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gl_{n+1} ALGEBRA OF MATRIX DIFFERENTIAL OPERATORS AND MATRIX QUASI-EXACTLY-SOLVABLE PROBLEMS

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Abstract. The generators of the algebra gl_{n+1} in the form of differential operators of the first order acting on $\mathbf{R}^n$ with matrix coefficients are explicitly written. The algebraic Hamiltonians for matrix generalization of 3−body Calogero and Sutherland models are presented.

Keywords: algebra of differential operators, exactly-solvable problems.

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1. Introduction

This work has a certain history related to Miloslav Havlicek. On the important occasion of Miloslav's 75th birthday, we think this story should be revealed. About 25 years ago, when quasi-exactly-solvable Schroedinger equations with the hidden algebra sl_2 were discovered [1], one of the present authors (AVT) approached Israel M. Gelfand and asked about the existence of the algebra gl_{n+1} of matrix differential operators. Instead of giving an answer, Israel Moiseevich said that M. Havlicek knows the answer and that he must be asked. A set of Dubna preprints was given (see [2, 3] and reference therein). Then AVT studied them for many years, at first separately and then together with the first author (YuFS), who also happened to have the same set of preprints. The results of these studies are presented below. While carrying out these studies, we always kept in mind that a constructive answer exists and is known to Miloslav. Thus, we are certain that at least some of results presented here are known to Miloslav. Having difficulty to understand what is written in the texts we did not know what he really knew, and were therefore unable to indicate it in our text. Our main goal is to find a mixed representation of the algebra gl_{n+1} which contains both matrices and differential operators in a non-trivial way. Then to generalize it to a polynomial algebra which we call $g^{(m)}$ (see below, Section 4). Another goal is to apply the obtained representations for a construction of the algebraic forms of (quasi)-exactly-solvable matrix Hamiltonians.

2. The algebra gl_n in mixed representation

Let us take the algebra gl_n and consider the vector field representation

$$\tilde{E}_{ij} = x_i \partial_j, \quad i,j = 1,\dots n, x \in \mathbf{R}^n. \tag{1}$$

It obeys the canonical commutation relations

$$[\tilde{E}_{ij}, \tilde{E}_{kl}] = \delta_{jk}\tilde{E}_{il} - \delta_{il}\tilde{E}_{kj}. \tag{2}$$

On the other hand, let us consider another representation M_{pm}, $p,m = 1,\dots,n$ of the algebra gl_n in terms of some operators (matrix, finite-difference, etc) with the condition that all 'cross-commutators' between these two representations vanish

$$[\tilde{E}_{ij}, M_{pm}] = 0. \tag{3}$$

Let us choose M_{pm} to obey the canonical commutation relations

$$[M_{ij}, M_{kl}] = \delta_{jk}M_{il} - \delta_{il}M_{kj}, \tag{4}$$

(cf. (2)). It is evident that the sum of these two representations is also the representation,

$$E_{ij} \equiv \tilde{E}_{ij} + M_{ij} \in gl_n. \tag{5}$$

Now we consider an embedding of $gl_n \subset gl_{n+1}$ trying to complement the representation (1) of the algebra gl_n up to the representation of the algebra gl_{n+1}. In principle, this can be done due to the existence of the Weyl-Cartan decomposition,

$$gl_{n+1} = L \oplus (gl_n \oplus \mathbf{I}) \oplus U$$

with the property

$$gl_{n+1} = L \rtimes (gl_n \oplus \mathbf{I}) \ltimes U, \tag{6}$$

where $L(U)$ is the commutative algebra of the lowering (raising) generators with the property $[L,U] = gl_n \oplus \mathbf{I}$. Thus, it realizes a property of the Gauss decomposition of gl_{n+1}. It is worth emphasizing that $\dim(L) = \dim(U) = n$.

Obviously, the lowering generators (of negative grading) from L can be given by derivations

$$T_i^- = \partial_i, \quad i = 1,\dots,n, \quad \partial_i \equiv \frac{\partial}{\partial x_i}, \tag{7}$$

(see e.g. [5]) when assuming that all commutators

$$[T_i^-, M_{pm}] = 0, \tag{8}$$

vanish. This probably implies that the only possible choice for M_{pm} exists when they are either given by matrices or act in a space which is a complement to $x \in \mathbf{R}^n$. It is easy to check that

$$[E_{ij}, T_k^-] = -\delta_{ik} T_j^- .$$

Now we have to add the Euler-Cartan generator of the gl_n algebra, see (6)

$$-E_0 = \sum_{j=1}^{n} x_j \partial_j - k, \tag{9}$$

where k is arbitrary constant. Raising generators from U are chosen as

$$\begin{aligned} -T_i^+ &= -x_i E_0 + \sum_{j=1}^{n} x_j M_{ij} \\ &= x_i \Big(\sum_{j=1}^{n} x_j \partial_j - k \Big) + \sum_{j=1}^{n} x_j M_{ij}, \quad i = 1, \dots, n. \end{aligned} \tag{10}$$

(cf. for instance [5]). Needless to say that one can check explicitly that T_i^-, E_{ij}, E_0, T_i^+ span the algebra gl_{n+1}. In particular,

$$[E, T^+] = T^+,$$

and

$$[T_i^+, T_j^-] = E_{ii} - \delta_{ij} E_0.$$

If parameter k takes non-negative integer the algebra gl_{n+1} spanned by the generators (5), (7), (9), (10) appears in a finite-dimensional representation. There exists a linear finite-dimensional space of polynomials of finite-order in the space of columns/spinors of finite length which is a common invariant subspace for all generators (5), (7), (9), (10). This finite-dimensional representation is irreducible.

