integer can be regarded as an ordinal and such ordinals are called as finite ordinals and the least infinite ordinal is denoted by ω (which is equivalent to the set of all non-negative integers).

On an ordered set X, with more than one element, let B be the collection of all sets of the following types: (i) all open intervals (a, b) in X, (ii) all intervals of the form $[a_0, b)$, where a_0 is the smallest element (if any) of X and (iii) all intervals of the form $(a, b_0]$, where b_0 is the largest element (if any) of X. The collection B is a basis for a topology on X, which is called the order topology. In this paper, we consider the order topology on ordinal spaces. The set of limit points of an ordinal α is precisely the set of *limit ordinals* less than α. Zero and the successor ordinals less than α are isolated points in α. It can be seen that any ordinal of finite derived length can be written as $\omega^n \cdot m_n + \omega^{n-1} \cdot m_{n-1} + ... + \omega \cdot m_1 + m_0$, $m_i \in \mathbb{N}_0$. Hereafter, we use the notation $O^{(n)} = \omega^n \cdot m_n + \omega^{n-1} \cdot m_{n-1} + ... + \omega \cdot m_1 + m_0$. The notions of limit ordinal and arithmetic of ordinals can be seen in [10]. We also use the notation $|W|$ to denote the cardinality of a set W.

2. Partition of $O^{(n)}$

Here, a partition is induced on the ordinal $O^{(n)}$. The definition of this partition involves only elementary operations: intersection, complementation and formation of derived set. This partition is done with respect to a given subset of $O^{(n)}$ i.e., every subset $S \subset O^{(n)}$ gives rise to a partition $\mathcal{P}_S$ of $O^{(n)}$.

This is finer than the partition of $O^{(n)}$ in to the different levels of limit points. To define the partition $\mathcal{P}_S$, the following notations are introduced.

1. Let X be a topological space. For a subset $Z \subset X$, $D(Z)$ denotes the set of limit points of Z in X and for every $n \in \mathbb{N}$, define inductively $D^{n+1}(Z) = D(D^n(Z))$. Then, for $k \in \mathbb{N}$, denote by X_k, the set $D^k(X) \setminus D^{k+1}(X)$ and let X_0 denote the set of isolated points of X. X_k is called the set of k^{th} *level limit points* of X in X for every $k \in \mathbb{N}_0$.

2. Define a sequence $\mathcal{J}^{(l)}$, $l \in \mathbb{N}_0$ inductively as follows :
$\mathcal{J}^{(0)} = \{a, b\}$, $\mathcal{J}^{(l+1)} = (\mathcal{P}(\mathcal{J}^{(l)}) \setminus \{\varnothing\}) \times \mathcal{J}^{(0)}$ ($\mathcal{P}(X)$ stands for the set of all subsets of X).

Remark 2.1.

(1) Here, a and b can be any two distinct symbols. What is actually needed to define the partition is a set containing two elements, which we denote by $\mathcal{J}^{(0)}$.
(2) The letters P and $\mathcal{P}$ are used in three different contexts. $\mathcal{P}_S$ denotes the partition associated with the set S, $\mathcal{P}(X)$ stands for the collection of all subsets of X and $P(h)$ will be used to denote the set of periodic points of h.

The partition $\mathcal{P}_S$: Let $S \subset O^{(n)}$, $n \in \mathbb{N}$. For every $V \in \bigcup_{l=0}^{n} \mathcal{J}^{(l)}$, we associate a subset S_V of $O^{(n)}$ in the following way. Note that these S_V's are defined recursively.
Define $S_a = (O^{(n)} \setminus S) \cap (O^{(n)})_0$ and $S_b = S \cap (O^{(n)})_0$. If $V \in \mathcal{J}^{(k+1)}$ for some $0 \leq k \leq n-1$ and say $V = (W, i)$ ($i \in \{a, b\}$) for some $W \subset \mathcal{J}^{(k)}$, then define $S_V = [(\bigcap_{A \in W} \overline{S_A}) \setminus (\bigcup_{A \in \mathcal{J}^{(k)} \setminus W} \overline{S_A})] \cap S^i \cap (O^{(n)})_{k+1}$,
where $S^i = \begin{cases} O^{(n)} \setminus S, & i = a \\ S, & i = b \end{cases}$.

