

# THREE-VARIABLE ALTERNATING TRIGONOMETRIC FUNCTIONS AND CORRESPONDING FOURIER TRANSFORMS

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**ABSTRACT.** The common trigonometric functions admit generalizations to any higher dimension. In this paper, we restrict ourselves to three dimensional generalization only, focusing on alternating case in detail. Many specific properties of this new class of special functions are studied, such as the orthogonalities, both the continuous one and the discrete one on the 3D lattice of any density, discrete and continuous Fourier transforms, and others. Rapidly increasing precision of the interpolation with increasing density of the 3D lattice is shown in an example.

**KEYWORDS:** trigonometric functions; orthogonal polynomials.

## 1. INTRODUCTION

There are many applications of functions which are symmetric or antisymmetric with respect to the symmetry group  $S_n$  in mathematics. They appear for example in quantum theory or in theory of integrable systems. In [1] (anti)symmetric multivariate exponential functions were defined and corresponding Fourier transforms were given. These functions were examined in detail in dimension two and three in [2]. The natural question which arises is the restriction of the symmetry to the subgroup  $A_n$  of  $S_n$  consisting of transformations  $w$  with  $\det w = 1$ . Such functions were examined in [3] for  $A_n$  involving the general number  $n$ , and in [4] in greater detail concerning the smallest possible  $n$ , namely  $n = 3$ .

The exponential functions are not the only functions which admit symmetric, antisymmetric and alternating generalizations to higher dimensions. Their odd and even counterparts, sines and cosines, admit similar generalizations and fulfil many analogical properties as pure exponentials. Symmetric and antisymmetric case in general was covered in [5], alternating case was covered in [6]. Two-dimensional symmetric and antisymmetric generalizations were studied in detail in [7, 8].

The aim of this paper is to study in detail the alternating generalization of sine and cosine functions in dimension three.

(Anti)symmetric multivariate sine and cosine functions, considered in [5, 7, 8], as well as alternating multivariate sine and cosine functions, studied in [6] and in this paper, are closely related to symmetric and antisymmetric orbit functions studied in [9–11]. Because the definition of the considered functions and orbit functions are similar (roughly speaking, the exponentials are replaced by sines and cosines), they satisfy the same or similar properties, namely symmetry relations, reductions to the dominant or semidominant forms, periodicity etc. The discrete Fourier transforms of sine and cosine functions can be derived with the help of relations valid for exponential functions.[8] The restriction of the functions to three variables allows us to be more specific about the details of their properties in the following sections. This is most notable when describing their discretization and orthogonality relations.

The paper consists of several parts. After the first part, devoted to the definition of three dimensional alternating sine and cosine functions and their basic properties, come the parts describing their continuous orthogonality relations, their decomposition rules and the two parts treating four kinds of discrete cosine and sine transforms.

## 2. DEFINITION

Three dimensional alternating sine functions  $\sin_{(\lambda,\mu,\nu)} : \mathbb{R}^3 \rightarrow \mathbb{R}$  have the following explicit form [6]:

$$\begin{aligned} \sin_{(\lambda,\mu,\nu)}(x, y, z) &= \frac{1}{2} \begin{vmatrix} \sin 2\pi\lambda x & \sin 2\pi\lambda y & \sin 2\pi\lambda z \\ \sin 2\pi\mu x & \sin 2\pi\mu y & \sin 2\pi\mu z \\ \sin 2\pi\nu x & \sin 2\pi\nu y & \sin 2\pi\nu z \end{vmatrix} + \frac{1}{2} \begin{vmatrix} \sin 2\pi\lambda x & \sin 2\pi\lambda y & \sin 2\pi\lambda z \\ \sin 2\pi\mu x & \sin 2\pi\mu y & \sin 2\pi\mu z \\ \sin 2\pi\nu x & \sin 2\pi\nu y & \sin 2\pi\nu z \end{vmatrix}^+ \\ &= \sin 2\pi\lambda x \sin 2\pi\mu y \sin 2\pi\nu z + \sin 2\pi\lambda z \sin 2\pi\mu x \sin 2\pi\nu y + \sin 2\pi\lambda y \sin 2\pi\mu z \sin 2\pi\nu x, \\ & \qquad \qquad \qquad x, y, z, \lambda, \mu, \nu \in \mathbb{R}, \quad (1) \end{aligned}$$

where the second determinant with superscript  $+$  stands for *permanent* [12], which is symmetric with respect to permutations of its rows and columns.

Three dimensional alternating cosine functions  $\cos_{(\lambda,\mu,\nu)} : \mathbb{R}^3 \rightarrow \mathbb{R}$  are defined similarly:

$$\begin{aligned} \cos_{(\lambda,\mu,\nu)}(x, y, z) &= \frac{1}{2} \begin{vmatrix} \cos 2\pi\lambda x & \cos 2\pi\lambda y & \cos 2\pi\lambda z \\ \cos 2\pi\mu x & \cos 2\pi\mu y & \cos 2\pi\mu z \\ \cos 2\pi\nu x & \cos 2\pi\nu y & \cos 2\pi\nu z \end{vmatrix} + \frac{1}{2} \begin{vmatrix} \cos 2\pi\lambda x & \cos 2\pi\lambda y & \cos 2\pi\lambda z \\ \cos 2\pi\mu x & \cos 2\pi\mu y & \cos 2\pi\mu z \\ \cos 2\pi\nu x & \cos 2\pi\nu y & \cos 2\pi\nu z \end{vmatrix}^+ \\ &= \cos 2\pi\lambda x \cos 2\pi\mu y \cos 2\pi\nu z + \cos 2\pi\lambda z \cos 2\pi\mu x \cos 2\pi\nu y + \cos 2\pi\lambda y \cos 2\pi\mu z \cos 2\pi\nu x, \end{aligned}$$