The non-negative integer parameter k has the meaning of the length of the first row of the Young tableau of gl_{n+1}, describing a totally symmetric representation (see below). All other parameters are coded in M_{ij}, which corresponds to an arbitrary Young tableau of gl_n. Thus, we have some peculiar splitting of the Young tableau.

Each representation is characterized by the Gelfand-Tseitlin signature, $[m_{1,n}, \dots m_{nn}]$, where $m_{in} \geq m_{i+1,n}$ and their difference is positive integer. Each basic vector is characterized by the Gelfand-Tseitlin scheme. An explicit form of the representation is given by the Gelfand-Tseitlin formulas [4].

It can be demonstrated that all Casimir operators of gl_{n+1} in this realization (5), (7), (9), (10) are expressed in M_{ij}, and thus do not depend on x. They coincide with the Casimir operators of the gl_n-subalgebra realized by matrices M_{ij}.

3. Example: the algebra gl_3 in mixed representation

In the case of the algebra gl_3, the generators (5), (7), (9), (10) take the form

$$\begin{aligned} &E_{11} = x_1 \partial_1 + M_{11}, &&E_{22} = x_2 \partial_2 + M_{22}, \\ &E_{12} = x_1 \partial_2 + M_{12}, &&E_{21} = x_2 \partial_1 + M_{21}, \\ &E_0 = k - x_1 \partial_1 - x_2 \partial_2, && \\ &T_1^- = \partial_1, &&T_2^- = \partial_2, \\ &T_1^+ = x_1(k - x_1\partial_1 - x_2\partial_2) - x_1 M_{11} - x_2 M_{12}, && \\ &T_2^+ = x_2(k - x_1\partial_1 - x_2\partial_2) - x_1 M_{21} - x_2 M_{22}. && \end{aligned} \tag{11}$$

The Casimir operators of gl_3 in this realization are given by

$$\begin{aligned} C_1 &= E_{11} + E_{22} + E_0 = k + M_{11} + M_{22} = k + C_1(M), \\ C_2 &= E_{12}E_{21} + E_{21}E_{12} + T_1^+T_1^- + T_1^-T_1^+ + T_2^+T_2^- \\ &\quad + T_2^-T_2^+ + E_{11}^2 + E_{22}^2 + E_0^2 = k(k+2) \\ &\quad + M_{11}^2 + M_{22}^2 + M_{12}M_{21} + M_{21}M_{12} \\ &\quad - M_{11} - M_{22} = k(k+2) + C_2(M) - C_1(M), \end{aligned}$$

and, finally,

$$C_3 = -\frac{1}{2} C_1^3 + \frac{3}{2} C_1 C_2 + 3C_2 - 2C_1^2 - 2C_1.$$

In this realization, the Casimir operator C_3 is algebraically dependent on C_1 and C_2. In fact, C_1 and C_2 are nothing but the Casimir operators of the gl_2 sub-algebra. Therefore, the center of the gl_3 universal enveloping algebra in realization (11) is generated by the Casimir operators of the gl_2 sub-algebra realized by M_{ij}. Thus, it seems natural that these reps are irreducible.

Now we consider concrete matrix realizations of the gl_2-subalgebra in our scheme.

3.1. Reps in 1×1 matrices

This corresponds to the trivial representation of gl_2,

$$M_{11} = M_{12} = M_{21} = M_{22} = 0.$$

This is $[k, 0]$ or, in other words, a symmetric representation (the Young tableau has two rows of length k and 0, correspondingly). We also can call it a *scalar* representation, since the generators

$$\begin{aligned} &E_{11} = x_1 \partial_1, &&E_{22} = x_2 \partial_2, \\ &E_{12} = x_1 \partial_2, &&E_{21} = x_2 \partial_1, \\ &E_0 = k - x_1\partial_1 - x_2\partial_2, && \\ &T_1^- = \partial_1, &&T_2^- = \partial_2, \\ &T_1^+ = x_1(k - x_1\partial_1 - x_2\partial_2), && \\ &T_2^+ = x_2(k - x_1\partial_1 - x_2\partial_2), && \end{aligned} \tag{12}$$

act on one-component spinors or, in other words, on scalar functions (see e.g. [5]). The Casimir operators are:

$$C_1 = k, \qquad C_2 = k(k+2).$$

If parameter k takes non-negative integer the algebra gl_3 spanned by the generators (12) appears in finite-dimensional representation. Its finite-dimensional representation space is a space of polynomials

$$\mathcal{P}_{k,0} = \langle x_1{}^{p_1} x_2{}^{p_2} \mid 0 \le p_1+p_2 \le k \rangle, \quad k = 0,1,2,\dots. \tag{13}$$

Namely in this representation (12), the algebra gl_3 appears as the hidden algebra of the 3-body Calogero and Sutherland models [5], BC_2 rational and trigonometric, and G_2 rational models [6] and even of the BC_2 elliptic model [7].