Remark 2.2. In the above partition, $O^{(n)}$ is partitioned in to $(O^{(n)})_k$'s and each $(O^{(n)})_k$ is partitioned into $S \cap (O^{(n)})_k$ and $(O^{(n)} \setminus S) \cap (O^{(n)})_k$. This partition is further refined in such a way that the common limit points of some and only those partition classes in $(O^{(n)})_{k-1}$ constitute the partition classes of $(O^{(n)})_k$; these are denoted by S_V, $V \in \bigcup_{l=0}^{n} \mathcal{J}^{(l)}$. In other words, for every partition class $S_V \subset (O^{(n)})_k$, there exist partition classes $S_{V_1}, S_{V_2}, \ldots S_{V_m}$ (m depends on V) in $(O^{(n)})_{k-1}$ such that S_V is precisely the set $\cap_{i=1}^{m} D(S_{V_i})$.

Thus every subset $S \subset O^{(n)}$ gives rise to a partition $\mathcal{P}_S$ of $O^{(n)}$. This is finer than the partition of $O^{(n)}$ in to the different levels of limit points. $\mathcal{P}_S$ helps to prove the following four different results.

- A subset $S \subset O^{(n)}$ arises as the set of periodic points of a homeomorphism on $O^{(n)}$ if and only if every finite partition class in $\mathcal{P}_S$ is contained in S.
- Using this partition $\mathcal{P}_S$, it is proved that in a separable metric space X with finite derived length, there are finitely many orbits under the

action of the group $G_S = \{h : X \to X : h$ is a homeomorphism such that $h(S) = S\}$, for any $S \subset X$.

- $\mathcal{P}_S$ is the smallest partition of ω^n such that
  (i)S is a union of partition classes and
  (ii)if $A \subset \omega^n$ is a union of partition classes, then $D(A)$ is also a union of partition classes.
  $\mathcal{P}_S$ is the smallest partition in the sense that any finer refinement of $\mathcal{P}_S$ will not satisfy the above conditions (i) and (ii).
- Starting from a subset $S \subset \omega^n$, go on forming many other sets using the operation of derived set, complement and union i.e, S, $\omega^n \setminus S$, $\overline{S}$, $\overline{\omega^n \setminus S}$ and so on. In other words, if n sets are already formed, the $(n+1)^{th}$ set can be the derived set of any of the n sets, or the complement of any of the n sets, or the union of any two of these n sets. Among the distinct subsets that can be formed in this way, the minimal ones are precisely the partition classes in $\mathcal{P}_S$ i.e., in the collection of sets formed as above, there is no set which is strictly contained in any of the partition classes of the partition $\mathcal{P}_S$.

The first two results form the main part of this paper and the last two follow from these. The last result is similar to, and is a natural sequel to the closure-complementation problem described in [7] and further investigated in [4].

3. Sets of periodic points

3.1. Invariance of partition classes.

Proposition 3.1. *If X is a topological space with finite derived length n and h is a homeomorphism on X, then X_k is $h-$invariant for every $0 \leq k \leq n$.*

Proof. Since, being an isolated point is a topological property, X_0 is $h-$invariant for any homeomorphism h on X. For any $1 \leq m < n$, by the definition of X_{m+1}, there is no sequence in $\cup_{i=m+1}^{n} X_i$ that converges to a point in X_{m+1}; because, the limit of such a sequence would be a limit point of a level larger than $m+1$. Thus the points of X_{m+1} consists of the isolated points of $X \setminus \cup_{i=0}^{m} X_i$ i.e., $X_{m+1} = (X \setminus \cup_{i=0}^{m} X_i)_0$. In particular, $X_1 = (X \setminus X_0)_0$. Since X_0 is $h-$invariant, h is a self homeomorphism on $X \setminus X_0$. Thus by the above argument, it follows that X_1 is $h-$invariant. Now let $1 \leq k < n$. Suppose that X_l is $h-$invariant for every $0 \leq l \leq k$. Then, h is a self homeomorphism on $X \setminus \bigcup_{i=0}^{k} X_i$ and since $X_{k+1} = (X \setminus \cup_{i=0}^{k} X_i)_0$, X_{k+1} is $h-$invariant. Hence X_k is $h-$invariant for every $0 \leq k \leq n$. □

