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On the generalized asymptotically nonspreading mappings in convex metric spaces

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Abstract

In this article, we propose a new class of nonlinear mappings, namely, generalized asymptotically nonspreading mapping, *and prove the existence of fixed points for such mapping in convex metric spaces. Furthermore, we also obtain the demiclosed principle and a* Δ*-convergence theorem of Mann iteration for generalized asymptotically nonspreading mappings in CAT(0) spaces.*

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1. Introduction

Throughout this paper, we denote the set of positive integers by $\mathbb{N}$. Let T be a mapping on a nonempty subset C of a Banach space X. We denote by $F(T)$ the set of fixed points of T, *i.e.*, $F(T) = \{x \in C : Tx = x\}$.

In 2008, Kohsaka and Takahashi [14] introduced a nonlinear mapping called *nonspreading mapping* in a smooth, strictly convex, and reflexive Banach space X as follows: Let C be a nonempty closed convex subset of X. A mapping

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$T : C \to C$ is said to be *nonspreading* if

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(Ty, x) \tag{1.1}$$

for all $x, y \in C$, where $\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ for all $x, y \in X$ and J is the duality mapping on C. Observe that if X is a real Hilbert space, then J is the identity mapping and $\phi(x, y) = \|x - y\|^2$ for all $x, y \in X$. So, a nonspreading mapping T in a real Hilbert space X is defined as follows:

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2$$

for all $x, y \in C$.

Since then, some fixed point theorems of such mapping has been studied by many researchers; see, for example, [9, 10, 11].

Later in 2013, Naraghirad [17] introduced a new class of nonspreading-type mappings in a real Banach space, called an *asymptotically nonspreading mapping*, as follows: A mapping $T : C \to C$ is called *asymptotically nonspreading* if

$$\|T^n x - T^n y\|^2 \leq \|x - y\|^2 + 2\langle x - T^n x, J(y - T^n y)\rangle \tag{1.2}$$

for all $x, y \in C$ and $n \in \mathbb{N}$, where J is the normalized duality mapping of C. In the case when X is a real Hilbert space, we know that J is the identity mapping. So, an asymptotically nonspreading mapping T in a real Hilbert space X is defined as follows:

$$\|T^n x - T^n y\|^2 \leq \|x - y\|^2 + 2\langle x - T^n x, y - T^n y\rangle \tag{1.3}$$

for all $x, y \in C$ and $n \in \mathbb{N}$. In a real Hilbert space, it is easy to show that (1.3) is equivalent to

$$2\|T^n x - T^n y\|^2 \leq \|T^n x - y\|^2 + \|T^n y - x\|^2$$

for all $x, y \in C$ and $n \in \mathbb{N}$.

Naraghirad [17] proved weak and strong convergence theorems of the iterative sequences generated by an asymptotically nonspreading mapping in a real Banach space.

Motivated by the above works, we define a new class of nonlinear mappings which contains the class of asymptotically nonspreading mappings in convex metric spaces, called a *generalized asymptotically nonspreading mapping*, and prove some existence theorems for such mapping in convex metric spaces. Furthermore, we also obtain the demiclosed principle and a Δ-convergence theorem of Mann iteration for generalized asymptotically nonspreading mappings in CAT(0) spaces.

2. Preliminaries

In the sequel, we recall some definitions, notations, and conclusions which will be needed in proving our main results.

Let (X, d) be a metric space. A mapping $W : X \times X \times [0, 1] \to X$ is said to be a *convex structure* [22] on X if for each $x, y \in X$ and $\lambda \in [0, 1]$,

$$d(z, W(x, y, \lambda)) \leq \lambda d(z, x) + (1 - \lambda)d(z, y)$$

for all $z \in X$. A metric space (X, d) together with a convex structure W is called a *convex metric space* which will be denoted by (X, d, W). A nonempty subset C of X is said to be *convex* if $W(x, y, \lambda) \in C$ for all $x, y \in C$ and $\lambda \in [0, 1]$. It is easy to see that open and closed balls are convex and the intersection of a family of convex subsets of a convex metric space X is also convex, see [22]. Clearly, a normed space and each of its convex subsets are convex metric spaces, but the converse does not hold.

