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ALGEBRAIC DESCRIPTION OF SHAPE INVARIANCE REVISITED

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Abstract. We revisit the algebraic description of shape invariance method in one-dimensional quantum mechanics. In this note we focus on four particular examples: the Kepler problem in flat space, the Kepler problem in spherical space, the Kepler problem in hyperbolic space, and the Rosen–Morse potential problem. Following the prescription given by Gangopadhyaya *et al.*, we first introduce certain nonlinear algebraic systems. We then show that, if the model parameters are appropriately quantized, the bound-state problems can be solved solely by means of representation theory.

Keywords: exactly solvable models; shape invariance; representation theory.

1. Introduction

The purpose of this note is to revisit a couple of one-dimensional quantum-mechanical bound-state problems that can be solved exactly. In this note we shall focus on four particular examples: the Kepler problem in flat space, the Kepler problem in spherical space [1–3], the Kepler problem in hyperbolic space [4, 5], and the Rosen–Morse potential problem [6, 7], all of whose bound-state spectra are known to be exactly calculable. Hamiltonians of these problems[1] are respectively given by

$$\begin{aligned} H_{\text{Kepler}} &= -\frac{d^2}{dx^2} + \frac{j(j-1)}{x^2} - \frac{2g}{x}, & H_{\text{spherical Kepler}} &= -\frac{d^2}{dx^2} + \frac{j(j-1)}{\sin^2 x} - 2g\cot x, \\ H_{\text{hyperbolic Kepler}} &= -\frac{d^2}{dx^2} + \frac{j(j-1)}{\sinh^2 x} - 2g\coth x, & H_{\text{Rosen–Morse}} &= -\frac{d^2}{dx^2} - \frac{j(j-1)}{\cosh^2 x} - 2g\tanh x, \end{aligned} \tag{1.1}$$

where j and g are real parameters. The potential energies and bound-state spectra are depicted in Figure 1.

There exist several methods to solve the eigenvalue problems of these Hamiltonians (1.1). Among them is the shape invariance method [9],[2] which is based on the factorization of Hamiltonian and the Darboux transformation. And, as discussed by Gangopadhyaya *et al.* [11] (see also the reviews [12, 13]), the shape invariance can always be translated into the (Lie-)algebraic description—the so-called potential algebra.[3] The spectral problem can then be solved by means of representation theory. However, as far as we noticed, the representation theory of potential algebra has not been fully analyzed yet. In particular, the spectral problems of the above Hamiltonians have not been solved in terms of potential algebra. The purpose of this note is to fill this gap. As we will see below, these very old spectral problems require to introduce rather nontrivial nonlinear algebraic systems. The goal of this note is to show that these bound-state problems can be solved by representation theory of the operators $\{J_3, J_+, J_-\}$ that satisfy the linear commutation relations between J_3 and $J_\pm$

$$[J_3, J_\pm] = \pm J_\pm, \tag{1.2}$$

and the nonlinear commutation relations between J_+ and J_-

$$\text{(Kepler)} \quad [J_+, J_-] = -\frac{g^2}{J_3^2} + \frac{g^2}{(J_3-1)^2}, \qquad \text{(spherical Kepler)} \quad [J_+, J_-] = J_3^2 - \frac{g^2}{J_3^2} - (J_3-1)^2 + \frac{g^2}{(J_3-1)^2},$$

$$\text{(hyperbolic Kepler \& Rosen–Morse)} \quad [J_+, J_-] = -J_3^2 - \frac{g^2}{J_3^2} + (J_3-1)^2 + \frac{g^2}{(J_3-1)^2}. \tag{1.3}$$