Theorem 3.2. *If $S \subset O^{(n)}$ is $h-$invariant for some homeomorphism h on $O^{(n)}$, then S_V is $h-$invariant $\forall V \in \bigcup_{k=0}^{n} \mathcal{J}^{(k)}$.*

Proof. From the above proposition, it is clear that S_a and S_b are $h-$invariant. We now prove that if S_W is $h-$invariant $\forall W \in \mathcal{J}^{(k)}$ for some $0 \leq k \leq n-1$, then S_V is $h-$invariant $\forall V \in \mathcal{J}^{(k+1)}$.

Consider a partition class S_V, where $V \in \mathcal{J}^{(k+1)}$. Then $V = (W, i)$ for some $W \subset \mathcal{J}^{(k)}$ and $i \in \{a, b\}$. By definition, $S_V = (\bigcap_{A \in W} \overline{S_A} \setminus \bigcup_{A \in \mathcal{J}^{(k)} \setminus W} \overline{S_A}) \cap$

$S^i \cap (O^{(n)})_{k+1}$. Since h is a homeomorphism, observe that for any $A \in \mathcal{J}^{(k)}$, $h(\overline{S_A}) = \overline{h(S_A)} = \overline{S_A}$. Then $h(\bigcap_{A\in W} \overline{S_A}) = \bigcap_{A\in W} \overline{S_A}$ and $h(\bigcup_{A\in\mathcal{J}^{(k)}\setminus W} \overline{S_A}) = \bigcup_{A\in\mathcal{J}^{(k)}\setminus W} \overline{S_A}$. Hence S_V is $h-$invariant. $\square$

3.2. **Periodic points.**

Lemma 3.3. *Let $n \in \mathbb{N}$ and $S \subset \omega^{n+1}$. If $1 \le k \le n$ and $(\omega^{n+1} \setminus S) \cap S_A$ is either empty or infinite for every $A \in \mathcal{J}^{(k-1)} \cup \mathcal{J}^{(k)}$ then any bijection f on $(\omega^{n+1})_k$ such that*

1. *f has no orbit of even length*
2. *S_V is $f-$invariant $\forall V \in \mathcal{J}^{(k)}$ and*
3. *$P(f) = S \cap (\omega^{n+1})_k$*

can be extended to a homeomorphism h on $(\omega^{n+1})_{k-1} \cup (\omega^{n+1})_k$ such that $P(h) = S \cap ((\omega^{n+1})_{k-1} \cup (\omega^{n+1})_k)$.

Proof. f being a bijection on $(\omega^{n+1})_k$ with no orbits of even length, defines three types of orbits in $(\omega^{n+1})_k$:

1. an infinite orbit ($\{x_l : l \in \mathbb{Z}$ and $f(x_i) = x_{i+1}\}$),
2. a periodic orbit with odd length greater than 1 ($\{x_l : -m \le l \le m$ for some fixed $m \in \mathbb{N}$; $f(x_i) = x_{i+1}\ \forall i < m$ and $f(x_m) = x_{-m}\}$) and
3. a singleton orbit ($\{x : f(x) = x\}$) i.e., a fixed point.

Since $P(f) = S \cap (\omega^{n+1})_k$, $(\omega^{n+1})_k \setminus S$ is a union of infinite orbits and $S \cap (\omega^{n+1})_k$ is a union of finite orbits.

We now define a homeomorphism h on $(\omega^{n+1})_{k-1} \cup (\omega^{n+1})_k$ with $P(h) = S \cap ((\omega^{n+1})_{k-1} \cup (\omega^{n+1})_k)$ in such a way that $h = f$ on $(\omega^{n+1})_k$ and moreover the orbit of each point in $(\omega^{n+1})_{k-1}$ falls in to one of the three types. To cover the general case, we assume that each point in $(\omega^{n+1})_k$ is in both $\overline{S}$ as well as $\overline{(\omega^{n+1})_{k-1} \setminus S}$; and the same method works for other cases also.

