Some Formulas for Legendre Functions Induced by the Poisson Transform

I. A. Shilin, A. I. Nizhnikov

Abstract

Using the Poisson transform, which maps any homogeneous and infinitely differentiable function on a cone into a corresponding function on a hyperboloid, we derive some integral representations of the Legendre functions.

Keywords: Legendre functions, Lorentz group, Poisson transform.

1 Introduction

Let us assume that the linear space $\mathbb{R}^{n+1}$ is endowed with the quadratic form

$$q(x) := x_0^2 - x_1^2 - \ldots - x_n^2.$$

We denote the polar bilinear form for q by $\hat{q}$. The Lorentz group $SO(n,1)$ preserves this form and divides $\mathbb{R}^{n+1}$ into orbits. We will deal with two kinds of these orbits. One of them is

$$C := \{x \mid q(x) = 0\};$$

it is a cone. The second kind of orbits consist of two-sheet hyperboloids

$$H(r) := \{x \mid q(x) = r^2\}$$

for any $r > 0$.

The group $SO(n,1)$ has 2 connected components. One of them contains the identity and will be under our consideration further. We denote this subgroup by symbol G. The action $x \longmapsto g^{-1}x$ of the group G is transitive on C. Let $\sigma \in \mathbb{C}$ and D_σ be a linear subspace in $C^\infty(C)$ consisting of σ-homogeneous functions. It is useful to suppose throughout this paper that $-n+1 < \mathrm{re}\,\sigma < 0$. We define the representation T_σ in D_σ by left shifts:

$$T_\sigma(g)[f(x)] := f(g^{-1}x).$$

Suppose that γ is a contour on C intersecting all generatrices (i.e. all lines containing the origin). Every point $x \in \gamma$ depends on $n-1$ parameters, so every point $x \in C$ can be represented as

$$x_i = \{tF_i(\xi_1, \ldots, \xi_{n-1}), \qquad i = 1, \ldots, n+1.$$

Denoting by $\tilde{G}$ the subgroup of G which acts transitively on γ, we have

$$dx = t^{n-3}\, dt\, d\gamma, \tag{1}$$

where $d\gamma$ is the $\tilde{G}$-invariant measure on γ.

For any pair $(D_\sigma, D_{\tilde{\sigma}})$, we define the bilinear functionals $\mathsf{F}_\gamma : (D_\sigma, D_{\tilde{\sigma}}) \longrightarrow \mathbb{C}$,

$$(f_1, f_2) \longmapsto \int_\gamma f_1(x) f_2(x)\, d\gamma.$$

The functional F_γ does not depend on γ if $\tilde{\sigma} = -\sigma - n + 1$, because, first, we have formula (1), and, second, f_1 and f_2 are both homogeneous functions, and, third, the G-invariant measure on C can be represented in the form

$$dx = \frac{dx_{\zeta(1)} \ldots dx_{\zeta(n)}}{|x_{\zeta(n+1)}|}, \tag{2}$$

where $\zeta \in \underset{n}{\mathbf{S}}$ and $\underset{n+1}{\mathbf{S}}$ is the permutation group of the set $\{1, \ldots, n+1\}$.

Let $f \in D_\sigma$ and $y \in H(1)$. We refer to the integral transform

$$\Pi(f)(y) := \mathsf{F}_\gamma(\hat{q}^{-\sigma-n+1}(y,x), f)$$

as the Poisson transform [1].

2 Formulas related to sphere and paraboloid

Let γ_1 be the intersection of the cone C and the plane $x_0 = 1$. Each point $x \in \gamma_1$ depends on spherical parameters $\phi_1, \ldots, \phi_{n-1}$ by the formula

$$x_s = \prod_{i=1}^{n-s} \sin\phi_i \cdot \cos\phi_{n-s+1}, \qquad s \neq 0,$$

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if angle ϕ_{n-s+1} exists. Here $\phi_{n-1} \in [0; 2\pi)$ and $\phi_1, \ldots, \phi_{n-2} \in [0; \pi)$.