$x, y, z, \lambda, \mu, \nu \in \mathbb{R}. \quad (2)$

From (1) we immediately see the (rotational) symmetry of  $\sin_{(\lambda,\mu,\nu)}(x, y, z)$  with respect to cyclic permutations of variables  $(x, y, z)$  and  $(\lambda, \mu, \nu)$ :

$$\sin_{(\lambda,\mu,\nu)}(x, y, z) = \sin_{(\lambda,\mu,\nu)}(z, x, y) = \sin_{(\lambda,\mu,\nu)}(y, z, x) = \sin_{(\nu,\lambda,\mu)}(x, y, z) = \sin_{(\mu,\nu,\lambda)}(x, y, z).$$

From (2) we have similar symmetries for  $\cos_{(\lambda,\mu,\nu)}(x, y, z)$ :

$$\cos_{(\lambda,\mu,\nu)}(x, y, z) = \cos_{(\lambda,\mu,\nu)}(z, x, y) = \cos_{(\lambda,\mu,\nu)}(y, z, x) = \cos_{(\nu,\lambda,\mu)}(x, y, z) = \cos_{(\mu,\nu,\lambda)}(x, y, z).$$

The rotational symmetries above allow us to put any real triple  $(\lambda, \mu, \nu)$  to the *canonical* form. For example, we may restrict ourselves to the functions  $\sin_{(\lambda,\mu,\nu)}$  and  $\cos_{(\lambda,\mu,\nu)}$  with so called *semidominant*  $(\lambda, \mu, \nu)$ , that is, triples  $(\lambda, \mu, \nu)$  where  $\lambda \geq \mu \geq \nu$  or  $\mu > \lambda > \nu$  holds.

The functions  $\sin_{(k,l,m)}$  and  $\cos_{(k,l,m)}$  with  $k, l, m \in \mathbb{Z}$  have additional symmetries induced by the periodicity of sine and cosine functions:

$$\begin{aligned} \sin_{(k,l,m)}(x + r, y + s, z + t) &= \sin_{(k,l,m)}(x, y, z), \quad r, s, t \in \mathbb{Z}, \\ \cos_{(k,l,m)}(x + r, y + s, z + t) &= \cos_{(k,l,m)}(x, y, z), \quad r, s, t \in \mathbb{Z}. \end{aligned} \quad (3)$$

Another symmetries of  $\sin_{(\lambda,\mu,\nu)}$  and  $\cos_{(\lambda,\mu,\nu)}$  follow from the well known fact that ordinary sine and cosine functions of one variable are odd and even functions, respectively. Therefore for all  $\lambda, \mu, \nu$  and  $x, y, z$  we have

$$\begin{aligned} \sin_{(\lambda,\mu,\nu)}(-x, y, z) &= \sin_{(\lambda,\mu,\nu)}(x, -y, z) = \sin_{(\lambda,\mu,\nu)}(x, y, -z) = -\sin_{(\lambda,\mu,\nu)}(x, y, z), \\ \cos_{(\lambda,\mu,\nu)}(-x, y, z) &= \cos_{(\lambda,\mu,\nu)}(x, -y, z) = \cos_{(\lambda,\mu,\nu)}(x, y, -z) = \cos_{(\lambda,\mu,\nu)}(x, y, z). \end{aligned} \quad (4)$$

There is also duality of  $\sin_{(\lambda,\mu,\nu)}$  and  $\cos_{(\lambda,\mu,\nu)}$  functions,

$$\sin_{(\lambda,\mu,\nu)}(x, y, z) = \sin_{(x,y,z)}(\lambda, \mu, \nu), \cos_{(\lambda,\mu,\nu)}(x, y, z) = \cos_{(x,y,z)}(\lambda, \mu, \nu). \quad (5)$$

Moreover,

$$\sin_{(k,l,m)}(x, y, z) = 0 \quad \text{when} \quad k = 0 \text{ or } l = 0 \text{ or } m = 0, \quad (6)$$

therefore we can exclude  $\sin_{(k,l,m)}$  with at least one index being zero from our considerations.

The relations (3)–(6) imply that it is sufficient to restrict to the following families of functions:

$$\sin_{(k,l,m)}, \quad k, l, m \in \mathbb{N}, k \geq l \geq m \text{ or } l > k > m,$$

and

$$\cos_{(k,l,m)}, \quad k, l, m \in \mathbb{Z}, k \geq l \geq m \geq 0 \text{ or } l > k > m \geq 0,$$

and to consider them on the closure of the *fundamental domain*  $F(\tilde{A}_3^{\text{aff}})$  [6] only. This fundamental domain of the dual affine group of  $A_3$  can be chosen in 3D to be equal to the part of the cube shown in Fig. 1.

$$F(\tilde{A}_3^{\text{aff}}) = \{(x, y, z) \in (0, \frac{1}{2}) \times (0, \frac{1}{2}) \times (0, \frac{1}{2}) | x > z, y > z\}.$$

### 3. CONTINUOUS ORTHOGONALITY

The alternating trigonometric functions  $\sin_{(k,l,m)}$  resp.  $\cos_{(k,l,m)}$  are pairwise orthogonal on  $F(\tilde{A}_3^{\text{aff}})$ . Let us denote

$$G_{klm} = \begin{cases} 3 & \text{if } k = l = m, \\ 1 & \text{otherwise.} \end{cases}$$

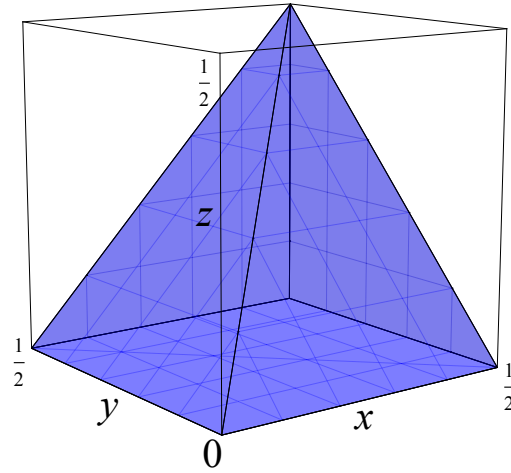


FIGURE 1. Fundamental domain  $F(\tilde{A}_3^{\text{aff}})$

Then we have

$$\int_{F(\tilde{A}_3^{\text{aff}})} \sin_{(k,l,m)}(x, y, z) \sin_{(k',l',m')}(x, y, z) \, dx \, dy \, dz = \frac{G_{klm}}{64} \delta_{kk'} \delta_{ll'} \delta_{mm'},$$

where  $k, l, m, k', l', m' \in \mathbb{N}$ ,  $k \geq l \geq m$  or  $l > k > m$ ,  $k' \geq l' \geq m'$  or  $l' > k' > m'$ , and

$$\int_{F(\tilde{A}_3^{\text{aff}})} \cos_{(k,l,m)}(x, y, z) \cos_{(k',l',m')}(x, y, z) \, dx \, dy \, dz = \frac{G_{klm}}{64} \delta_{kk'} \delta_{ll'} \delta_{mm'},$$

where  $k, l, m, k', l', m' \in \{0, 1, 2, \dots\}$ ,  $k \geq l \geq m$  or  $l > k > m$ ,  $k' \geq l' \geq m'$  or  $l' > k' > m'$ .