We first define maps on some clopen neighbourhoods of points of $(\omega^{n+1})_k$. This is done in one of the three different ways depending on the type of orbit in which the point lies.

Type I : $\{x_i : i \in \mathbb{Z}$ and $f(x_i) = x_{i+1}\}$
Here, $x_i \in S_V\ \forall i \in \mathbb{Z}$ for some $V \in \mathcal{J}^{(k)}$. Let B_i and B'_i be two deleted clopen neighborhoods of x_i (i.e., $x_i \notin B_i \cup B'_i$ and $B_i \cup \{x_i\}$, $B'_i \cup \{x_i\}$ are clopen neighborhoods of x_i) such that $B_i \subset S$ and $B'_i \subset (\omega^{n+1} \setminus S)$. Choose B_i's and B'_i's in such a way that $B_i \cap S_W$ and $B'_i \cap S_W$ are either empty or infinite $\forall W \in \mathcal{J}^{(k-1)}$. Since all the x_i's are in the same S_V, we can assume that for any $W \in \mathcal{J}^{(k-1)}$, $|B_i \cap S_W| = |B_j \cap S_W|$ and $|B'_i \cap S_W| = |B'_j \cap S_W|$, $\forall i, j \in \mathbb{Z}$. Say $B_i \cap (\omega^{n+1})_{k-1} = \{x_{ij} : j \in \mathbb{N}\}$ and $B'_i \cap (\omega^{n+1})_{k-1} = \{x'_{ij} : j \in \mathbb{N}\}$.
Define a bijection h_I on $(\bigcup_{i\in\mathbb{Z}}(B_i \cup B'_i)) \cap (\omega^{n+1})_{k-1}$ as

$$h_I(x_{ij}) = \begin{cases} x_{(i+1)j} & \text{if } -j \le i < j \\ x_{(-j)j} & \text{if } i = j \\ x_{ij} & \text{if } |i| > j \end{cases}$$

and $h_I(x'_{ij}) = x'_{(i+1)j}$.
Extend h_I to $(\bigcup_{i\in\mathbb{Z}} B_i) \bigcup (\bigcup_{i\in\mathbb{Z}} B'_i) \cup \{x_i : i \in \mathbb{Z}\}$ by defining $h_I(x_i) = f(x_i)$.

Type II : $\{x_i : -m \leq i \leq m$ for some fixed $m \in \mathbb{N}$ and $f(x_i) = x_{i+1}\ \forall i < m$ and $f(x_m) = x_{-m}\}$
Let B_i and B'_i be two deleted clopen neighborhoods of x_i such that $B_i \subset S$ and $B'_i \subset (\omega^{n+1} \setminus S)$. Choose B_i's and B'_i's in such a way that $B_i \cap S_W$ and $B'_i \cap S_W$ are either empty or infinite $\forall W \in \mathcal{J}^{(k-1)}$. Here again, $x_i \in S_V\ \forall -m \leq i \leq m$ for some $V \in \mathcal{J}^{(k)}$. Also, $|B_i \cap S_W| = |B_j \cap S_W|$ and $|B'_i \cap S_W| = |B'_j \cap S_W|$, $\forall -m \leq i, j \leq m$ and for any $W \in \mathcal{J}^{(k-1)}$. Say $B_i \cap (\omega^{n+1})_{k-1} = \{x_{ij} : j \in \mathbb{N}\}$ and $B'_i \cap (\omega^{n+1})_{k-1} = \{x'_{ij} : j \in \mathbb{N}\}$.
Define a bijection h_{II} on $[\bigcup_{-m \leq i \leq m}(B_i \cup B'_i)] \cap (\omega^{n+1})_{k-1}$ as $h_{II}(x_{ij}) = x_{\psi(i)j}$ and
$h_{II}(x'_{ij}) = x'_{\psi(i)\phi(j)}$, where ψ is a bijection on $\{-m, -m+1, .., 0, 1, ..., m\}$ defined as
$\psi(i) = \begin{cases} i+1 & \forall i < m \\ -m & \text{if } i = m \end{cases}$
and $\phi : \mathbb{N} \to \mathbb{N}$ is defined as $\phi(2i-1) = 2i+1\ \forall i \in \mathbb{N}$, $\phi(2i) = 2i-2\ \forall i \geq 2$ and $\phi(2) = 1$.
Extend h_{II} to $(\bigcup_{-m \leq i \leq m} B_i) \bigcup (\bigcup_{-m \leq i \leq m} B'_i) \cup \{x_i : -m \leq i \leq m\}$ by defining $h_{II}(x_i) = f(x_i)$.
Type III : $\{x : f(x) = x\}$ (i.e., x is a fixed point).
Let B and B' be two deleted clopen neighborhoods of x such that $B \subset S$ and $B' \subset (\omega^{n+1} \setminus S)$. Say $B = \{x_i : i \in \mathbb{N}\}$ and $B' = \{x'_i : i \in \mathbb{N}\}$. Define h_{III} on $(B \cup B') \cap (\omega^{n+1})_{k-1}$ as $h_{III}(x_i) = x_i$ and $h_{III}(x'_i) = \phi(x'_i)$, where ϕ is defined as above. Extend h_{III} to $B \cup B' \cup \{x\}$ by defining $h_{III}(x) = x$.