The subgroup $H_1 \simeq SO(n)$ acts transitively on γ_1, and any permutate $\zeta \in \underset{n+1}{\mathbf{S}}$ defines the H_1-invariant measure

$$\mathrm{d}\gamma_1 = \frac{\mathrm{d}\gamma_{\zeta(2)} \ldots \mathrm{d}\gamma_{\zeta(n)}}{|x_{\zeta(n+1)}|}.$$

The invariant measure in spherical coordinates is given by 9.1.1.(9) [2]

Let γ_2 be the intersection of cone C and the hyperplane $x_0 + x_n = 1$. We describe every point $x \in \gamma_2$ by the coordinates $r, \phi_1, \ldots, \phi_{n-2}$ according to the formulas

$$x_0 = \frac{1+r^2}{2}, \qquad x_n = \frac{1-r^2}{2},$$

$$x_s = r \prod_{i=1}^{n-s-1} \sin\phi_i \, \cos\phi_{n-s}, \qquad s \notin \{0, n\}$$

(if angle ϕ_{n-s} exists), where $r \geq 0, \phi_{n-2} \in [0; 2\pi)$ and $\phi_1, \ldots, \phi_{n-3} \in [0; \pi)$.

We denote as H_2 the subgroup of G acting transitively on γ_2. H_2 consists of the matrices

$$n(b) = \begin{pmatrix} \mathrm{diag}\,(\underbrace{1, \ldots, 1}_{n-1}) & b^{\mathrm{T}} & b^{\mathrm{T}} \\ -b & 1 - b^* & -b^* \\ b & b^* & b^* \end{pmatrix},$$

where $b = (b_1, \ldots, b_{n-1})$ and $b^* = \frac{1}{2}(b_1^2 + \ldots + b_{n-1}^2)$.

It is not too hard to derive the H_2-invariant measure

$$d\gamma = r^{n-2} \, \mathrm{d}r \prod_{i=1}^{n-2} \sin^{n-i-2}\phi_i \, \mathrm{d}\phi_i$$

on γ_2.

Let $\lambda > 0$, $\mu \in \mathbb{R}$, $k_0 \geq k_1 \geq \ldots \geq k_{n-2} \geq 0$, $l_1 \geq \ldots \geq l_{n-2} \geq 0$, $m_1 \geq \ldots \geq m_{n-2} \geq 0$, $K = (k_0, k_1, \ldots, k_{n-3}, \pm k_{n-2})$, $L = (l_1, \ldots, l_{n-3}, \pm l_{n-2})$, $M = (m_1, \ldots, m_{n-3}, \pm m_{n-2})$.

We will now deal with two bases in D_σ. One of them consists of the functions

$$f_K^{\sigma 1}(x) = x_0^{\sigma - k_0} \, \Xi_K^n(x),$$

where $K = (k_0, k_1, \ldots, k_{n-3}, \pm k_{n-2}) \in \mathbb{Z}^{n-1}$, $k_i \geq k_{i+1} \geq 0$ and

$$\Xi_T^n(x) = \prod_{i=1}^{n-3} r_{n-i}^{t_i - t_{i+1}} \cdot C_{t_i - t_{i+1}}^{\frac{n-i}{2} - 1}\left(\frac{x_{n-i}}{r_{n-i}}\right) (x_2 \pm \mathbf{i} x_1)^{t_{n-2}}.$$

The second basis consists of the functions

$$f_{(L,\lambda)}^{\sigma 2}(x) = (x_0 + x_n)^{\sigma + \frac{n-3}{2}} \cdot \left(\frac{\lambda}{2}\right)^{l_1} \left(\frac{\lambda r_{n-1}}{2}\right)^{\frac{3-n}{2} - l_1} \cdot J_{l_1 + \frac{n-3}{2}}\left(\frac{\lambda r_{n-1}}{x_0 + x_n}\right) \Xi_L^{n-1}(x),$$

where $r_j^2 = x_1^2 + \ldots + x_j^2$, $L = (l_1, \ldots, l_{n-3}, \pm l_{n-2}) \in \mathbb{Z}^{n-2}$, $\lambda \geq 0$ and $l_i \geq l_{i+1} \geq 0$. Suppose, in addition, that the functions of the above bases are equipped with the normalizing factors defined by formulas [2, 9.4.1.7, 10.3.4.9].