3.2. Reps in 2×2 matrices

Take gl_2 in two-dimensional reps by 2×2 matrices,

$$M_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad M_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$
$$M_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad M_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

Then the generators (11) of gl_3 are:

$$T_1^- = \begin{pmatrix} \partial_1 & 0 \\ 0 & \partial_1 \end{pmatrix}, \qquad T_2^- = \begin{pmatrix} \partial_2 & 0 \\ 0 & \partial_2 \end{pmatrix},$$
$$E_{11} = \begin{pmatrix} x_1\partial_1+1 & 0 \\ 0 & x_1\partial_1 \end{pmatrix}, \quad E_{12} = \begin{pmatrix} x_1\partial_2 & 1 \\ 0 & x_1\partial_2 \end{pmatrix},$$
$$E_{21} = \begin{pmatrix} x_2\partial_1 & 0 \\ 1 & x_2\partial_1 \end{pmatrix}, \quad E_{22} = \begin{pmatrix} x_2\partial_2 & 0 \\ 0 & x_2\partial_2+1 \end{pmatrix},$$
$$E_0 = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix},$$
$$T_1^+ = \begin{pmatrix} x_1(A-1) & -x_2 \\ 0 & x_1 A \end{pmatrix},$$
$$T_2^+ = \begin{pmatrix} x_2 A & 0 \\ -x_1 & x_2(A-1) \end{pmatrix}, \tag{14}$$

where $A = k - x_1\partial_1 - x_2\partial_2$. This is $[k,1]$-representation (the Young tableau has two rows of length k and 1, correspondingly), and their Casimir operators are:

$$C_1 = k+1, \qquad C_2 = (k+1)^2.$$

If parameter k takes non-negative integer the algebra gl_3 spanned by the generators (14) appears in finite-dimensional representation.

Let us consider several different values of k in detail.

The case $k = 1$. Then three-dimensional representation space $V_1^{(2)}$ appears to be spanned by:

$$P_- = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad P_+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad Y_1 = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}. \tag{15}$$

This corresponds to antiquark multiplet in standard (fundamental) representation. The Newton polygon is a triangle with points $P_\pm$ as vortices at the base.

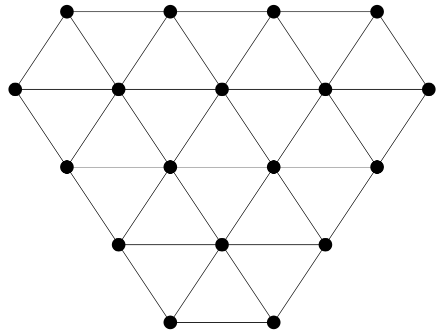

Figure 1. Newton hexagon for the representation space $V_4^{(2)}$ of the $[4,1]$-representation of dimension 24.

The case $k = 2$. Then eight-dimensional representation space $V_2^{(2)}$ appears to be spanned by:

$$P_- = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad P_+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad P_-^{(1)} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix},$$
$$Y_1^{(1)} = \begin{bmatrix} 0 \\ x_1 \end{bmatrix}, \quad Y_1^{(2)} = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}, \quad P_+^{(1)} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix},$$
$$Y_2 = \begin{bmatrix} x_2^2 \\ -x_1x_2 \end{bmatrix}, \quad Y_3 = \begin{bmatrix} x_1x_2 \\ -x_1^2 \end{bmatrix}. \tag{16}$$

This corresponds to octet in standard (fundamental) representation. Space $V_2^{(2)}$ contains $V_1^{(2)}$ as a subspace, $V_1^{(2)} \subset V_2^{(2)}$. It should be mentioned that $Y_1 = -Y_1^{(1)} + Y_1^{(2)}$. Now the Newton polygon is a hexagon where the central point is doubled, being presented by $Y_1^{(1,2)}$, and the lower (upper) base has length two being given by $P_\pm$ $(Y_{2,3})$.

The case $k = 3$. The representation space $V_3^{(2)}$ is 15-dimensional. In addition to $P_\pm, P_\pm^{(1)}$ and $Y_1^{(1,2)}$ (see (15) and (16)), it contains several vectors more, namely,

$$P_-^{(2)} = \begin{bmatrix} 0 \\ x_2^2 \end{bmatrix}, \quad P_+^{(2)} = \begin{bmatrix} x_1^2 \\ 0 \end{bmatrix}, \tag{17}$$

which are situated on the $\pm$-sides of the Newton hexagon, doubling the points corresponding to $Y_{2,3}$ (see (16))

$$Y_2^{(1)} = \begin{bmatrix} 0 \\ x_1x_2 \end{bmatrix}, \quad Y_2^{(2)} = \begin{bmatrix} x_2^2 \\ 0 \end{bmatrix},$$
$$Y_3^{(1)} = \begin{bmatrix} 0 \\ x_1^2 \end{bmatrix}, \quad Y_3^{(2)} = \begin{bmatrix} x_1x_2 \\ 0 \end{bmatrix}, \tag{18}$$

plus three extra vectors on the boundary

$$Y_8 = \begin{bmatrix} x_2^3 \\ -x_1x_2^2 \end{bmatrix}, \quad Y_9 = \begin{bmatrix} x_1x_2^2 \\ -x_1^2x_2 \end{bmatrix}, \quad Y_{10} = \begin{bmatrix} x_1^2x_2 \\ -x_1^3 \end{bmatrix}. \tag{19}$$

It is clear that $V_1^{(2)} \subset V_2^{(2)} \subset V_3^{(2)}$. All internal points of the Newton hexagon are double points, while the points on the boundary are single ones.