The neighborhoods considered above will form a partition of a clopen set, say $X \subset (\omega^{n+1})_{k-1}$ and we can assume that for every $W \in \mathcal{J}^{(k-1)}$, $S_W \subset X$ or $S_W \subset (\omega^{n+1})_{k-1} \setminus X$. Thus $(\omega^{n+1})_{k-1} \setminus X$ is a union of S_U's for some $U \in \mathcal{J}^{(k-1)}$ such that $D(S_U) = \varnothing$. Thus any bijection on $(\omega^{n+1})_{k-1} \setminus X$ will be a homeomorphism on it. So, we define a bijection on $(\omega^{n+1})_{k-1} \setminus X$ so that S_U is a single infinite orbit (type I) if $S_U \subset (\omega^n \setminus S)$ or otherwise each point of S_U is fixed. Since the neighborhoods defined above for the three types form a partition of X,, the domains of h_I, h_{II} and h_{III} are mutually disjoint and further each of them is a clopen set. Hence h_I, h_{II} and h_{III} can be pasted to get a homeomorphism on X and by pasting this homeomorphism and the bijection on $(\omega^{n+1})_{k-1} \setminus X$, we get a homeomorphism h on $(\omega^{n+1})_k \cup (\omega^{n+1})_{k-1}$ such that $P(h) = S \cap ((\omega^{n+1})_{k-1} \cup (\omega^{n+1})_k)$. □

Remark 3.4. Similar to the bijection f, the extended homeomorphism h satisfies the following :

(1) h has no orbit of even length
(2) S_V is $h-$invariant $\forall V \in \mathcal{J}^{(k)} \cup \mathcal{J}^{(k-1)}$ and
(3) $P(h) = S \cap ((\omega^{n+1})_{k-1} \cup (\omega^{n+1})_k)$.

Theorem 3.5. *Let $S \subset \omega^n$ for some $n \in \mathbb{N}$. The following are equivalent:*

(1) *$S = P(h)$ for some self-homeomorphism h on ω^n.*
(2) *$S_V \cap (\omega^n \setminus S)$ is either empty or infinite for every $V \in \bigcup_{k=0}^{n-1} \mathcal{J}^{(k)}$.*

Proof. If $S = P(h)$, then S is $h-$invariant and thus by Theorem 3.2, S_V is $h-$invariant for every V. Also, either $S_V \subset S$ or $S_V \subset (\omega^n \setminus S)$. Since a finite non-empty invariant set certainly contains a periodic point, it follows that $S_V \cap (\omega^n \setminus S)$ is either empty or infinite for every V.