Let us consider the distribution

$$f_K^{\sigma 1}(x) = \sum_L \int_0^{+\infty} c_{K,(L,\lambda)}^{\sigma 12} \, f_{(L,\lambda)}^{\sigma 2} \, \mathrm{d}\lambda. \tag{3}$$

From the orthogonality of the functions Ξ_T^n, we obtain the property

$$\mathsf{F}_\gamma(f_K^{\sigma 1}, f_{-\tilde{K}}^{-\sigma-n-1,1}) = \delta_{K\tilde{K}}.$$

From this property, it immediately follows that

$$c_{K,(L,\lambda)}^{\sigma 12} = \mathsf{F}_\gamma(f_K^{\sigma 1}, f_{(L,\lambda)}^{-\sigma-n-1,2}).$$

Let $\gamma = \gamma_1$. Then from the formula

$$\int_{-1}^{1} (1 - x^2)^{\nu - \frac{1}{2}} \, C_m^\nu(x) \, C_n^\nu(x) \, \mathrm{d}x = 0,$$

where $m \neq n$, re $\nu > -\frac{1}{2}$, we derive

Lemma 1. *If* $\sum_{i=1}^{n-2}(k_i - l_i)^2 \neq 0$, *then* $c_{K,(L,\lambda)}^{\sigma 12} = 0$.

Let us assume another situation.

Lemma 2. *If* $\sum_{i=1}^{n-2}(k_i - l_i)^2 = 0$, *then*

$$c_{K,(L,\lambda)}^{\sigma 12} = 2^{-\sigma+n+3k_1-3} \, \pi^{-1} \, \mathbf{i}^{k_1} \, (n + 2k_0 - 2)^{\frac{1}{2}} \cdot \sqrt{(k_0 - k_1)!} \, \lambda^{k_1} \, \Gamma\left(\frac{n-1}{2}\right) \Gamma\left(\frac{n-1}{2} + k_1\right) \cdot \Gamma\left(\frac{n}{2} + k_1 - 1\right) \Gamma^{\frac{1}{2}}\left(\frac{n-1}{2}\right) \Gamma^{-1}(n + 2k_1 - 2) \cdot \Gamma^{-\frac{1}{2}}(n + k_0 + k_1 - 2) \sum_{m=0}^{k_0 - k_1} (-1)^m \, (m!)^{-1} \cdot \Gamma(n + k_0 + k_1 + m - 2) \, \Gamma^{-1}\left(\frac{n-1}{2} + k_1 + m\right) \cdot \Gamma^{-1}(k_0 - k_1 - m - 1) \, \Gamma^{-1}(-\sigma + k_1 + m) \cdot G_{13}^{21}\left(\frac{\lambda^2}{4} \,\middle|\, \begin{matrix} -m \\ -\sigma + k_1 - 1, \frac{n-3}{2} + k_1 \end{matrix}\right).$$

Proof. Suppose $\gamma = \gamma_2$. Then we obtain the integral

$$\int_0^{+\infty} r^{\frac{n-1}{2}+l_1}\,(r^2+1)^{\sigma-k_1}\cdot C_{k_0-k_1}^{\frac{n}{2}-k_1-1}\left(\frac{1-r^2}{1+r^2}\right) J_{\frac{n-3}{2}+l_1}(\lambda r)\,\mathrm{d}r,$$

which can be solved explicitly after replacing

$$r^k\,J_k(\lambda r) = 2^k\,\lambda^{-k}\,G_{02}^{10}\left(\left(\frac{\lambda r}{2}\right)^2 \middle|\, \begin{matrix} 0 \\ k, 0\end{matrix}\right)$$

according to formulas [3, 8.932.1, 8.932.2] and [4, 20.5.4].⋄

Theorem 1.

$$P_{-\sigma-\frac{n}{2}}^{-\frac{n}{2}+1}(\cosh\alpha) = 2^{2n-\frac{9}{2}}\,\pi^{-\frac{3}{2}}\sqrt{n-1}\cdot \sinh^{\frac{n}{2}-1}\alpha\, e^{(\sigma+n-1)\alpha}\,\Gamma\left(\frac{n}{2}-1\right)\Gamma\left(\frac{n+1}{2}\right)\cdot \Gamma^{-1}(-\sigma)\,\Gamma^{-\frac{1}{2}}(n-1)\int_0^{+\infty}\lambda^{-n+3}\cdot G_{13}^{21}\left(\frac{\lambda^2}{4}\,\middle|\, \begin{matrix} 0 \\ -\sigma-1, 0, \frac{n-3}{2}\end{matrix}\right)\cdot G_{13}^{21}\left(\frac{(\lambda e^{-\alpha})^2}{4}\,\middle|\, \begin{matrix} 0 \\ \sigma-\frac{n-1}{2}, 0, \frac{n-3}{2}\end{matrix}\right)\mathrm{d}\lambda.$$

Proof. Suppose that the condition $k_1 = l_1, \ldots, k_{n-2} = l_{n-2}$ holds. From the distribution (3), we obtain

$$\Pi(f_K^{\sigma 1}) = \int_0^{+\infty} c_{K,(L,\lambda)}^{\sigma 12}\,\Pi(f_{(L,\lambda)}^{\sigma 2})\,\mathrm{d}\lambda.$$