In 1996, Shimizu and Takahashi [21] introduced the concept of uniform convexity in convex metric spaces and studied the properties of these spaces.

Definition 2.1. A convex metric space (X, d, W) is said to be *uniformly convex* if for any $\varepsilon > 0$, there exists $\delta(\varepsilon) \in (0, 1]$ such that for all $r > 0$ and $x, y, z \in X$ with $d(z, x) \leq r$, $d(z, y) \leq r$ and $d(x, y) \geq r\varepsilon$, imply that

$$d\left(z, W\left(x, y, \frac{1}{2}\right)\right) \leq (1 - \delta(\varepsilon))\, r.$$

Obviously, uniformly convex Banach spaces are uniformly convex metric spaces.

One of the special spaces of uniformly convex metric spaces is a CAT(0) space (see more details in [3]). The useful inequality of CAT(0) space is *(CN) inequality* [4], that is, if z, x, y are points in a CAT(0) space and if m is the midpoint of the geodesic segment $[x, y]$, then the CAT(0) inequality implies

$$\text{(CN)} \qquad d(z, m)^2 \leq \frac{1}{2}d(z, x)^2 + \frac{1}{2}d(z, y)^2 - \frac{1}{4}d(x, y)^2.$$

By using the (CN) inequality, it is easy to see that CAT(0) spaces are uniformly convex. Moreover, if X is a CAT(0) space and $x, y \in X$, then for any $\lambda \in [0, 1]$, there exists a unique point $\lambda x \oplus (1 - \lambda)y \in [x, y]$ such that

$$d(z, \lambda x \oplus (1 - \lambda)y) \leq \lambda d(z, x) + (1 - \lambda)d(z, y),$$

for any $z \in X$. It follows that CAT(0) spaces have a convex structure $W(x, y, \lambda) := \lambda x \oplus (1 - \lambda)y$. Existence theorems and convergence theorems in convex metric spaces and CAT(0) spaces have been studied and investigated, see, for examples, [13, 5, 16, 12, 18, 15, 19, 1, 2].

The notion of the asymptotic center can be introduced in the general setting of a CAT(0) space X as follows: Let $\{x_n\}$ be a bounded sequence in X. For $x \in X$, we define a mapping $r(\cdot, \{x_n\}) : X \to [0, \infty)$ by

$$r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n).$$

The *asymptotic radius* of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\},$$

and the *asymptotic center* of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known by [7] that in a CAT(0) space, the asymptotic center $A(\{x_n\})$ consists of exactly one point.

We now give the definition and collect some basic properties of the Δ-convergence which will be used in the sequel.

Definition 2.2 ([13]). A sequence $\{x_n\}$ in a CAT(0) space X is said to *Δ-converge* to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_{n\to\infty} x_n = x$ and call x the Δ-limit of $\{x_n\}$.

We now collect some basic properties of the Δ-convergence which will be used in the sequel.

Lemma 2.3 ([13]). *Every bounded sequence in a CAT(0) space has a Δ-convergent subsequence.*

Lemma 2.4 ([6]). *Let C be a nonempty closed convex subset of a CAT(0) space X. If $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C.*

Lemma 2.5 ([8]). *Let $\{x_n\}$ be a sequence in a CAT(0) space X with $A(\{x_n\}) = \{x\}$. If $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and $\{d(x_n, u)\}$ converges, then $x = u$.*

The following lemma is a generalization of the (CN) inequality which can be found in [8].

Lemma 2.6. *Let X be a CAT(0) space. Then*

$$d(z, \lambda x \oplus (1-\lambda)y)^2 \leq \lambda d(z,x)^2 + (1-\lambda)d(z,y)^2 - \lambda(1-\lambda)d(x,y)^2,$$

for any $\lambda \in [0,1]$ and $x, y, z \in X$.