The general case. The finite-dimensional representation space $V_k^{(2)}$ has dimension $k(k+2)$ and is presented by the Newton hexagon, which contains $(k+1)$ horizontal layers. The lower base has length two, while the upper base has length k (see Fig. 1 as an illustration for $k=4$). All internal points of the Newton hexagon are double points, while the points on the boundary are single ones. Except for k vectors of the last (highest) layer of the Newton hexagon, the remaining $k(k+1)$ vectors span the space of all possible two-component spinors with components given by the inhomogeneous polynomials in x_1, x_2 of degree not higher than $(k-1)$. We denote this space as $\tilde{V}_k^{(2)} \subset V_k^{(2)}$. The non-trivial task is to describe k vectors of the last (highest) layer of the hexagon. After some analysis one can find that they have the form

$$Y_{k(k+1)+i} = \begin{bmatrix} x_2^{k-i} x_1^i \\ -x_2^{k-i-1} x_1^{i+1} \end{bmatrix}, \qquad i = 0,1,2,\ldots,(k-1), \tag{20}$$

hence they span a non-trivial k-dimensional subspace of spinors with components given by specific homogeneous polynomials of degree k.

3.3. Reps in 3×3 matrices

Take gl_2 in three-dimensional reps by 3×3 matrices,

$$M_{11} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad M_{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

$$M_{12} = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}, \quad M_{21} = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}.$$

Then the generators (11) of gl_3 are:

$$T_1^- = \begin{pmatrix} \partial_1 & 0 & 0 \\ 0 & \partial_1 & 0 \\ 0 & 0 & \partial_1 \end{pmatrix}, \quad T_2^- = \begin{pmatrix} \partial_2 & 0 & 0 \\ 0 & \partial_2 & 0 \\ 0 & 0 & \partial_2 \end{pmatrix},$$

$$E_{11} = \begin{pmatrix} x_1\partial_1 + 2 & 0 & 0 \\ 0 & x_1\partial_1 + 1 & 0 \\ 0 & 0 & x_1\partial_1 \end{pmatrix},$$

$$E_{12} = \begin{pmatrix} x_1\partial_2 & \sqrt{2} & 0 \\ 0 & x_1\partial_2 & \sqrt{2} \\ 0 & 0 & x_1\partial_2 \end{pmatrix},$$

$$E_{21} = \begin{pmatrix} x_2\partial_1 & 0 & 0 \\ \sqrt{2} & x_2\partial_1 & 0 \\ 0 & \sqrt{2} & x_2\partial_1 \end{pmatrix},$$

$$E_{22} = \begin{pmatrix} x_2\partial_2 & 0 & 0 \\ 0 & x_2\partial_2 + 1 & 0 \\ 0 & 0 & x_2\partial_2 + 2 \end{pmatrix},$$

$$E_0 = \begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{pmatrix},$$

$$T_1^+ = \begin{pmatrix} x_1(A-2) & -\sqrt{2}x_2 & 0 \\ 0 & x_1(A-1) & -\sqrt{2}x_2 \\ 0 & 0 & x_1 A \end{pmatrix},$$

$$T_2^+ = \begin{pmatrix} x_2 A & 0 & 0 \\ -\sqrt{2}x_1 & x_2(A-1) & 0 \\ 0 & -\sqrt{2}x_1 & x_2(A-2) \end{pmatrix}, \tag{21}$$

where $A = k - x_1\partial_1 - x_2\partial_2$. This is $[k,2]$-representation (the Young tableau has two rows of length k and 2, correspondingly) and their Casimir operators are:

$$C_1 = k+2, \qquad C_2 = (k+1)^2 + 3.$$

As an illustration let us explicitly show finite-dimensional representation spaces for $k=2,3$.

The case $k=2$. Then the six-dimensional representation space $V_2^{(3)}$ appears to be spanned by:

$$P_- = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad P_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad P_+ = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$$Y_1 = \begin{pmatrix} 0 \\ x_2 \\ -\sqrt{2}x_1 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} -\sqrt{2}x_2 \\ x_1 \\ 0 \end{pmatrix},$$

$$Y_3 = \begin{pmatrix} x_2^2 \\ -\sqrt{2}x_1x_2 \\ x_1^2 \end{pmatrix}. \tag{22}$$

This corresponds to 'di-antiquark' multiplet.

The case $k=3$. Then 15-dimensional representation space $V_3^{(3)}$ appears to be spanned by:

$$P_- = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad P_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad P_+ = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$$Y_1^{(1)} = \begin{pmatrix} 0 \\ x_2 \\ 0 \end{pmatrix}, \quad Y_1^{(2)} = \begin{pmatrix} 0 \\ 0 \\ x_1 \end{pmatrix}, \quad Y_2^{(1)} = \begin{pmatrix} x_2 \\ 0 \\ 0 \end{pmatrix},$$

$$Y_2^{(2)} = \begin{pmatrix} 0 \\ x_1 \\ 0 \end{pmatrix}, \quad P_-^{(1)} = \begin{pmatrix} 0 \\ 0 \\ x_2 \end{pmatrix}, \quad P_+^{(1)} = \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix},$$

$$Y_3^{(1)} = \begin{pmatrix} -\sqrt{2}x_2^2 \\ x_1x_2 \\ 0 \end{pmatrix}, \qquad Y_3^{(2)} = \begin{pmatrix} 0 \\ x_1x_2 \\ -\sqrt{2}x_1^2 \end{pmatrix},$$

$$Y_4 = \begin{pmatrix} 0 \\ -\sqrt{2}x_2^2 \\ 2x_1x_2 \end{pmatrix}, \qquad Y_5 = \begin{pmatrix} 2x_1x_2 \\ -\sqrt{2}x_1^2 \\ 0 \end{pmatrix},$$

$$Y_6 = \begin{pmatrix} x_2^3 \\ -\sqrt{2}x_1x_2^2 \\ x_1^2x_2 \end{pmatrix}, \quad Y_7 = \begin{pmatrix} x_1x_2^2 \\ -\sqrt{2}x_1^2x_2 \\ x_1^3 \end{pmatrix}. \tag{23}$$