Let us have a function  $f: \mathbb{R}^3 \rightarrow \mathbb{R}$  with the following properties: rotational symmetry, that is,

$$f(x, y, z) = f(y, z, x) = f(z, x, y);$$

periodic, that is

$$f(x + r, y + s, z + t) = f(x, y, z) \text{ for all } r, s, t \in \mathbb{Z},$$

and odd in each variable, that is,

$$f(-x, y, z) = f(x, -y, z) = f(x, y, -z) = -f(x, y, z).$$

Let  $f$  be sufficiently smooth. Then it can be expanded in terms of the alternating sine functions, respectively, using formulas

$$f(x, y, z) = \sum_{\substack{k,l,m \geq 1 \\ k \geq l \geq m \text{ or } l > k > m}} c_{klm} \sin_{(k,l,m)}(x, y, z),$$

$$c_{klm} = 64G_{klm}^{-1} \int_{F(\tilde{A}_3^{\text{aff}})} f(x, y, z) \sin_{(k,l,m)}(x, y, z) \, dx \, dy \, dz.$$

Similarly, any function  $\tilde{f}: \mathbb{R}^3 \rightarrow \mathbb{R}$  with the symmetries

$$\tilde{f}(x, y, z) = f(y, z, x) = f(z, x, y),$$

$$\tilde{f}(x + r, y + s, z + t) = f(x, y, z) \text{ for all } r, s, t \in \mathbb{Z},$$

$$\tilde{f}(-x, y, z) = f(x, -y, z) = f(x, y, -z) = f(x, y, z),$$

and sufficiently smooth can be expanded in terms of the alternating cosine functions. The expansion is done using the formulas

$$\tilde{f}(x, y, z) = \sum_{\substack{k,l,m \geq 0 \\ k \geq l \geq m \text{ or } l > k > m}} \tilde{c}_{klm} \cos_{(k,l,m)}(x, y, z),$$

$$\tilde{c}_{klm} = 64G_{klm}^{-1} \int_{F(\tilde{A}_3^{\text{aff}})} \tilde{f}(x, y, z) \cos_{(k,l,m)}(x, y, z) \, dx \, dy \, dz.$$

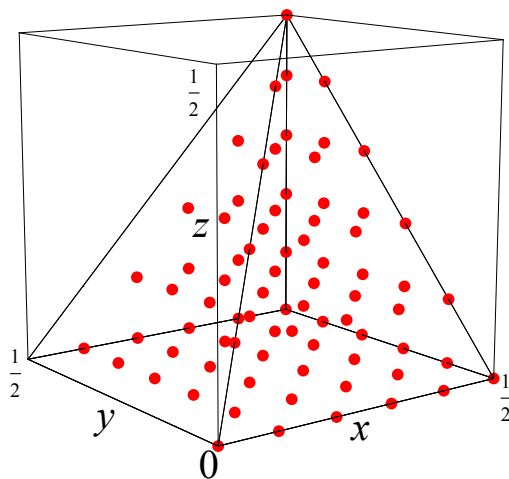


FIGURE 2. Finite grid of points in  $F(\tilde{A}_3^{\text{aff}})$

### 4. PRODUCT DECOMPOSITION

The product of two alternating cosines,  $\cos_{(\lambda, \mu, \nu)} \cos_{(\lambda', \mu', \nu')}$  (we omit general argument  $(x, y, z)$  here) can be decomposed into the sum of alternating cosines using the following formulas. The decomposition can serve e. g. for the derivation of recursion relations for the considered functions.

We have

$$\begin{aligned} \cos_{(\lambda, \mu, \nu)} \cos_{(\lambda', \mu', \nu')} &= \frac{1}{8} (\cos_{(\lambda - \lambda', \mu - \mu', \nu - \nu')} + \cos_{(\lambda - \lambda', \mu - \mu', \nu + \nu')} + \cos_{(\lambda - \lambda', \mu + \mu', \nu - \nu')} \\ &+ \cos_{(\lambda - \lambda', \mu + \mu', \nu + \nu')} + \cos_{(\lambda + \lambda', \mu - \mu', \nu - \nu')} + \cos_{(\lambda + \lambda', \mu - \mu', \nu + \nu')} + \cos_{(\lambda + \lambda', \mu + \mu', \nu - \nu')} + \cos_{(\lambda + \lambda', \mu + \mu', \nu + \nu')} \\ &+ \cos_{(\lambda - \mu', \mu - \nu', \nu - \lambda')} + \cos_{(\lambda - \mu', \mu - \nu', \lambda' + \nu)} + \cos_{(\lambda - \mu', \mu + \nu', \nu - \lambda')} + \cos_{(\lambda - \mu', \mu + \nu', \lambda' + \nu)} + \cos_{(\lambda + \mu', \mu - \nu', \nu - \lambda')} \\ &+ \cos_{(\lambda + \mu', \mu - \nu', \lambda' + \nu)} + \cos_{(\lambda + \mu', \mu + \nu', \nu - \lambda')} + \cos_{(\lambda + \mu', \mu + \nu', \lambda' + \nu)} + \cos_{(\lambda - \nu', \mu - \lambda', \nu - \mu')} + \cos_{(\lambda - \nu', \mu - \lambda', \mu' + \nu)} \\ &+ \cos_{(\lambda - \nu', \lambda' + \mu, \nu - \mu')} + \cos_{(\lambda - \nu', \lambda' + \mu, \mu' + \nu)} + \cos_{(\lambda + \nu', \mu - \lambda', \nu - \mu')} + \cos_{(\lambda + \nu', -\lambda' + \mu, \mu' + \nu)} + \cos_{(\lambda + \nu', \lambda' + \mu, \nu - \mu')} + \\ &\cos_{(\lambda + \nu', \lambda' + \mu, \mu' + \nu)}). \end{aligned} \tag{7}$$