3. Main results

In this section, we study the existence and convergence theorems for a generalized asymptotically nonspreading mapping in both convex metric spaces and CAT(0) spaces. We first define a generalized asymptotically nonspreading mapping in convex metric spaces.

Definition 3.1. Let C be a nonempty subset of a convex metric space (X, d, W). A mapping $T : C \to C$ is called *generalized asymptotically nonspreading* if there exist two functions $f, g : C \to [0, \gamma]$, $\gamma < 1$ such that

(C1) $d(T^n x, T^n y)^2 \leq f(x)d(T^n x, y)^2 + g(x)d(T^n y, x)^2$ for all $x, y \in C$ and $n \in \mathbb{N}$;

(C2) $0 < f(x) + g(x) \leq 1$ for all $x \in C$.

Remark 3.2. The class of generalized asymptotically nonspreading mappings contains the class of asymptotically nonspreading mappings. Indeed, we know that if $f(x) = g(x) = \frac{1}{2}$ for all $x \in C$, then T is an asymptotically nonspreading mapping.

The next example shows that there is a generalized asymptotically nonspreading mapping which is not asymptotically nonspreading.

Example 3.3. Define a mapping $T : [0, \infty) \to [0, \infty)$ by

$$Tx = \begin{cases} 0.9, & \text{if } x \geq 1, \\ 0, & \text{if } x \in [0, 1). \end{cases}$$

Then T is not an asymptotically nonspreading mapping. Indeed, if $x = 1.2$ and $y = 0.7$, then $Tx = 0.9$, $Ty = 0$, and

$$2d(Tx, Ty)^2 = 1.62 > 1.48 = 0.04 + 1.44 = d(Tx, y)^2 + d(Ty, x)^2.$$

However, T is a generalized asymptotically nonspreading mapping. Indeed, let $f, g : [0, \infty) \to [0, 0.9)$ be defined by

$$f(x) = \begin{cases} 0, & \text{if } x \geq 1, \\ 0.81, & \text{if } x \in [0, 1), \end{cases}$$

and

$$g(x) = \begin{cases} 0.81, & \text{if } x \geq 1, \\ 0, & \text{if } x \in [0, 1). \end{cases}$$

Now, we only need to consider the following two cases:

(i) If $x \geq 1$ and $y \in [0, 1)$, then $Tx = 0.9, Ty = 0, T^n x = T^n y = 0 \ \forall n \geq 2$, $f(x) = 0$, and $g(x) = 0.81$. So, we have

$$d(Tx, Ty)^2 = 0.81 \leq g(x)x^2 = f(x)d(Tx, y)^2 + g(x)d(Ty, x)^2.$$

On the other hand, for any $n \geq 2$, we have

$$d(T^n x, T^n y)^2 = 0 \leq f(x)d(T^n x, y)^2 + g(x)d(T^n y, x)^2.$$

(ii) If $x \in [0, 1)$ and $y \geq 1$, then $Tx = 0, Ty = 0.9, T^n x = T^n y = 0 \ \forall n \geq 2$, $f(x) = 0.81$, and $g(x) = 0$. So, we have

$$d(Tx, Ty)^2 = 0.81 \leq f(x)y^2 = f(x)d(Tx, y)^2 + g(x)d(Ty, x)^2.$$

On the other hand, for any $n \geq 2$, we have

$$d(T^n x, T^n y)^2 = 0 \leq f(x)d(T^n x, y)^2 + g(x)d(T^n y, x)^2.$$

Therefore, T is a generalized asymptotically nonspreading mapping.

3.1. **Existence theorems.** We now prove existence theorems for generalized asymptotically nonspreading mappings in complete convex metric spaces.