It is worth mentioning that as a consequence of a particular realization of the generators (11) of the gl_3 algebra there exist a certain relations between generators other than those given by the Casimir operators. The first observation is that there are no linear relations between generators of such a type. Some time ago nine quadratic relations were found

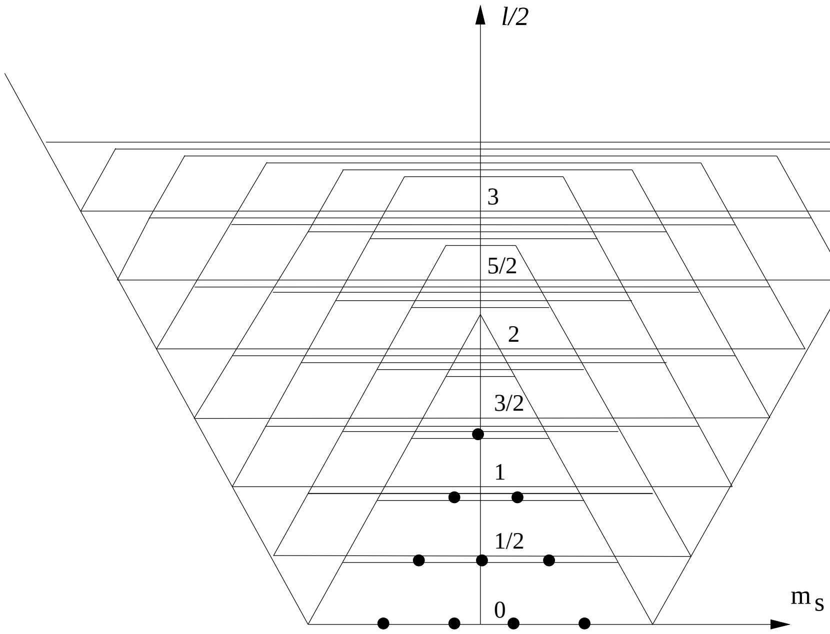

Figure 2. Verma module with the lowest weight (057) for $s = 5/2$.

between gl_3 generators taken in scalar representation (12) other than Casimir operators [8]. Surprisingly, certain modifications of these relations also exist for $[kn]$ mixed representations (11),

$$\begin{aligned}
-T_1^+E_{22} + T_2^+E_{12} &= x_1\big[M_{22}x_1\partial_1 \\
&+ M_{11}x_2\partial_2 + (M_{11}-k)M_{22} - M_{21}M_{12}\big] \\
&- x_2(x_1\partial_1 - k - 1)M_{12} - M_{21}x_1^2\partial_2 \equiv -\tilde{T}_1^+,
\end{aligned} \tag{24}$$

$$\begin{aligned}
-T_2^+E_{11} + T_1^+E_{21} &= x_2\big[M_{22}x_1\partial_1 \\
&+ M_{11}x_2\partial_2 + (M_{22}-k)M_{11} - M_{12}M_{21}\big] \\
&- x_1(x_2\partial_2 - k - 1)M_{21} - M_{12}x_2^2\partial_1 \equiv -\tilde{T}_2^+,
\end{aligned} \tag{25}$$

$$\begin{aligned}
&-E_{12}(E_0+1) + T_1^+T_2^- \\
&= M_{12}(x_1\partial_1 - k - 1) - M_{11}x_1\partial_2 \equiv -\tilde{E}_{12},
\end{aligned} \tag{26}$$

$$\begin{aligned}
&-E_{21}(E_0+1) + T_2^+T_1^- \\
&= M_{21}(x_2\partial_2 - k - 1) - M_{22}x_2\partial_1 \equiv -\tilde{E}_{21},
\end{aligned} \tag{27}$$

$$\begin{aligned}
T_1^+T_1^- - E_{11}(1+E_0) &= M_{11}x_2\partial_2 \\
&- M_{12}x_2\partial_1 - (k+1)M_{11} \equiv -\tilde{E}_{11},
\end{aligned} \tag{28}$$

$$\begin{aligned}
T_2^+T_2^- - E_{22}(1+E_0) &= M_{22}x_1\partial_1 \\
&- M_{21}x_1\partial_2 - (k+1)M_{22} \equiv -\tilde{E}_{22},
\end{aligned} \tag{29}$$

$$\begin{aligned}
E_{12}E_{21} - E_{11}E_{22} - E_{11} &= M_{12}x_2\partial_1 + M_{21}x_1\partial_2 \\
&- M_{22}x_1\partial_1 - M_{11}x_2\partial_2 + M_{12}M_{21} \\
&- M_{11}M_{22} - M_{11} \equiv -\hat{E}_{11},
\end{aligned} \tag{30}$$

$$E_{22}T_1^- - E_{21}T_2^- = M_{22}\partial_1 - M_{21}\partial_2 \equiv \tilde{T}_1^-, \tag{31}$$

$$E_{12}T_1^- - E_{11}T_2^- = M_{12}\partial_1 - M_{11}\partial_2 \equiv -\tilde{T}_2^-. \tag{32}$$

Not all these relations are independent. It can be shown that one relation is linearly dependent, since the sum of (28) + (29) + (30) gives the second Casimir operator C_2.