Similarly, the product of two alternating sines,  $\sin_{(\lambda, \mu, \nu)} \sin_{(\lambda', \mu', \nu')}$  can be decomposed into the sum of alternating cosines using the following formula:

$$\begin{aligned} \sin_{(\lambda, \mu, \nu)} \sin_{(\lambda', \mu', \nu')} &= \frac{1}{8} (\cos_{(\lambda - \lambda', \mu - \mu', \nu - \nu')} - \cos_{(\lambda - \lambda', \mu - \mu', \nu + \nu')} - \cos_{(\lambda - \lambda', \mu + \mu', \nu - \nu')} \\ &+ \cos_{(\lambda - \lambda', \mu + \mu', \nu + \nu')} - \cos_{(\lambda + \lambda', \mu - \mu', \nu - \nu')} + \cos_{(\lambda + \lambda', \mu - \mu', \nu + \nu')} + \cos_{(\lambda + \lambda', \mu + \mu', \nu - \nu')} - \cos_{(\lambda + \lambda', \mu + \mu', \nu + \nu')} \\ &+ \cos_{(\lambda - \mu', \mu - \nu', \nu - \lambda')} - \cos_{(\lambda - \mu', \mu - \nu', \lambda' + \nu)} - \cos_{(\lambda - \mu', \mu + \nu', \nu - \lambda')} + \cos_{(\lambda - \mu', \mu + \nu', \lambda' + \nu)} - \cos_{(\lambda + \mu', \mu - \nu', \nu - \lambda')} \\ &+ \cos_{(\lambda + \mu', \mu - \nu', \lambda' + \nu)} + \cos_{(\lambda + \mu', \mu + \nu', \nu - \lambda')} - \cos_{(\lambda + \mu', \mu + \nu', \lambda' + \nu)} + \cos_{(\lambda - \nu', \mu - \lambda', \nu - \mu')} - \cos_{(\lambda - \nu', \mu - \lambda', \mu' + \nu)} \\ &- \cos_{(\lambda - \nu', \lambda' + \mu, \nu - \mu')} + \cos_{(\lambda - \nu', \lambda' + \mu, \mu' + \nu)} - \cos_{(\lambda + \nu', \mu - \lambda', \nu - \mu')} + \cos_{(\lambda + \nu', \mu - \lambda', \mu' + \nu)} + \cos_{(\lambda + \nu', \lambda' + \mu, \nu - \mu')} \\ &- \cos_{(\lambda + \nu', \lambda' + \mu, \mu' + \nu)}). \end{aligned} \tag{8}$$

For the sake of completeness, one can decompose also the product  $\sin_{(\lambda, \mu, \nu)} \cos_{(\lambda', \mu', \nu')}$ . This can be however easily obtained from the decomposition (7) by substitution  $\cos \rightarrow \sin$  on the right hand side.

### 5. DISCRETE COSINE TRANSFORMS

To each of standard discrete cosine transforms, denoted in the literature often as DCT-1, DCT-2, DCT-3 and DCT-4 [13], there corresponds an alternating three-dimensional discrete cosine transforms. We denote these corresponding transforms as AMDCT-1, AMDCT-2, AMDCT-3, AMDCT-4.

#### 5.1. AMDCT-1

For given  $N \in \mathbb{N}$  let us define a lattice

$$F_N = \left\{ \left( \frac{r}{2N}, \frac{s}{2N}, \frac{t}{2N} \right) \mid 0 \leq r, s, t \leq N, r \geq s \geq t \text{ or } s > r > t \right\}. \tag{9}$$

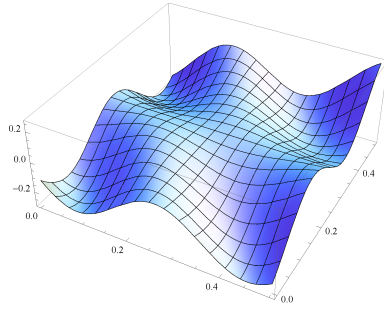


FIGURE 3. The function (10) used in the interpolation example.

| $N$ | $\int_{F(\tilde{A}_3^{\text{aff}})}  f - f_N ^2$ |
|-----|--|
| 1   | 0.0486711  |
| 2   | 0.0391407  |
| 3   | 0.0096493  |
| 4   | 0.0029516  |

TABLE 1. Integral error estimates for the approximations in Fig. 4.

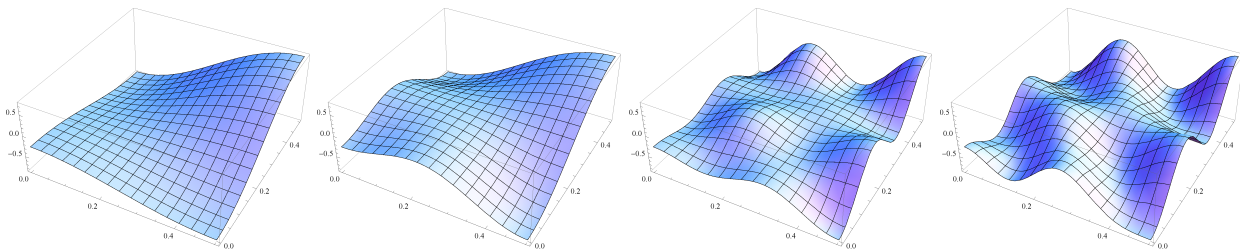


FIGURE 4. Alternating cosines approximations of (10) for  $N = 1, 2, 3, 4$ .