Theorem 3.4. *Let C be a nonempty closed convex subset of a complete convex metric space (X, d, W) such that $A(\{x_n\})$ is singleton for all bounded sequence $\{x_n\}$ in C and $T : C \to C$ be a generalized asymptotically nonspreading mapping. Then the following assertions are equivalent:*

(i) *$F(T)$ is nonempty;*
(ii) *there exists a bounded sequence $\{x_n\}$ in C such that*

$$\liminf_{n\to\infty} d(x_n, Tx_n) = 0.$$

Proof. Since it is obvious that (i) implies (ii), we show that (ii) implies (i). Suppose that there exists a bounded sequence $\{x_n\}$ in C such that

$$\liminf_{n\to\infty} d(x_n, Tx_n) = 0.$$

Consequently, there is a bounded subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\lim_{k\to\infty} d(x_{n_k}, Tx_{n_k}) = 0.$$

Suppose $A(\{x_{n_k}\}) = \{z\}$. Since T is a generalized asymptotically nonspreading mapping, we have

$$\begin{aligned}
d(x_{n_k}, Tz)^2 &\leq \left(d(x_{n_k}, Tx_{n_k}) + d(Tz, Tx_{n_k})\right)^2 \\
&= d(x_{n_k}, Tx_{n_k})^2 + d(Tz, Tx_{n_k})^2 + 2d(x_{n_k}, Tx_{n_k})d(Tx_{n_k}, Tz) \\
&\leq d(x_{n_k}, Tx_{n_k})^2 + f(z)d(Tz, x_{n_k})^2 + g(z)d(Tx_{n_k}, z)^2 \\
&\quad + 2d(x_{n_k}, Tx_{n_k})d(Tx_{n_k}, Tz) \\
&\leq d(x_{n_k}, Tx_{n_k})^2 + f(z)d(Tz, x_{n_k})^2 + g(z)(d(Tx_{n_k}, x_{n_k}) + d(x_{n_k}, z))^2 \\
&\quad + 2d(x_{n_k}, Tx_{n_k})d(Tx_{n_k}, Tz) \\
&= (1 + g(z))d(x_{n_k}, Tx_{n_k})^2 + f(z)d(Tz, x_{n_k})^2 + g(z)d(x_{n_k}, z)^2 \\
&\quad + 2g(z)d(Tx_{n_k}, x_{n_k})d(x_{n_k}, z) + 2d(x_{n_k}, Tx_{n_k})d(Tx_{n_k}, Tz).
\end{aligned}$$

This implies that

$$\begin{aligned}
(1 - f(z))d(x_{n_k}, Tz)^2 \leq{}& (1 + g(z))d(x_{n_k}, Tx_{n_k})^2 + g(z)d(x_{n_k}, z)^2 \\
&+ 2M(1 + g(z))d(x_{n_k}, Tx_{n_k}),
\end{aligned}$$

where $M = \sup_{k\in\mathbb{N}}\{d(x_{n_k}, z), d(Tx_{n_k}, Tz)\}$. Taking lim sup on both sides of the above inequality, we get

$$\begin{aligned}
(1 - f(z)) \limsup_{k\to\infty} d(x_{n_k}, Tz)^2 &\leq g(z) \limsup_{k\to\infty} d(x_{n_k}, z)^2 \\
&\leq (1 - f(z)) \limsup_{k\to\infty} d(x_{n_k}, z)^2.
\end{aligned}$$

So, we have

$$\limsup_{k\to\infty} d(x_{n_k}, Tz) \leq \limsup_{k\to\infty} d(x_{n_k}, z).$$

It implies that

$$r(Tz, \{x_{n_k}\}) = \limsup_{k\to\infty} d(x_{n_k}, Tz) \leq \limsup_{k\to\infty} d(x_{n_k}, z) = r(z, \{x_{n_k}\}).$$

This shows that $Tz \in A(\{x_{n_k}\})$. By the uniqueness of asymptotic center, we conclude that $Tz = z$. □

Theorem 3.5. *Let C be a nonempty closed convex subset of a complete convex metric space (X, d, W) such that $A(\{x_n\})$ is singleton for all bounded sequence $\{x_n\}$ in C and $T : C \to C$ be a generalized asymptotically nonspreading mapping. If $\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0$ for all $x \in C$, then the following assertions are equivalent:*