In scalar case, at least, we can assign a natural (vectorial) grading to the generators. The above relations also reflect a certain decomposition of the gradings,

$$\begin{aligned}
(1,0)(0,0) &= (0,1)(1,-1) \\
(0,1)(0,0) &= (1,0)(-1,0)
\end{aligned}$$

for the first two relations,

$$\begin{aligned}
(1,-1)(0,0) &= (1,0)(0,-1) \\
(-1,1)(0,0) &= (0,1)(-1,0)
\end{aligned}$$

for the second two,

$$\begin{aligned}
(1,0)(-1,0) &= (0,0)(0,0) \\
(0,1)(0,-1) &= (0,0)(0,0) \\
(1,-1)(-1,1) &= (0,0)(0,0)
\end{aligned}$$

for three before the last two, and

$$\begin{aligned}
(0,0)(-1,0) &= (-1,1)(0,-1) \\
(0,0)(0,-1) &= (1,-1)(-1,1)
\end{aligned}$$

for the last two.

4. Algebra $g^{(m)}$ in mixed representation

The basic property which was used to construct the mixed representation of the algebra gl_{n+1} is the existence of the Weyl-Cartan decomposition $gl_{n+1} = L \oplus (gl_n \oplus \mathbf{I}) \oplus U$ with property (6). One can pose a question about the existence of other algebras than gl_{n+1} for which the Weyl-Cartan decomposition with property (6) holds. The answer is affirmative. Let us consider the important particular case of the Cartan algebra $gl_2 \oplus \mathbf{I}$, and construct a realization of a new algebra denoted $g^{(m)}$ with the property

$$g^{(m)} = L_{m+1} \rtimes (gl_2 \oplus \mathbf{I}) \ltimes U_{m+1}, \qquad (33)$$

where $L_m(U_m)$ is the commutative algebra of the lowering (raising) generators with the property $[L_m, U_m] = P_{m-1}(gl_2 \oplus \mathbf{I})$ with P_{m-1} as a polynomial of degree $(m-1)$ in generators of $gl_2 \oplus \mathbf{I}$. Thus, it realizes a property of the generalized Gauss decomposition. The emerging algebra is a *polynomial* algebra. It is worth emphasizing that the realization we are going to construct appears at $\dim(L_k) = \dim(U_m) = m$. For $m = 1$ the algebra $g^{(1)} = gl_3$, see (6). Our final goal is to build the realization of (33) in terms of finite order differential operators acting on the plane $\mathbf{R}^2$.

The simplest realization of the algebra gl_2 by differential operators in two variables is the vector field representation, see (1) at $n = 2$. Exactly this representation was used to construct the representation of the gl_3 algebra acting of $\mathbf{R}^2$, see (11), (12). In this case $\dim(L_m) = \dim(U_m) = 2$. We are unable to find other algebras with $\dim(L_m) = \dim(U_m) > 2$. However, there exists another representation of the algebra gl_2 by the first order differential operators in two variables,

$$\begin{aligned} \tilde{J}_{12} &= \partial_x, \\ \tilde{J}_{11}^{(k)} &= -x\partial_x + \frac{k}{3}, \\ \tilde{J}_{22}^{(k)} &= -x\partial_x + sy\partial_y, \\ \tilde{J}_{21}^{(k)} &= x^2\partial_x + sxy\partial_y - kx, \end{aligned} \qquad (34)$$

(see S. Lie, [9] at $k = 0$ and A. González-Lopéz et al, [10] at $k \neq 0$ (Case 24)), where s, k are arbitrary numbers. These generators obey the standard commutation relations (2) of the algebra gl_2 in the vector field representation (1). It is evident that the sum of the two representations, $\tilde{J}_{ij}$ and the matrix representation M_{ij}, is also a representation,

$$J_{ij} \equiv \tilde{J}_{ij} + M_{ij} \in gl_2. \qquad (35)$$

(cf. (5)). It is worth mentioning that the gl_2 algebra commutation relations for M_{pm} are taken in a canonical form (4). The unity generator $\mathbf{I}$ in (33) is written in the form of a generalized Euler-Cartan operator

$$J_0^{(k)} = x\partial_x + sy\partial_y - k. \qquad (36)$$

Now let us assume that s is non-negative integer, $s = m$, $m = 0, 1, 2, \ldots$. Evidently, the lowering generators (of negative grading) from L_{m+1} can be given by

$$T_i^- = x^i\partial_y, \quad i = 0, 1, \ldots, m, \qquad (37)$$

forming commutative algebra

$$[T_i^-, T_j^-] = 0. \qquad (38)$$

(cf. [9, 10]). Eventually, the generators of the algebra $(gl_2 \oplus \mathbf{I}) \ltimes L_{m+1}$ take the form

$$\begin{aligned} J_{12} &= \partial_x + M_{12}, \\ J_{11}^{(k)} &= -x\partial_x + \frac{k}{3} + M_{11}, \\ J_{22}^{(k)} &= -x\partial_x + my\partial_y + M_{22}, \\ J_{21}^{(k)} &= x^2\partial_x + mxy\partial_y - kx + M_{21}, \end{aligned} \qquad (39)$$

with $J_0^{(k)}$ and T_i^- given by (36) and (37), respectively.

Let us consider two particular cases of the general construction of the raising generators for the commutative algebra U.