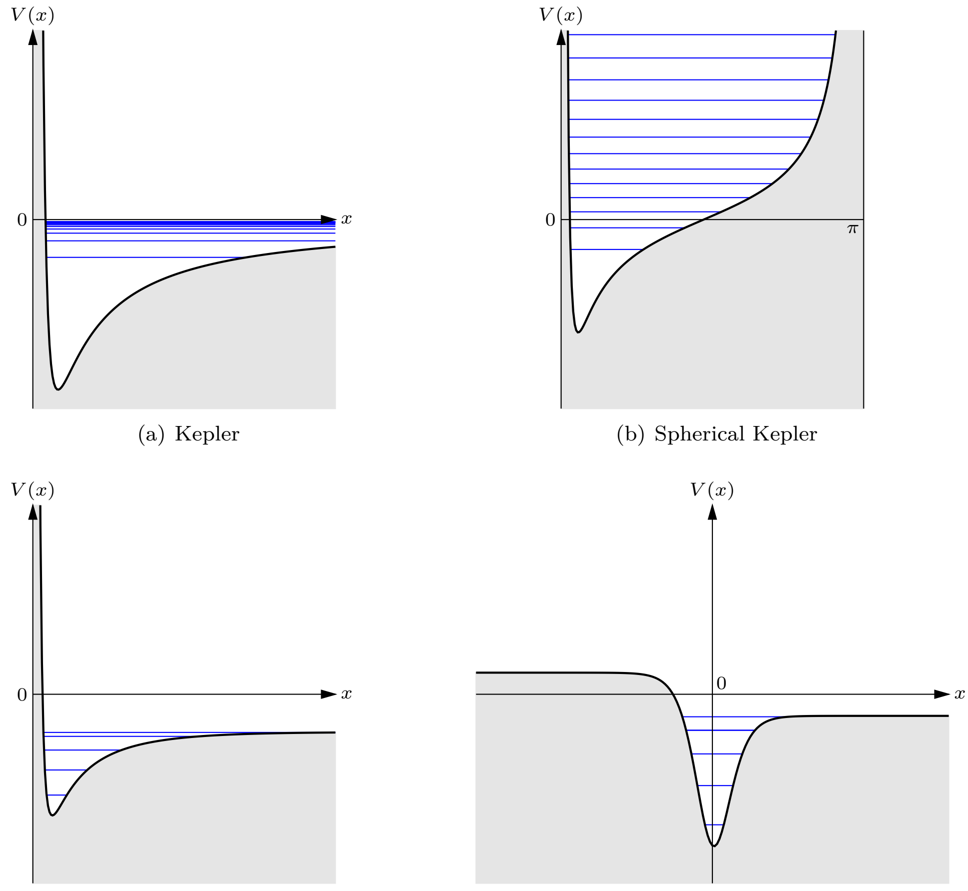
We will see that, if j is a half-integer, the bound-state problems of (1.1) can be solved from these operators.

The rest of the note is organized as follows: In Section 2 we introduce the potential algebra for the Kepler problem in flat space and solve the spectral problem by means of representation theory. In Sections 3 and 4

[1] These names for the Hamiltonians, though not so popular nowadays, are borrowed (with slight modifications) from Infeld and Hull [8]. Notice that these are different from those commonly used in the supersymmetric quantum mechanics literature [9].

[2] Recently it has been demonstrated that spectral intertwining relation provides a yet another scheme to solve the eigenvalue problems of H_{Kepler}, $H_{\text{spherical Kepler}}$, and $H_{\text{hyperbolic Kepler}}$ [10].

[3] A similar algebraic description for shape invariance has also been discussed by Balantekin [14].



(a) Kepler (b) Spherical Kepler

(c) Hyperbolic Kepler (d) Rosen–Morse

Figure 1. Potential energies (thick solid curves) and discrete energy levels (blue lines).

we generalize to the other problems. We shall see that the bound-state spectra of the hyperbolic Kepler and Rosen–Morse Hamiltonians just correspond to two distinct representations of the same algebraic system. We conclude in Section 5.

2. Kepler

Let us start with the Kepler problem in flat space. As is well known, the Kepler Hamiltonian H_{Kepler} in (1.1) can be factorized as follows:

$$H_{\mathrm{Kepler}} = A_-(j)A_+(j) - \frac{g^2}{j^2}, \tag{2.1}$$

where $A_\pm(j)$ are the first-order differential operators given by

$$A_\pm(j) = \pm\frac{d}{dx} - \frac{j}{x} + \frac{g}{j}. \tag{2.2}$$

Let us next introduce the potential algebra of this system. Following [11] with slight modifications, we first introduce an auxiliary periodic variable $\theta \in [0, 2\pi)$, then upgrade the parameter j to an operator $J_3 = -i\partial_\theta$, and then replace $A_+(j)$ and $A_-(j)$ to $J_+ = \mathrm{e}^{i\theta}A_+(J_3)$ and $J_- = A_-(J_3)\mathrm{e}^{-i\theta}$. The resultant operators that we wish to study are thus as follows:

$$J_3 = -i\partial_\theta, \qquad J_+ = \mathrm{e}^{i\theta}\Big(\partial_x - \frac{J_3}{x} + \frac{g}{J_3}\Big), \qquad J_- = \Big(-\partial_x - \frac{J_3}{x} + \frac{g}{J_3}\Big)\mathrm{e}^{-i\theta}. \tag{2.3}$$