(i) $F(T)$ *is nonempty;*
(ii) *there exists* $x \in C$ *such that* $\{T^n x\}$ *is bounded.*

Proof. Since it is obvious that (i) implies (ii), we show that (ii) implies (i). Suppose that there exists $x \in C$ such that $\{T^n x\}$ is bounded. Setting $y_n = T^n x$ for all $n \in \mathbb{N}$. Then we have

$$\lim_{n\to\infty} d(Ty_n, y_n) = \lim_{n\to\infty} d(T^{n+1}x, T^n x) = 0.$$

Since $\{y_n\}$ is bounded, it implies by Theorem 3.4 that $F(T)$ is nonempty. $\square$

It follows from the fact that, in a complete uniformly convex metric space, the asymptotic center of a bounded sequence with respect to a closed convex subset is singleton; see [20]. So, we have the following results.

Theorem 3.6. *Let* C *be a nonempty closed convex subset of a complete uniformly convex metric space* (X, d, W) *and* $T : C \to C$ *be a generalized asymptotically nonspreading mapping. Then the following assertions are equivalent:*

(i) $F(T)$ *is nonempty;*
(ii) *there exists a bounded sequence* $\{x_n\}$ *in* C *such that*

$$\liminf_{n\to\infty} d(x_n, Tx_n) = 0.$$

Theorem 3.7. *Let* C *be a nonempty closed convex subset of a complete uniformly convex metric space* (X, d, W) *and* $T : C \to C$ *be a generalized asymptotically nonspreading mapping. If* $\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0$ *for all* $x \in C$, *then the following assertions are equivalent:*

(i) $F(T)$ *is nonempty;*
(ii) *there exists* $x \in C$ *such that* $\{T^n x\}$ *is bounded.*

Since every CAT(0) space is a uniformly convex metric space, the following results can be obtained from Theorems 3.6 and 3.7 immediately.

Theorem 3.8. *Let* C *be a nonempty closed convex subset of a complete CAT(0) space* (X, d) *and* $T : C \to C$ *be a generalized asymptotically nonspreading mapping. Then the following assertions are equivalent:*

(i) $F(T)$ *is nonempty;*
(ii) *there exists a bounded sequence* $\{x_n\}$ *in* C *such that*

$$\liminf_{n\to\infty} d(x_n, Tx_n) = 0.$$

Theorem 3.9. *Let* C *be a nonempty closed convex subset of a complete CAT(0) space* (X, d) *and* $T : C \to C$ *be a generalized asymptotically nonspreading mapping. If* $\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0$ *for all* $x \in C$, *then the following assertions are equivalent:*

(i) $F(T)$ *is nonempty;*
(ii) *there exists* $x \in C$ *such that* $\{T^n x\}$ *is bounded.*

Remark 3.10. Theorems 3.4-3.9 improve and extend the main results of Naraghirad [17] from an asymptotically nonspreading mapping to a generalized asymptotically nonspreading mapping and from a Banach space to a complete convex metric space.

3.2. **Δ-convergence theorems.** In this section, we study Δ-convergence theorems for a generalized asymptotically nonspreading mapping in complete CAT(0) spaces.

The following theorem show the demiclosed principle for a generalized asymptotically nonspreading mapping on complete CAT(0) spaces.

Theorem 3.11. *Let C be a nonempty closed convex subset of a complete CAT(0) space (X,d) and $T: C \to C$ be a generalized asymptotically nonspreading mapping. If $\{x_n\}$ is a bounded sequence in C that Δ-converges to z and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, then $z \in F(T)$.*