Further we assume $\Pi(f_K^{\sigma 1}) = \mathsf{F}_{\gamma_1}(\hat{q}^{-\sigma-n+1}(y,x), f_K^{\sigma 1})$ and $\Pi(f_{(L,\lambda)}^{\sigma 2}) = \mathsf{F}_{\gamma_2}(\hat{q}^{-\sigma-n+1}(y,x), f_{(L,\lambda)}^{\sigma 2})$, then for the case $y = (\cosh\alpha, 0, \ldots, 0, \sinh\alpha)$ and put $K = (0, \ldots, 0)$.⋄

Consider the case $SO(2,1)$ of the group $SO(n,1)$. In this case, $K \equiv k$ and $(L,\lambda) \equiv \lambda$. The following theorem is related to this case.

Theorem 2. *If* $-1 < \mathrm{re}\,\sigma < 0$ *and* $\alpha \neq 0$*, then*

$$P_{\sigma+\frac{1}{2}}^{-l+\frac{1}{2}}(\cosh\alpha) = (-1)^{l-1}\,2^{-\sigma-\frac{l}{2}-\frac{9}{4}}\,\pi^{-\frac{1}{2}}\times e^{-\alpha}\sin(-\pi\sigma)\sinh^{l+\frac{1}{2}}\alpha\cdot \left(\frac{\cosh\alpha+1}{\cosh\alpha-1}\right)^{\frac{l}{2}+\frac{1}{4}}\Gamma(\sigma-l+1)\,\Gamma\left(l-\frac{3}{2}\right)\cdot \Gamma^{-1}\left(l+\frac{1}{2}\right)\int_0^{\infty}\rho^{-\sigma-1}K_{\sigma+1}(\rho e^{-\alpha})\cdot \sum_{s=0}^{\infty}(-1)^n\,\Gamma^{-2}(s+1)\,\Gamma^{-1}(s-\sigma)\cdot G_{13}^{21}\left(\frac{\rho^2}{4}\,\middle|\, \begin{matrix} -s \\ -\sigma-1,\ 0\end{matrix}\right)\mathrm{d}\rho \tag{4}$$

Proof. After repeating the proof of the previous theorem, we derive the following representation of the Gauss hypergeometric function:

$${}_2F_1\left(-\sigma-\frac{1}{2}, \sigma+\frac{3}{2}; \frac{1}{2}+l; \frac{1-\cosh\alpha}{2}\right) = (-1)^{l-1}\,2^{-\sigma-\frac{5}{2}}\,\pi^{-\frac{1}{2}}\,e^{-\alpha}\sinh\alpha\,\sin(-\pi\sigma)\cdot \left(\frac{\cosh\alpha+1}{\cosh\alpha-1}\right)^{\frac{l}{2}+\frac{1}{4}}\Gamma(\sigma+1-l)\,\Gamma\left(l-\frac{3}{2}\right)\cdot \int_0^{\infty}\lambda^{-\sigma-1}K_{\sigma+1}(\lambda e^{-\alpha})\sum_{s=0}^{\infty}(-1)^n\,\Gamma^{-2}(s+1)\cdot \Gamma^{-1}(s-\sigma)\,G_{13}^{21}\left(\frac{\lambda^2}{4}\,\middle|\, \begin{matrix} -s \\ -\sigma-1, 0, 0\end{matrix}\right)\mathrm{d}\lambda.$$

Now we use the formula [5, 7.3.1.88] for $l = 0$.⋄

3 Formulas related to paraboloid and hyperboloid

Let γ_{3+} be the intersection of cone C and the plane $x_n = 1$. We denote as γ_{3-} the intersection of C and the plane $x_n = -1$. Let $\gamma_3 := \gamma_{3+} \cup \gamma_{3-}$. The contour γ_3 is a homogeneous space with respect to the subgroup $H_3 \simeq SO(n-1,1)$. If x belongs to γ_3, then

$$x_n = \pm 1, \qquad x_0 = \cosh t,$$
$$x_s = \sinh t \prod_{i=1}^{n-s-1}\sin\phi_i\cdot\cos\phi_{n-s}, \qquad s \notin \{0, n\}$$

(if angle ϕ_{n-s} exists), where $t \in \mathbb{R}$, $\phi_{n-2} \in [0; 2\pi)$ and $\phi_1, \ldots, \phi_{n-3} \in [0; \pi)$.