This lattice is chosen such way that it fulfills the whole space when symmetries (3) and (4) are applied to it. Evidently, we have a partial freedom which parts of boundary to include in the lattice. The lattice (9) contains  $\frac{1}{3}(N^3 + 3N^2 + 5N + 3)$  points. For example,

$$F_2 = \left\{ (0, 0, 0), \left(\frac{1}{4}, 0, 0\right), \left(\frac{1}{4}, \frac{1}{4}, 0\right), \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right), \left(\frac{1}{4}, \frac{1}{2}, 0\right), \left(\frac{1}{2}, 0, 0\right), \left(\frac{1}{2}, \frac{1}{4}, 0\right), \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right), \left(\frac{1}{2}, \frac{1}{2}, 0\right), \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{4}\right), \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \right\}.$$

Another example, for  $N = 5$ , is shown in Fig. 2.

The alternating cosines  $\cos_{(k,l,m)}$  are pairwise orthogonal on this lattice, namely we have

$$\sum_{\substack{0 \leq r,s,t \leq N, \\ r \geq s \geq t \text{ or } s > r > t}} G_{rst}^{-1} c_r c_s c_t \cos_{(k,l,m)}\left(\frac{r}{2N}, \frac{s}{2N}, \frac{t}{2N}\right) \cos_{(k',l',m')}\left(\frac{r}{2N}, \frac{s}{2N}, \frac{t}{2N}\right) = N^3 \tilde{c}_k \tilde{c}_l \tilde{c}_m G_{klm} \delta_{(k,l,m),(k',l',m')},$$

where

$$c_\alpha = \begin{cases} \frac{1}{2} & \text{if } \alpha = 0 \text{ or } \alpha = N, \\ 1 & \text{otherwise,} \end{cases} \quad \tilde{c}_\alpha = \begin{cases} 1 & \text{if } \alpha = 0 \text{ or } \alpha = N, \\ \frac{1}{2} & \text{otherwise,} \end{cases}$$

and  $0 \leq k, l, m \leq N$ ,  $k \geq l \geq m$  or  $l > k > m$ , and  $0 \leq k', l', m' \leq N$ ,  $k' \geq l' \geq m'$  or  $l' > k' > m'$ .

Let  $f$  be a real function defined on  $F_N$ . Then it can be expanded into a sum of alternating cosines,

$$f(x, y, z) = \sum_{\substack{0 \leq k,l,m \leq N, \\ k \geq l \geq m \text{ or } l > k > m}} a_{(k,l,m)} \cos_{(k,l,m)}(x, y, z),$$

where the coefficients  $a_{(k,l,m)}$  are given by the formula

$$a_{(k,l,m)} = \frac{1}{N^3 c_k c_l c_m G_{klm}} \sum_{\substack{0 \leq r,s,t \leq N, \\ r \geq s \geq t \text{ or } s > r > t}} \frac{\tilde{c}_r \tilde{c}_s \tilde{c}_t}{G_{rst}} f\left(\frac{r}{2N}, \frac{s}{2N}, \frac{t}{2N}\right) \cos_{(k,l,m)}\left(\frac{r}{2N}, \frac{s}{2N}, \frac{t}{2N}\right).$$

As an interpolation example, we take

$$f(x, y, z) = \cos 2\pi x \cos 2\pi y \cos 2\pi z \cos 10(x + y + z), \tag{10}$$

and compute four interpolations  $f_N$  for  $N = 1, 2, 3, 4$  using alternating cosines. The graph of the function (10) is shown in Fig. 3 and four approximations in Fig. 4.

By visual inspection, approximations  $f_N$  get closer to  $f$  as  $N$  increases. This is verified by computing integral error estimates shown in Table 1.

### 5.2. AMDCT-2

Alternating discrete cosine transform of second kind uses a lattice

$$\tilde{F}_N = \left\{ \left( \frac{r+1/2}{2N}, \frac{s+1/2}{2N}, \frac{t+1/2}{2N} \right) \mid 0 \leq r, s, t \leq N-1, r \geq s \geq t \text{ or } s > r > t \right\}. \tag{11}$$

This lattice contains  $\frac{1}{3}N(N^2 + 2)$  points. For example,

$$\tilde{F}_2 = \left\{ \left( \frac{1}{8}, \frac{1}{8}, \frac{1}{8} \right), \left( \frac{3}{8}, \frac{1}{8}, \frac{1}{8} \right), \left( \frac{3}{8}, \frac{3}{8}, \frac{1}{8} \right), \left( \frac{3}{8}, \frac{3}{8}, \frac{3}{8} \right) \right\}.$$

The alternating cosines  $\cos_{(k,l,m)}$  are pairwise orthogonal on this lattice, namely we have

$$\sum_{\substack{0 \leq r,s,t \leq N-1, \\ r \geq s \geq t \text{ or } s > r > t}} G_{rst}^{-1} c_r c_s c_t \cos_{(k,l,m)} \left( \frac{r+1/2}{2N}, \frac{s+1/2}{2N}, \frac{t+1/2}{2N} \right) \cos_{(k',l',m')} \left( \frac{r+1/2}{2N}, \frac{s+1/2}{2N}, \frac{t+1/2}{2N} \right) = N^3 \tilde{c}_k \tilde{c}_l \tilde{c}_m G_{klm} \delta_{(k,l,m),(k',l',m')},$$

where  $0 \leq k, l, m \leq N-1, k \geq l \geq m$  or  $l > k > m$ , and  $0 \leq k', l', m' \leq N-1, k' \geq l' \geq m'$  or  $l' > k' > m'$ .