Proof. Suppose that $\{x_n\}$ is a bounded sequence in C such that $\Delta\text{-}\lim_{n\to\infty} x_n = z$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. By the definition of T, we have

$$\begin{aligned}
d(x_n, Tz)^2 &\leq (d(x_n, Tx_n) + d(Tz, Tx_n))^2 \\
&= d(x_n, Tx_n)^2 + 2d(x_n, Tx_n)d(Tz, Tx_n) + d(Tz, Tx_n)^2 \\
&\leq d(x_n, Tx_n)^2 + 2d(x_n, Tx_n)d(Tz, Tx_n) + f(z)d(Tz, x_n)^2 \\
&\quad + g(z)d(Tx_n, z)^2 \\
&\leq d(x_n, Tx_n)^2 + 2d(x_n, Tx_n)d(Tz, Tx_n) + f(z)d(Tz, x_n)^2 \\
&\quad + g(z)(d(Tx_n, x_n) + d(x_n, z))^2 \\
&= (1 + g(z))d(x_n, Tx_n)^2 + 2d(x_n, Tx_n)d(Tz, Tx_n) + f(z)d(Tz, x_n)^2 \\
&\quad + 2g(z)d(Tx_n, x_n)d(x_n, z) + g(z)d(x_n, z)^2.
\end{aligned}$$

This implies that

$$\begin{aligned}
(1 - f(z))d(x_n, Tz)^2 \leq (1 + g(z))d(x_n, Tx_n)^2 + 2M(1 + g(z))d(x_n, Tx_n) \\
+ g(z)d(x_n, z)^2,
\end{aligned}$$

where $M = \sup_{n\in\mathbb{N}}\{d(x_n, z), d(Tx_n, Tz)\}$. Taking lim sup on both sides of the above inequality, we get

$$\begin{aligned}
(1 - f(z)) \limsup_{n\to\infty} d(x_n, Tz)^2 &\leq g(z) \limsup_{n\to\infty} d(x_n, z)^2 \\
&\leq (1 - f(z)) \limsup_{n\to\infty} d(x_n, z)^2.
\end{aligned}$$

Thus, we have

$$\limsup_{n\to\infty} d(x_n, Tz) \leq \limsup_{n\to\infty} d(x_n, z).$$

By the uniqueness of asymptotic centers, we have $Tz = z$. Hence $z \in F(T)$. $\square$

By using Theorem 3.11, we obtain the following Δ-convergence theorem for a generalized asymptotically nonspreading mapping in complete CAT(0) spaces.

Theorem 3.12. *Let C be a nonempty closed convex subset of a complete CAT(0) space (X,d) and $T : C \to C$ be a generalized asymptotically nonspreading mapping with $F(T)$ is nonempty. Assume that $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $0 < a \leq \alpha_n \leq b < 1$. Let $\{x_n\}$ be a sequence in C generated by*

$$x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n T^n x_n, \text{ for all } n \in \mathbb{N}. \tag{3.1}$$

Then the sequence $\{x_n\}$ Δ-converges to a fixed point of T.

Proof. Let $z \in F(T)$. Since T is generalized asymptotically nonspreading, we have

$$d(z, T^n x_n)^2 \leq f(z)d(z,x_n)^2 + g(z)d(T^n x_n, z)^2.$$

This implies that

$$(1-g(z))d(z,T^n x_n)^2 \leq f(z)d(z,x_n)^2.$$

Since $0 < f(z) + g(z) \leq 1$, we have

$$d(z, T^n x_n) \leq d(z, x_n). \tag{3.2}$$

In view of Lemma 2.6, (3.1), and (3.2),

$$\begin{aligned} d(x_{n+1}, z)^2 &\leq (1-\alpha_n)d(x_n,z)^2 + \alpha_n d(T^n x_n, z)^2 - \alpha_n(1-\alpha_n)d(x_n, T^n x_n)^2 \\ &\leq d(x_n,z)^2 - \alpha_n(1-\alpha_n)d(x_n,T^n x_n)^2 \\ &\leq d(x_n,z)^2 - a(1-b)d(x_n,T^n x_n)^2. \qquad (3.3) \end{aligned}$$