Any permutation $\zeta \in \underset{n}{\mathbf{S}}$ determines the H_3-invariant measure

$$\mathrm{d}\gamma_4 = \frac{\mathrm{d}x_{\zeta(1)}\ldots\mathrm{d}x_{\zeta(n-1)}}{|x_{\zeta(n)}|}$$

on γ_3, so

$$\mathrm{d}\gamma_3 = \cosh^{n-2} t\,\mathrm{d}t\prod_{i=1}^{n-2}\sin^{n-i-2}\phi_i\,\mathrm{d}\phi_i.$$

Let us now consider the basis consisting of the functions

$$f_{(M,\mu,\pm)}^{\sigma 2}(x) = (x_n)_\pm^{\sigma+\frac{n-3}{2}}\, r_{n-1}^{\frac{3-n}{2}-m_1}\cdot P_{-\frac{1}{2}+\mathbf{i}\mu}^{\frac{3-n}{2}-m_1}\left(\frac{x_0}{x_n}\right)\Xi_M^{n-1}(x),$$

where $(x_n)_\pm^{\sigma+\frac{n-3}{2}}$ is the generalized function defined as

$$(x_n)_\pm^{\sigma+\frac{n-3}{2}} = \begin{cases} |x_n|^{\sigma+\frac{n-3}{2}}, & \text{if sign } x_n = \pm 1, \\ 0, & \text{if sign } x \neq \pm 1, \end{cases}$$

$M = (m_1, \ldots, m_{n-3}, \pm m_{n-2}) \in \mathbb{Z}^{n-2}$, $m_i \geq m_{i+1} \geq 0$ and $\mu \in \mathbb{R}$.

By analogy with the previous case, we can obtain the coefficients $c_{K,(M,\mu,+)}$. Let us suppose that $n = 3$ and $K = (l, s), M \equiv m$. From the distribution

$$f^{\sigma 3}_{m,\mu,+}(x) = \sum_{l=0}^{\infty} \sum_{s=-|l|}^{|l|} c_{l,s,m,\mu,+} \, f^{\sigma 1}_{l,s}(x),$$

we have

$$f^{\sigma 3}_{-s,\mu,+}(x) = \sum_{l=0}^{\infty} c_{l,s,-s,\mu,+} \, f^{\sigma 1}_{l,s}(x)$$

and, therefore,

$$\Pi(f^{\sigma 3}_{-s,\mu,+}) = \sum_{l=0}^{\infty} \sum_{s=-|l|}^{|l|} c_{l,s,-s,\mu,+} \, \Pi(f^{\sigma 1}_{l,s}). \quad (5)$$

We choose γ_3 (in fact, γ_{3+}) on the left side of equality (5) and γ_1 on the opposite side. In accordance with our choice, we use two parametrizations of a point $y \in H(1)$:

$$y(v) = \left(\frac{v + v^{-1}}{2}, 0, \ldots, 0, \frac{v^{-1} - v}{2} \right)$$

and $y(t) = (\cosh t, 0, \ldots, 0, \sinh t)$ respectively, so $v = e^{-t}$. After integration we have

$$\sin[\pi(\sigma + 1)] \cosh^{-1} t \, \Gamma\left(\mathbf{i}\mu - \sigma - \frac{1}{2} \right) \cdot$$
$$\Gamma\left(-\frac{3}{2} - \sigma - \mathbf{i}\mu \right) P^{\sigma+1}_{-\frac{1}{2}+\mathbf{i}\mu}(\tanh t) =$$
$$\sqrt{2}\, \pi^{\frac{3}{2}} \sum_{l=0}^{\infty} (-1)^l \, (l!)^{-1} A_l \sinh^{\frac{1}{2}} t \cdot$$
$$\Gamma(l + 1)\, \Gamma^{-1}(\sigma - l + 1)\, P^{-\frac{1}{2}-l}_{\sigma+\frac{1}{2}}(\cosh t),$$

where A_l is the normalizing factor of the function $f^{\sigma 1}_{l,s}(x)$.

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Ilya Shilin
Dept of Higher Mathematics
M. Scholokhov Moscow State
University for the Humanities
Verhnya Radishevskaya 16–18
Moscow 109240, Russia
Dept 311
Moscow Aviation Institute
Volokolamskoe shosse 4, Moscow 125993, Russia

Aleksandr Nizhnikov
Moscow Pedagogical State University
M. Pirogovskaya 1, Moscow 119991, Russia