Let  $\tilde{f}$  be a real function defined on  $\tilde{F}_N$ . Then it can be expanded into a sum of alternating cosines,

$$\tilde{f}(x, y, z) = \sum_{\substack{0 \leq k,l,m \leq N-1, \\ k \geq l \geq m \text{ or } l > k > m}} b_{(k,l,m)} \cos_{(k,l,m)}(x, y, z),$$

where the coefficients  $b_{(k,l,m)}$  are given by the formula

$$b_{(k,l,m)} = \frac{1}{N^3 c_k c_l c_m G_{klm}} \sum_{\substack{0 \leq r,s,t \leq N-1, \\ r \geq s \geq t \text{ or } s > r > t}} \frac{\tilde{c}_r \tilde{c}_s \tilde{c}_t}{G_{rst}} \tilde{f} \left( \frac{r+1/2}{2N}, \frac{s+1/2}{2N}, \frac{t+1/2}{2N} \right) \cos_{(k,l,m)} \left( \frac{r+1/2}{2N}, \frac{s+1/2}{2N}, \frac{t+1/2}{2N} \right).$$

### 5.3. AMDCT-3

Alternating discrete cosine transform of the third kind uses the lattice

$$F'_N = \left\{ \left( \frac{r}{2N}, \frac{s}{2N}, \frac{t}{2N} \right) \mid 0 \leq r, s, t \leq N-1, r \geq s \geq t \text{ or } s > r > t \right\}.$$

Instead of alternating cosines  $\cos_{(k,l,m)}$ , shifted cosines  $\cos_{(k+1/2,l+1/2,m+1/2)}$  are used. They have similar properties as normal alternating cosines with integer arguments. For scalar product we obtain

$$\sum_{\substack{0 \leq r,s,t \leq N-1, \\ r \geq s \geq t \text{ or } s > r > t}} G_{rst}^{-1} c_r c_s c_t \cos_{(k+1/2,l+1/2,m+1/2)} \left( \frac{r}{2N}, \frac{s}{2N}, \frac{t}{2N} \right) \cos_{(k'+1/2,l'+1/2,m'+1/2)} \left( \frac{r}{2N}, \frac{s}{2N}, \frac{t}{2N} \right) = N^3 \tilde{c}_k \tilde{c}_l \tilde{c}_m G_{klm} \delta_{(k,l,m),(k',l',m')},$$

where  $0 \leq k, l, m \leq N-1, k \geq l \geq m$  or  $l > k > m$ , and  $0 \leq k', l', m' \leq N-1, k' \geq l' \geq m'$  or  $l' > k' > m'$ .

Let  $g$  be a real function defined on  $F'_N$ . Then it can be expanded into a sum of shifted alternating cosines,

$$g(x, y, z) = \sum_{\substack{0 \leq k,l,m \leq N-1, \\ k \geq l \geq m \text{ or } l > k > m}} c_{(k,l,m)} \cos_{(k+1/2,l+1/2,m+1/2)}(x, y, z),$$

where the coefficients  $c_{(k,l,m)}$  are given by the formula

$$c_{(k,l,m)} = \frac{1}{N^3 c_k c_l c_m G_{klm}} \sum_{\substack{0 \leq r,s,t \leq N-1, \\ r \geq s \geq t \text{ or } s > r > t}} \frac{\tilde{c}_r \tilde{c}_s \tilde{c}_t}{G_{rst}} g \left( \frac{r}{2N}, \frac{s}{2N}, \frac{t}{2N} \right) \cos_{(k+1/2,l+1/2,m+1/2)} \left( \frac{r}{2N}, \frac{s}{2N}, \frac{t}{2N} \right).$$

### 5.4. AMDCT-4

Alternating discrete cosine transform of the fourth kind uses the lattice  $\tilde{F}_N$  and shifted cosines as in the third case. We have

$$\sum_{\substack{0 \leq r,s,t \leq N-1, \\ r \geq s \geq t \text{ or } s > r > t}} G_{rst}^{-1} c_r c_s c_t \cos_{(k+1/2,l+1/2,m+1/2)} \left( \frac{r+1/2}{2N}, \frac{s+1/2}{2N}, \frac{t+1/2}{2N} \right) \times \cos_{(k'+1/2,l'+1/2,m'+1/2)} \left( \frac{r+1/2}{2N}, \frac{s+1/2}{2N}, \frac{t+1/2}{2N} \right) = N^3 \tilde{c}_k \tilde{c}_l \tilde{c}_m G_{klm} \delta_{(k,l,m),(k',l',m')},$$

where  $0 \leq k, l, m \leq N-1, k \geq l \geq m$  or  $l > k > m$ , and  $0 \leq k', l', m' \leq N-1, k' \geq l' \geq m'$  or  $l' > k' > m'$ .

Let  $\tilde{f}$  be a real function defined on  $\tilde{F}_N$ . Then it can be expanded into a sum

$$\tilde{f}(x, y, z) = \sum_{\substack{0 \leq k, l, m \leq N-1, \\ k \geq l \geq m \text{ or } l > k > m}} d_{(k, l, m)} \cos_{(k+1/2, l+1/2, m+1/2)}(x, y, z),$$

where the coefficients  $d_{(k, l, m)}$  are given by the formula

$$d_{(k, l, m)} = \frac{1}{N^3 c_k c_l c_m G_{klm}} \sum_{\substack{0 \leq r, s, t \leq N-1, \\ r \geq s \geq t \text{ or } s > r > t}} \frac{\tilde{c}_r \tilde{c}_s \tilde{c}_t}{G_{rst}} \tilde{f}\left(\frac{r+1/2}{2N}, \frac{s+1/2}{2N}, \frac{t+1/2}{2N}\right) \cos_{(k+1/2, l+1/2, m+1/2)}\left(\frac{r+1/2}{2N}, \frac{s+1/2}{2N}, \frac{t+1/2}{2N}\right).$$

### 6. DISCRETE SINE TRANSFORMS

Analogical formulas for discrete sine transforms 1–4 can be derived. To each of standard discrete sine transforms, denoted in the literature often as DST-1, DST-2, DST-3 and DST-4, there corresponds an alternating three-dimensional discrete sine transforms. We denote these corresponding transforms as AMDST-1, AMDST-2, AMDST-3, AMDST-4.