Thus, we have

$$d(x_{n+1}, z) \leq d(x_n, z).$$

This implies that $\lim_{n\to\infty} d(x_n, z)$ exists for all $z \in F(T)$ and hence $\{x_n\}$ is bounded. It follows by (3.3) that

$$a(1-b)d(x_n, T^n x_n)^2 \leq d(x_n,z)^2 - d(x_{n+1},z)^2,$$

which yields that

$$\lim_{n\to\infty} d(x_n, T^n x_n) = 0. \tag{3.4}$$

In view of (3.1), we see that

$$d(x_{n+1}, x_n) \leq \alpha_n d(T^n x_n, x_n) \leq b d(T^n x_n, x_n).$$

Then we have

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{3.5}$$

Since T is generalized asymptotically nonspreading, we obtain that

$$\begin{aligned} d(T^{n+1}x_n, T^{n+1}x_{n+1})^2 &\leq f(x_n)d(T^{n+1}x_n, x_{n+1})^2 + g(x_n)d(T^{n+1}x_{n+1}, x_n)^2 \\ &\leq f(x_n)(d(T^{n+1}x_n, T^{n+1}x_{n+1}) + d(T^{n+1}x_{n+1}, x_{n+1}))^2 \\ &\quad + g(x_n)(d(T^{n+1}x_{n+1}, x_{n+1}) + d(x_{n+1}, x_n))^2 \\ &= f(x_n)d(T^{n+1}x_n, T^{n+1}x_{n+1})^2 + f(x_n)d(T^{n+1}x_{n+1}, x_{n+1})^2 \\ &\quad + 2f(x_n)d(T^{n+1}x_n, T^{n+1}x_{n+1})d(T^{n+1}x_{n+1}, x_{n+1}) \\ &\quad + g(x_n)d(T^{n+1}x_{n+1}, x_{n+1})^2 + g(x_n)d(x_{n+1}, x_n)^2 \\ &\quad + 2g(x_n)d(T^{n+1}x_{n+1}, x_{n+1})d(x_{n+1}, x_n) \\ &\leq \gamma d(T^{n+1}x_n, T^{n+1}x_{n+1})^2 + \gamma d(T^{n+1}x_{n+1}, x_{n+1})^2 \\ &\quad + 2\gamma d(T^{n+1}x_n, T^{n+1}x_{n+1})d(T^{n+1}x_{n+1}, x_{n+1}) \\ &\quad + \gamma d(T^{n+1}x_{n+1}, x_{n+1})^2 + \gamma d(x_{n+1}, x_n)^2 \\ &\quad + 2\gamma d(T^{n+1}x_{n+1}, x_{n+1})d(x_{n+1}, x_n). \end{aligned}$$

Thus, we have

$$\begin{aligned} &(1-\gamma)d(T^{n+1}x_n, T^{n+1}x_{n+1})^2 \\ &\qquad \leq 2\gamma d(T^{n+1}x_{n+1}, x_{n+1})^2 + 2\gamma M_1 d(T^{n+1}x_{n+1}, x_{n+1}) \\ &\qquad\quad + \gamma d(x_{n+1}, x_n)^2 + 2\gamma d(T^{n+1}x_{n+1}, x_{n+1})d(x_{n+1}, x_n), \end{aligned}$$

where $M_1 = \sup_{n\in\mathbb{N}}\{d(T^{n+1}x_n, T^{n+1}x_{n+1})\}$. This implies from (3.4) and (3.5) that

$$\lim_{n\to\infty} d(T^{n+1}x_n, T^{n+1}x_{n+1}) = 0. \tag{3.6}$$

Consider

$$d(x_n, T^{n+1}x_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1}x_{n+1}) + d(T^{n+1}x_{n+1}, T^{n+1}x_n),$$

then, by (3.4), (3.5), and (3.6), we have

$$\lim_{n\to\infty} d(x_n, T^{n+1}x_n) = 0. \tag{3.7}$$

Since $d(T^n x_n, T^{n+1}x_n) \leq d(T^n x_n, x_n) + d(x_n, T^{n+1}x_n)$, it implies by (3.6) and (3.7) that