Case 1. For the first case we take the trivial matrix representation of the gl_2,

$$M_{11} = M_{12} = M_{21} = M_{22} = 0.$$

One can check that one of the raising generators is given by

$$U_0 = y\partial_x^m, \qquad (40)$$

while all other raising generators are multiple commutators of $J_{21}^{(k)}$ with U_0,

$$\begin{aligned} U_i &\equiv \underbrace{[J_{21}^{(k)}, [J_{21}^{(k)}, [\cdots J_{21}^{(k)}, T_0] \cdots]]}_{i} \\ &= y\partial_x^{m-i} J_0^{(k)}(J_0^{(k)} + 1)\ldots(J_0^{(k)} + i - 1)\ , \end{aligned} \qquad (41)$$

at $i = 1, \ldots m$. All of them are differential operators of fixed degree m. The procedure for construction of the operators U_i has the property of nilpotency:

$$U_i = 0, \quad i > m.$$

In particular, for $m = 1$,

$$U_0 = y\partial_x, \quad U_1 = yJ_0^{(k)} = y(x\partial_x + y\partial_y - k).$$

Inspecting the generators $T_{0,1}^-$, J_{ij}, $J^{(n)}$, $U_{0,1}$ one can see that they span the algebra gl_3, see (12). Hence, the algebra $g^{(1)} \equiv gl_3$.

If parameter k takes non-negative integer the algebra $g^{(m)}$ spanned by the generators (39), (40), (41) appears in finite-dimensional representation. Its finite-dimensional representation space is a triangular space of polynomials

$$\mathcal{P}_{k,0} = \langle x^{p_1}y^{p_2} \mid 0 \le p_1 + mp_2 \le k \rangle, \qquad k = 0, 1, 2, \ldots. \qquad (42)$$

Namely in this representation, the algebra $g^{(m)}$ appears as a hidden algebra of the 3-body G_2 trigonometric model [6] at $m = 2$ and of the so-called TTW model at integer m, in particular, of the dihedral $I_2(m)$ rational model [11].

Case 2. The second case is a certain evident extension when generators M_{ij} are of an arbitrary matrix representation of the algebra gl_2. Raising generators (40), (41) remain raising generators even if Cartan generators are given by (39) with arbitrary $M_{ij} \in gl_2$. However, the algebra is not closed: $[T, U] \neq P(gl_2 \oplus \mathbf{I})$. It can be fixed, at least, for the case $m = 1$. If M_{ij} are generators of gl_2 subalgebra of gl_3. By adding to T, U generators (38), (40), (41) the appropriate matrix generators from gl_3, the algebra gets closed. We end up with the gl_3 algebra of matrix differential operators other than (11). We are not aware of a solution to this problem for the case of $m \neq 1$ except for the case of trivial matrix representation, see Case 1.

5. Extension of the 3-body Calogero Model

The first *algebraic* form for the 3-body Calogero Hamiltonian [12] appears after gauge rotation with the ground state function, separation of the center-of-mass and changing the variables to elementary symmetric polynomials of the translationally-symmetric coordinates [5],

$$h_{\mathrm{Cal}} = -2\tau_2 \partial^2_{\tau_2\tau_2} - 6\tau_3 \partial^2_{\tau_2\tau_3} + \frac{2}{3}\tau_2^2 \partial^2_{\tau_3\tau_3} - \big[4\omega\tau_2 + 2(1+3\nu)\big]\partial_{\tau_2} - 6\omega\tau_3\partial_{\tau_3}. \quad (43)$$

These new coordinates are polynomial invariants of the A_2 Weyl group. Its eigenvalues are

$$-\epsilon_p = 2\omega(2p_1 + 3p_2), \quad p_{1,2} = 0, 1, \ldots . \quad (44)$$

As is shown in Ruhl and Turbiner [5], the operator (43) can be rewritten in a Lie-algebraic form in terms of $gl(3)$-algebra generators of the representation $[k, 0]$. The corresponding expression is

$$h_{\mathrm{Cal}} = -2E_{11}T_1^- - 6E_{22}T_1^- + \frac{2}{3}E_{12}E_{12} - 4\omega E_{11} - 2(1+3\nu)T_1^- - 6\omega E_{22}\ . \quad (45)$$

Now we can substitute the generators of the representation $[k, n]$ in the form (11)

$$\begin{aligned}\tilde{h}_{\mathrm{Cal}} = {} & -2\tau_2 \partial^2_{\tau_2\tau_2} - 6\tau_3 \partial^2_{\tau_2\tau_3} + \frac{2}{3}\tau_2^2 \partial^2_{\tau_3\tau_3} \\ & - 2\big[2\omega\tau_2 + (1+3\nu) + (n - 2M_{22})\big]\partial_{\tau_2} \\ & - \Big(6\omega\tau_3 - \frac{4}{3}M_{12}\tau_2\Big)\partial_{\tau_3} + \frac{2}{3}M_{12}M_{12} \\ & - 4\omega n - 2\omega M_{22}. \end{aligned} \quad (46)$$

This is an $n \times n$ matrix differential operator. It contains infinitely many finite-dimensional invariant subspaces which are nothing but finite-dimensional representation spaces of the algebra $gl(3)$. This operator remains exactly-solvable with the *same* spectra as the scalar Calogero operator.

This operator probably remains completely integrable. A higher-than-second-order integral is the differential operator of the sixth order ($\omega \neq 0$) or of the third order ($\omega = 0$), which takes an algebraic form after gauging away the ground state function in τ coordinates. It can be rewritten in terms of the $gl(3)$-algebra generators of the representation $[k, 0]$, which then can be replaced by the generators of the representation $[k, n]$. Under such a replacement the spectra of the integral remain unchanged and algebraic.