#### 6.1. AMDST-1

For given  $N \in \mathbb{N}$  let us define a lattice

$$F_N^{(S)} = \left\{ \left( \frac{r}{2N}, \frac{s}{2N}, \frac{t}{2N} \right) \mid 1 \leq r, s, t \leq N - 1, r \geq s \geq t \text{ or } s > r > t \right\}.$$

This lattice contains  $\frac{1}{3}(N^3 - 3N^2 + 5N - 3)$  points. For example,

$$F_3^{(S)} = \left\{ \left( \frac{1}{6}, \frac{1}{6}, \frac{1}{6} \right), \left( \frac{1}{3}, \frac{1}{6}, \frac{1}{6} \right), \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{6} \right), \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) \right\}.$$

The alternating sines  $\sin_{(k, l, m)}$  are pairwise orthogonal on this lattice, namely we have

$$\sum_{\substack{1 \leq r, s, t \leq N-1, \\ r \geq s \geq t \text{ or } s > r > t}} G_{rst}^{-1} \sin_{(k, l, m)}\left(\frac{r}{2N}, \frac{s}{2N}, \frac{t}{2N}\right) \sin_{(k', l', m')}\left(\frac{r}{2N}, \frac{s}{2N}, \frac{t}{2N}\right) = \left(\frac{N}{2}\right)^3 G_{klm} \delta_{(k, l, m), (k', l', m')},$$

where  $1 \leq k, l, m \leq N - 1, k \geq l \geq m$  or  $l > k > m$ , and  $1 \leq k', l', m' \leq N - 1, k' \geq l' \geq m'$  or  $l' > k' > m'$ .

Let  $f^{(S)}$  be a real function defined on  $F_N^{(S)}$ . Then it can be expanded into a sum of alternating sines,

$$f^{(S)}(x, y, z) = \sum_{\substack{1 \leq k, l, m \leq N-1, \\ k \geq l \geq m \text{ or } l > k > m}} a_{(k, l, m)} \sin_{(k, l, m)}(x, y, z),$$

where the coefficients  $a_{(k, l, m)}^{(S)}$  are given by the formula

$$a_{(k, l, m)}^{(S)} = \frac{8}{N^3 G_{klm}} \sum_{\substack{1 \leq r, s, t \leq N-1, \\ r \geq s \geq t \text{ or } s > r > t}} \frac{1}{G_{rst}} f^{(S)}\left(\frac{r}{2N}, \frac{s}{2N}, \frac{t}{2N}\right) \sin_{(k, l, m)}\left(\frac{r}{2N}, \frac{s}{2N}, \frac{t}{2N}\right).$$

As an interpolation example, we take

$$f^{(S)}(x, y, z) = \sin 2\pi x \sin 2\pi y \sin 2\pi z \sin 10(x + y + z), \tag{12}$$

and compute four interpolations  $f_N^{(S)}$  for  $N = 1, 2, 3, 4$  using alternating sines. Picture of (12) is shown in Fig. 5 and four approximations in Fig. 6.

By visual inspection, approximations  $f_N^{(S)}$  get closer to (12) as  $N$  increases. This is verified by computing integral error estimates shown in Table 2.

#### 6.2. AMDST-2

Alternating discrete sine transform of second kind uses the same lattice as AMDCT-2, that is

$$\tilde{F}_N^{(S)} = \tilde{F}_N, \tag{13}$$

where  $\tilde{F}_N$  is given by (11).

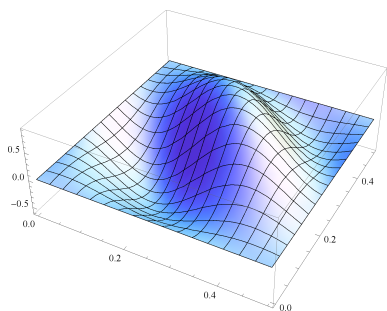


FIGURE 5. The function (12) used in interpolation example.

| N | $\int_{F(\tilde{A}_3^{\text{aff}})}  f^{(S)} - f_N^{(S)} ^2$ |
|---|--|
| 1 | 0.0482575  |
| 2 | 0.0282054  |
| 3 | 0.0078227  |
| 4 | 0.0006646  |

TABLE 2. Integral error estimates for the approximations in Fig. 6.

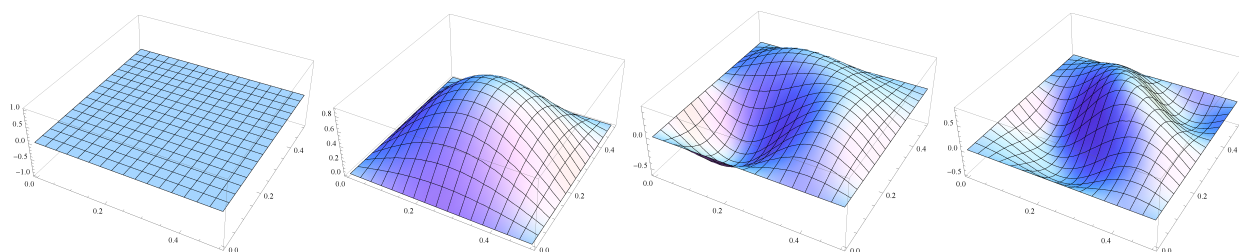


FIGURE 6. The alternating sines approximations of (12) for  $N = 1, 2, 3, 4$ .

The alternating sines  $\sin_{(k,l,m)}$  are pairwise orthogonal on this lattice, namely we have

$$\sum_{\substack{0 \leq r,s,t \leq N-1, \\ r \geq s \geq t \text{ or } s > r > t}} G_{rst}^{-1} \sin_{(k,l,m)}\left(\frac{r+1/2}{2N}, \frac{s+1/2}{2N}, \frac{t+1/2}{2N}\right) \sin_{(k',l',m')}\left(\frac{r+1/2}{2N}, \frac{s+1/2}{2N}, \frac{t+1/2}{2N}\right) = \left(\frac{N}{2}\right)^3 c_k c_l c_m G_{klm} \delta_{(k,l,m),(k',l',m')},$$

where  $c_p = \frac{1}{2}$  for  $p = N$  and  $c_p = 1$  otherwise and  $1 \leq k, l, m \leq N$ ,  $k \geq l \geq m$  or  $l > k > m$ , and  $1 \leq k', l', m' \leq N$ ,  $k' \geq l' \geq m'$  or  $l' > k' > m'$ .