For the converse, let $S \subset \omega^n$ such that $S_V \cap (\omega^n \setminus S)$ is either empty or infinite for every $V \in \bigcup_{k=0}^{n-1} \mathcal{J}^{(k)}$.
Suppose $V_1,\ V_2, ..., V_m \in \mathcal{J}^{(n-1)}$ such that $(\omega^n)_{n-1} \setminus S = \bigcup_{j=1}^{m} S_{V_j}$ and $S_{V_j} \neq \varnothing$ $\forall j \in \{1, 2, ..., m\}$. Say $S_{V_j} = \{x_{jl} : l \in \mathbb{Z}\}$.
Define $f : (\omega^n)_{n-1} \to (\omega^n)_{n-1}$ as $f(x) = \begin{cases} x & \text{if } x \in S \\ x_{j(l+1)} & \text{if } x = x_{jl} \in (\omega^n)_{n-1} \setminus S \end{cases}$.
If m does not exist i.e., if there is no $V \in \mathcal{J}^{(n-1)}$ such that $S_V \subset (\omega^n \setminus S)$, then define $f(x) = x\ \forall x \in (\omega^n)_{n-1}$. Then f is a bijection on $(\omega^n)_{n-1}$.

If $n = 1$, then f is a homeomorphism on ω such that $P(f) = S$. Otherwise, using the above Lemma, this f can be extended to a homeomorphism h_{n-2} on $(\omega^n)_{n-1} \cup (\omega^n)_{n-2}$ such that $P(h_{n-2}) = S \cap ((\omega^n)_{n-1} \cup (\omega^n)_{n-2})$.

h_{n-2} defines a bijection on $(\omega^n)_{n-2}$ which can be further extended to a homeomorphism h_{n-3} on $(\omega^n)_{n-2} \cup (\omega^n)_{n-3}$ such that $P(h) = S \cap ((\omega^n)_{n-2} \cup (\omega^n)_{n-3})$. By pasting h_{n-2} and h_{n-3}, we get a homeomorphism on $(\omega^n)_{n-1} \cup (\omega^n)_{n-2} \cup (\omega^n)_{n-3}$ with $S \cap ((\omega^n)_{n-1} \cup (\omega^n)_{n-2} \cup (\omega^n)_{n-3})$ as the set of periodic points. Continuing this way, we get a homeomorphism h on ω^n with $P(h) = S$. □

Now, we have the main theorem:

Theorem 3.6. *The following are equivalent for a subset $S \subset O^{(n)}$:*

(1) *$S = P(h)$ for some self-homeomorphism h on $O^{(n)}$.*
(2) *$S_V \cap (O^{(n)} \setminus S)$ is either empty or infinite for every $V \in \bigcup_{k=0}^{n} \mathcal{J}^{(k)}$.*

Proof. The first part of the proof is same as that in the above proof. For the converse, recall that $O^{(n)} = \omega^n \cdot m_n + \omega^{n-1} \cdot m_{n-1} + ... + \omega \cdot m_1 + m_0$ where $m_i \in \mathbb{N}_0$. Here, the highest level of limit points is n and since $(O^{(n)})_n$ is a finite set, $(O^{(n)})_n \subset S$. The rest of the proof follows by taking f to be the identity map on $(O^{(n)})_n$ in Lemma 3.3 and also using the ideas of Theorem 3.5. □

4. An equivalence class and a group action

Definition 4.1. Let Z be a topological space and $S \subset Z$. Let $x,\ y \in Z$.
x is said to be topologically same as y in Z with respect to S if there exists a homeomorphism h on Z such that $h(S) = S$ and $h(x) = y$.

Given $S \subset Z$, this definition induces an equivalence relation R_S on Z as : $x\ R_S\ y$ if x is topologically same as y with respect to S. It is easy to see that this is an equivalence relation on Z and the following theorem describes the equivalence classes of $O^{(n)}$.

Theorem 4.2. *The family $\{S_V : V \in \bigcup_{l=0}^{n} \mathcal{J}^{(l)},\ S_V \neq \varnothing\}$ is the set of equivalence classes of $O^{(n)}$ with respect to R_S.*