$$\lim_{n\to\infty} d(T^n x_n, T^{n+1}x_n) = 0. \tag{3.8}$$

Since T is generalized asymptotically nonspreading, we have

$$\begin{aligned} d(Tx_n, T^{n+1}x_n)^2 &\leq f(x_n)d(Tx_n, T^n x_n)^2 + g(x_n)d(T^{n+1}x_n, x_n)^2 \\ &\leq f(x_n)(d(Tx_n, T^{n+1}x_n) + d(T^{n+1}x_n, T^n x_n))^2 \\ &\quad + g(x_n)d(T^{n+1}x_n, x_n)^2 \\ &\leq f(x_n)d(Tx_n, T^{n+1}x_n)^2 + f(x_n)d(T^{n+1}x_n, T^n x_n)^2 \\ &\quad + 2f(x_n)d(Tx_n, T^{n+1}x_n)d(T^{n+1}x_n, T^n x_n) \\ &\quad + g(x_n)d(T^{n+1}x_n, x_n)^2 \\ &\leq \gamma d(Tx_n, T^{n+1}x_n)^2 + \gamma d(T^{n+1}x_n, T^n x_n)^2 \\ &\quad + 2\gamma d(Tx_n, T^{n+1}x_n)d(T^{n+1}x_n, T^n x_n) + \gamma d(T^{n+1}x_n, x_n)^2. \end{aligned}$$

This implies that

$$\begin{aligned} (1-\gamma)d(Tx_n, T^{n+1}x_n)^2 \leq \gamma d(T^{n+1}x_n, T^n x_n)^2 + 2\gamma M_2 d(T^{n+1}x_n, T^n x_n) \\ + \gamma d(T^{n+1}x_n, x_n)^2, \end{aligned}$$

where $M_2 = \sup_{n\in\mathbb{N}}\{d(Tx_n, T^{n+1}x_n)\}$. Thus, by (3.7) and (3.8), we have

$$\lim_{n\to\infty} d(Tx_n, T^{n+1}x_n) = 0. \tag{3.9}$$

From $d(Tx_n, x_n) \leq d(Tx_n, T^{n+1}x_n) + d(T^{n+1}x_n, x_n)$, it implies by (3.7) and (3.9) that

$$\lim_{n\to\infty} d(Tx_n, x_n) = 0. \tag{3.10}$$

We now let $\omega_\Delta(x_n) := \bigcup A(\{u_n\})$, where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. We claim that $\omega_\Delta(x_n) \subset F(T)$. Let $u \in \omega_\Delta(x_n)$. Then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. Since $\{u_n\}$ is bounded, it implies by Lemma 2.3 that there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\Delta\text{-}\lim_{k\to\infty} u_{n_k} = y \in C$. By (3.10) and Theorem 3.11, we have $y \in F(T)$. Then $\lim_{n\to\infty} d(x_n, y)$ exists. Suppose that $u \neq y$. By the uniqueness of asymptotic centers, we obtain that

$$\begin{aligned} \limsup_{k\to\infty} d(u_{n_k}, y) &< \limsup_{k\to\infty} d(u_{n_k}, u) \\ &\leq \limsup_{n\to\infty} d(u_n, u) \\ &< \limsup_{n\to\infty} d(u_n, y) \\ &= \limsup_{n\to\infty} d(x_n, y) \\ &= \limsup_{k\to\infty} d(u_{n_k}, y). \end{aligned}$$

This is a contradiction, hence $u = y \in F(T)$. This shows that $\omega_\Delta(x_n) \subset F(T)$.

Next, we show that $\omega_\Delta(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{z\}$. Since $u \in \omega_\Delta(x_n) \subset F(T)$, it implies that $\lim_{n\to\infty} d(x_n, u)$ exists. By Lemma 2.5, we get $z = u$. Hence, the sequence $\{x_n\}$ Δ-converges to a fixed point z of T. □

Remark 3.13. Theorem 3.12 improves and extends Theorem 4.1 of Naraghirad [17] to a generalized asymptotically nonspreading mapping and to a complete CAT(0) space.

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