Let  $\tilde{f}^{(S)}$  be a real function defined on  $\tilde{F}_N^{(S)}$ . Then it can be expanded into a sum of alternating sines,

$$\tilde{f}^{(S)}(x, y, z) = \sum_{\substack{1 \leq k,l,m \leq N, \\ k \geq l \geq m \text{ or } l > k > m}} b_{(k,l,m)}^{(S)} \sin_{(k,l,m)}(x, y, z),$$

where the coefficients  $b_{(k,l,m)}$  are given by the formula

$$b_{(k,l,m)}^{(S)} = \frac{8}{N^3 c_k c_l c_m G_{klm}} \sum_{\substack{0 \leq r,s,t \leq N-1, \\ r \geq s \geq t \text{ or } s > r > t}} \frac{1}{G_{rst}} \tilde{f}^{(S)}\left(\frac{r+1/2}{2N}, \frac{s+1/2}{2N}, \frac{t+1/2}{2N}\right) \sin_{(k,l,m)}\left(\frac{r+1/2}{2N}, \frac{s+1/2}{2N}, \frac{t+1/2}{2N}\right).$$

### 6.3. AMDST-3

Alternating discrete sine transform of the third kind uses the lattice

$$F_N^{(S)} = \left\{ \left( \frac{r}{2N}, \frac{s}{2N}, \frac{t}{2N} \right) \mid 1 \leq r, s, t \leq N, r \geq s \geq t \text{ or } s > r > t \right\}.$$

Instead of alternating sines  $\sin_{(k,l,m)}$ , shifted sines  $\sin_{(k+1/2,l+1/2,m+1/2)}$  are used. They have similar properties as normal alternating sines with integer arguments. For scalar product we obtain

$$\sum_{\substack{1 \leq r,s,t \leq N, \\ r \geq s \geq t \text{ or } s > r > t}} G_{rst}^{-1} c_r c_s c_t \sin_{(k+1/2,l+1/2,m+1/2)}\left(\frac{r}{2N}, \frac{s}{2N}, \frac{t}{2N}\right) \sin_{(k'+1/2,l'+1/2,m'+1/2)}\left(\frac{r}{2N}, \frac{s}{2N}, \frac{t}{2N}\right) = \left(\frac{N}{2}\right)^3 G_{klm} \delta_{(k,l,m),(k',l',m')},$$

where  $c_p = \frac{1}{2}$  for  $p = N$  and  $c_p = 1$  otherwise and  $0 \leq k, l, m \leq N - 1$ ,  $k \geq l \geq m$  or  $l > k > m$ , and  $0 \leq k', l', m' \leq N - 1$ ,  $k' \geq l' \geq m'$  or  $l' > k' > m'$ .



Let  $g^{(S)}$  be a real function defined on  $F'_N^{(S)}$ . Then it can be expanded into a sum of shifted alternating sines,

$$g^{(S)}(x, y, z) = \sum_{\substack{0 \leq k, l, m \leq N-1, \\ k \geq l \geq m \text{ or } l > k > m}} c_{(k, l, m)}^{(S)} \sin_{(k+1/2, l+1/2, m+1/2)}(x, y, z),$$

where the coefficients  $c_{(k, l, m)}^{(S)}$  are given by the formula

$$c_{(k, l, m)}^{(S)} = \frac{8}{N^3 G_{klm}} \sum_{\substack{1 \leq r, s, t \leq N, \\ r \geq s \geq t \text{ or } s > r > t}} \frac{c_r c_s c_t}{G_{rst}} g^{(S)}\left(\frac{r}{2N}, \frac{s}{2N}, \frac{t}{2N}\right) \sin_{(k+1/2, l+1/2, m+1/2)}\left(\frac{r}{2N}, \frac{s}{2N}, \frac{t}{2N}\right).$$

### 6.4. AMDST-4

Alternating discrete sine transform of the fourth kind uses the lattice  $\tilde{F}_N^{(S)}$  given by (13) and shifted sines as in the third case. We have

$$\sum_{\substack{0 \leq r, s, t \leq N-1, \\ r \geq s \geq t \text{ or } s > r > t}} G_{rst}^{-1} \sin_{(k+1/2, l+1/2, m+1/2)}\left(\frac{r+1/2}{2N}, \frac{s+1/2}{2N}, \frac{t+1/2}{2N}\right) \sin_{(k'+1/2, l'+1/2, m'+1/2)}\left(\frac{r+1/2}{2N}, \frac{s+1/2}{2N}, \frac{t+1/2}{2N}\right) = \left(\frac{N}{2}\right)^3 G_{klm} \delta_{(k, l, m), (k', l', m')},$$

where  $0 \leq k, l, m \leq N - 1, k \geq l \geq m$  or  $l > k > m$ , and  $0 \leq k', l', m' \leq N - 1, k' \geq l' \geq m'$  or  $l' > k' > m'$ .

Let  $\tilde{f}^{(S)}$  be a real function defined on  $\tilde{F}_N^{(S)}$ . Then it can be expanded into a sum

$$\tilde{f}^{(S)}(x, y, z) = \sum_{\substack{0 \leq k, l, m \leq N-1, \\ k \geq l \geq m \text{ or } l > k > m}} d_{(k, l, m)}^{(S)} \sin_{(k+1/2, l+1/2, m+1/2)}(x, y, z),$$

where the coefficients  $d_{(k, l, m)}^{(S)}$  are given by the formula

$$d_{(k, l, m)}^{(S)} = \frac{8}{N^3 G_{klm}} \sum_{\substack{0 \leq r, s, t \leq N-1, \\ r \geq s \geq t \text{ or } s > r > t}} \frac{1}{G_{rst}} \tilde{f}\left(\frac{r+1/2}{2N}, \frac{s+1/2}{2N}, \frac{t+1/2}{2N}\right) \sin_{(k+1/2, l+1/2, m+1/2)}\left(\frac{r+1/2}{2N}, \frac{s+1/2}{2N}, \frac{t+1/2}{2N}\right).$$

## 7. CONCLUSION

We have presented a detailed study of alternating three dimensional trigonometric functions and their properties including product decomposition, continuous and discrete orthogonality and interpolation problem. Practical computational aspects of the formalism presented in this paper need further investigation, namely a fast Fourier transform analog and comparing to the usual discrete Fourier transform in three dimensions.

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