6. Extension of the 3-body Sutherland Model

The first *algebraic* form for the 3-body Sutherland Hamiltonian [13] appears after gauge rotation with the ground state function, separation of the center-of-mass and changing the variables to elementary symmetric polynomials of the exponentials of translationally-symmetric coordinates [5],

$$\begin{aligned}h_{\mathrm{Suth}} = {} & -\Big(2\eta_2 + \frac{\alpha^2}{2}\eta_2^2 - \frac{\alpha^4}{24}\eta_3^2\Big)\partial^2_{\eta_2\eta_2} \\ & - \Big(6 + \frac{4\alpha^2}{3}\eta_2\Big)\eta_3\partial^2_{\eta_2\eta_3} + \Big(\frac{2}{3}\eta_2^2 - \frac{\alpha^2}{2}\eta_3^2\Big)\partial^2_{\eta_3\eta_3} \\ & - \Big[2(1+3\nu) + 2\Big(\nu + \frac{1}{3}\Big)\alpha^2\eta_2\Big]\partial_{\eta_2} - 2\Big(\nu + \frac{1}{3}\Big)\alpha^2\eta_3\partial_{\eta_3}, \end{aligned} \quad (47)$$

where α is the inverse radius of the circle on which the bodies are situated. These new coordinates are fundamental trigonometric invariants of the A_2 Weyl group.

As shown in [5], operator (47) can be rewritten in a *Lie-algebraic* form in terms of the $gl(3)$-algebra generators of the representation $[k, 0]$,

$$\begin{aligned}h_{\mathrm{Suth}} = {} & -2E_{11}T_1^- - 6E_{22}T_1^- + \frac{2}{3}E_{12}E_{12} \\ & - 2(1+3\nu)T_1^- + \frac{\alpha^4}{24}E_{21}E_{21} - \frac{\alpha^2}{6}\Big[3E_{11}E_{11} + 8E_{11}E_{22} \\ & + 3E_{22}E_{22} + (1+12\nu)(E_{11}+E_{22})\Big]. \end{aligned} \quad (48)$$

Now we can substitute the generators of the representation $[k, n]$ in the form (11)

$$\begin{aligned}\tilde{h}_{\mathrm{Suth}} = {} & -\Big(2\eta_2 + \frac{\alpha^2}{2}\eta_2^2 - \frac{\alpha^4}{24}\eta_3^2\Big)\partial^2_{\eta_2\eta_2} \\ & - \Big(6 + \frac{4\alpha^2}{3}\eta_2\Big)\eta_3\partial^2_{\eta_2\eta_3} + \Big(\frac{2}{3}\eta_2^2 - \frac{\alpha^2}{2}\eta_3^2\Big)\partial^2_{\eta_3\eta_3} \\ & - 2\Big[(1+3\nu) + \Big(\nu + \frac{1}{3}\Big)\alpha^2\eta_2 + (n - 2M_{22})\Big]\partial_{\eta_2} \\ & + \frac{\alpha^4}{24}M_{21}\eta_3\partial_{\eta_2} + \Big[2\Big(\nu + \frac{1}{3}\Big)\alpha^2\eta_3 - \frac{4}{3}M_{12}\eta_2\Big]\partial_{\eta_3} \\ & - \frac{\alpha^2}{3}\Big[3n(\eta_2\partial_{\eta_2} + \eta_3\partial_{\eta_3}) + M_{11}\eta_3\partial_{\eta_3} + M_{22}\eta_2\partial_{\eta_2}\Big] \\ & + \frac{2}{3}M_{12}M_{12} + \frac{\alpha^4}{24}M_{21}M_{21} \\ & - \frac{\alpha^2}{6}\Big[2M_{11}M_{22} + (1+12\nu+3n)n\Big]. \end{aligned} \quad (49)$$

This is an $n \times n$ matrix differential operator. It contains infinitely-many finite-dimensional invariant subspaces which are nothing but finite-dimensional representation spaces of the algebra $gl(3)$. This operator remains exactly-solvable with the *same* spectra as the scalar Sutherland operator.

The operator (49) probably remains completely integrable. A non-trivial integral is the differential operator of the third order, it takes the algebraic form after gauging away the ground state function in η coordinates. It can be rewritten in terms of the $gl(3)$-algebra generators of the representation $[k, 0]$, which then can be replaced by the generators of the representation $[k, n]$. Under such a replacement the spectra of the integral remain unchanged and algebraic.

7. Conclusions

The algebra gl_n of differential operators plays the role of a hidden algebra for *all* A_n, B_n, C_n, D_n, BC_n Calogero-Moser Hamiltonians, both rational and trigonometric, with the Weyl symmetry of classical root spaces (see [14] and references therein). We have described a procedure which, in our opinion, should carry the name of the *Havlicek procedure*, to construct the algebra gl_n of the matrix differential operators. The procedure is based on a mixed, matrix-differential operator realization of the Gauss decomposition diagram.

As for Hamiltonian reduction models with the exceptional Weyl symmetry group $G_2, F_4, E_{6,7,8}$, both rational and trigonometric, there exist hidden algebras of differential operators (see [14] and references therein). All these algebras are infinite-dimensional but finitely-generated. For generating elements of these algebras an analogue of the Weyl-Cartan decomposition exists but in the Gauss decomposition diagram, a commutator of the lowering and raising generators is a polynomial of the higher-than-one order in the Cartan generators. Matrix realizations of these algebras surely exist. Thus, the above mentioned procedure for building the mixed representations can be realized. It may lead to a new class of matrix exactly-solvable models with exceptional Weyl symmetry.

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