Proof. From the proof of Theorem 3.2, it follows that S_V is invariant for every $V \in \bigcup_{l=0}^{n} \mathcal{J}^{(l)}$, under any homeomorphism h on $O^{(n)}$ such that $h(S) = S$. So if $V \neq V'$, then $x \in S_V$ and $y \in S_{V'}$ are not related.
Now, let $x, y \in S_V$ for $V \in \mathcal{J}^{(k)}$ for some $k \in \{0, 1, ..., n\}$. Define a bijection h on $(O^n)_k$ such that $h(z) = \begin{cases} z & \text{if } z \notin \{x, y\} \\ y & \text{if } z = x \\ x & \text{if } z = y \end{cases}$.
Using the ideas of the proof of Lemma 3.3, this map can be extended to a homeomorphism h on $\bigcup_{i=0}^{k}(O^{(n)})_i$ such that

$$h(S \cap (\bigcup_{i=0}^{k}(O^{(n)})_i)) = S \cap (\bigcup_{i=0}^{k}(O^{(n)})_i),$$

which can be further extended to $O^{(n)}$ by defining

$$h(z) = z, \forall z \in O^{(n)} \setminus \bigcup_{i=0}^{k}(O^{(n)})_i.$$

Hence the family $\{S_V : V \in \bigcup_{k=0}^{n} \mathcal{J}^{(k)},\ S_V \neq \varnothing\}$ is the set of equivalence classes. □

If a group G is acting on a set X and $x \in X$, then the set $Gx = \{gx : g \in G\}$ is called the $G-$orbit of x. When a finite group acts on a finite set, one natural problem is to count the number of orbits. Burnside's lemma (see [2]) and Polya's theorem (see [9]) are in this direction. Occasionally, there are some infinite groups acting on infinite sets, but having only finitely many orbits. The problem of the present section has such background.

To every topological space X, we can associate a natural group, namely the group of self homeomorphisms on it. Here, we consider a subgroup of this group. Given a subset S of a topological space X, let $G_S = \{f : X \to X : f$ is a homeomorphism on X such that $f(S) = S\}$, i.e., the collection of self-homeomorphisms on X under which S is invariant. It can be easily seen that G_S is a group.

The following theorem gives a neat description of the G_S-orbits in $O^{(n)}$ in terms of S_V's. Its proof follows from Theorem 4.

Theorem 4.3. *The family $\{S_V : V \in \bigcup_{k=0}^{n} \mathcal{J}^{(k)},\ S_V \neq \varnothing\}$ is the collection of G_S-orbits in $O^{(n)}$ for any subset $S \subset O^{(n)}$.*

It follows from the above theorem that a compact metric space X with finite derived length has finitely many G_S-orbits for any subset $S \subset X$. In fact, any separable metric space with finite derived length has this property. This is proved in the following theorem.

Theorem 4.4. *If X is a separable metric space with finite derived length, then X has finitely many G_S-orbits for every subset $S \subset X$.*

Proof. Let X be a separable metric space of finite derived length. If X is compact, then the result follows from Theorem 4.3.
Now, suppose X is not compact. Since X is separable, it is known that X is homeomorphic to a subspace of ω^n for some n (See [8]). So, it is enough

to consider the number of G_S-orbits in X, where $S \subset X \subset \omega^n$. Let $H = \{h : \omega^n \to \omega^n : h(X) = X \text{ and } h(S) = S\}$. Then H is isomorphic to a subgroup of G_S. Thus the number of G_S-orbits in X cannot exceed the number of $H-$orbits in ω^n. It follows from Theorem 4.3 and the proof of Theorem 4.2, that the non-empty members of the family $\{S \cap X_V : V \in \bigcup_{k=0}^n \mathcal{J}^{(k)}\} \cup \{(\omega^n \setminus S) \cap X_V : V \in \bigcup_{k=0}^n \mathcal{J}^{(k)}\}$ are precisely the collection of $H-$orbits in ω^n. So, the number of $H-$orbits is atmost twice the number of X_V's, which are finite in number. Thus there are finitely many $H-$orbits in ω^n and hence finitely many G_S-orbits in X.

□

5. Conclusion

We conclude this article by posing a question. The problem of characterising the sets of periodic points of homeomorphisms on ω^n leads to a very natural and interesting question of characterising the same sets for continuous maps on ω^n. In another direction, the same problem can be extended to the problem of characterising the sets of periodic points of homeomorphisms on ordinals of infinite derived length. It is hoped that the ideas in this paper will be useful in discussing the later question.

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