Here one may wonder about the meaning of $1/J_3$. The operator $1/J_3$ would be defined as the spectral decomposition $1/J_3 = \sum_j (1/j)P_j$, where P_j stands for the projection operator onto the eigenspace of J_3 with eigenvalue j. This definition would be well-defined unless the spectrum of J_3 contains $j = 0$. An alternative way to give a meaning to $1/J_3$ would be the (formal) power series $\frac{1}{J_3} = \frac{1}{\lambda}\frac{1}{1-(1-J_3/\lambda)} = \frac{1}{\lambda}\sum_{n=0}^{\infty}(1 - \frac{J_3}{\lambda})^n$, where

λ is an arbitrary constant. This expression would be well-defined if the operator norm of $1 - J_3/\lambda$ satisfies $\|1 - J_3/\lambda\| < 1$. For the moment, however, we will proceed the discussion at the formal level.

It is not difficult to show that the operators (2.3) satisfy the following commutation relations:

$$[J_3, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = -\frac{g^2}{J_3^2} + \frac{g^2}{(J_3 - 1)^2}, \tag{2.4}$$

which follow from $\mathrm{e}^{\mp i\theta} J_3 \mathrm{e}^{\pm i\theta} = J_3 \pm 1$ or $J_3 \mathrm{e}^{\pm i\theta} = \mathrm{e}^{\pm i\theta}(J_3 \pm 1)$. It is also easy to check that the invariant operator of this algebraic system is given by

$$H = J_- J_+ - \frac{g^2}{J_3^2} = J_+ J_- - \frac{g^2}{(J_3 - 1)^2} = -\partial_x^2 + \frac{J_3(J_3 - 1)}{x^2} - \frac{2g}{x}, \tag{2.5}$$

which commutes with $J_\pm$ and J_3.[4] Notice that if $g = 0$ the commutation relations (2.4) just describe those for the Lie algebra $\mathfrak{iso}(2)$ of the two-dimensional Euclidean group. In this case the invariant operator H is nothing but the Casimir operator of the Lie algebra $\mathfrak{iso}(2)$.

Now, let $|E, j\rangle$ be a simultaneous eigenstate of H and J_3 that satisfies the eigenvalue equations

$$H|E, j\rangle = E|E, j\rangle, \qquad J_3|E, j\rangle = j|E, j\rangle, \tag{2.6}$$

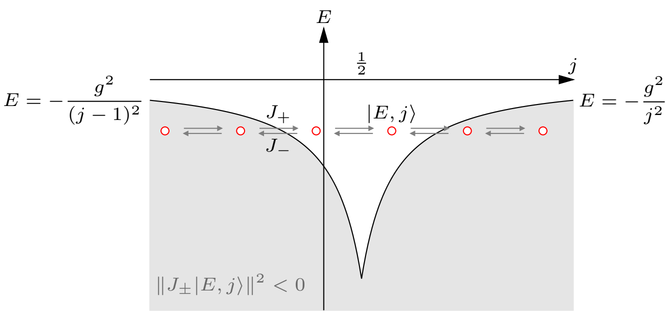
and the normalization condition $\||E, j\rangle\| = 1$. We wish to find the possible values of E and j. To this end, let us next consider the states $J_\pm|E, j\rangle$. As usual, the commutation relations (2.4) lead $J_3 J_\pm|E, j\rangle = (j \pm 1)J_\pm|E, j\rangle$, which implies $J_\pm$ raise and lower the eigenvalue j by ± 1:

$$J_\pm|E, j\rangle \propto |E, j \pm 1\rangle. \tag{2.7}$$

Proportional coefficients are determined by calculating the norms $\|J_\pm|E, j\rangle\|$. By using the relations $\|J_\pm|E, j\rangle\|^2 = \langle E, j|J_\mp J_\pm|E, j\rangle$, $J_- J_+ = H + g^2/J_3^2$, and $J_+ J_- = H + g^2/(J_3 - 1)^2$, we get

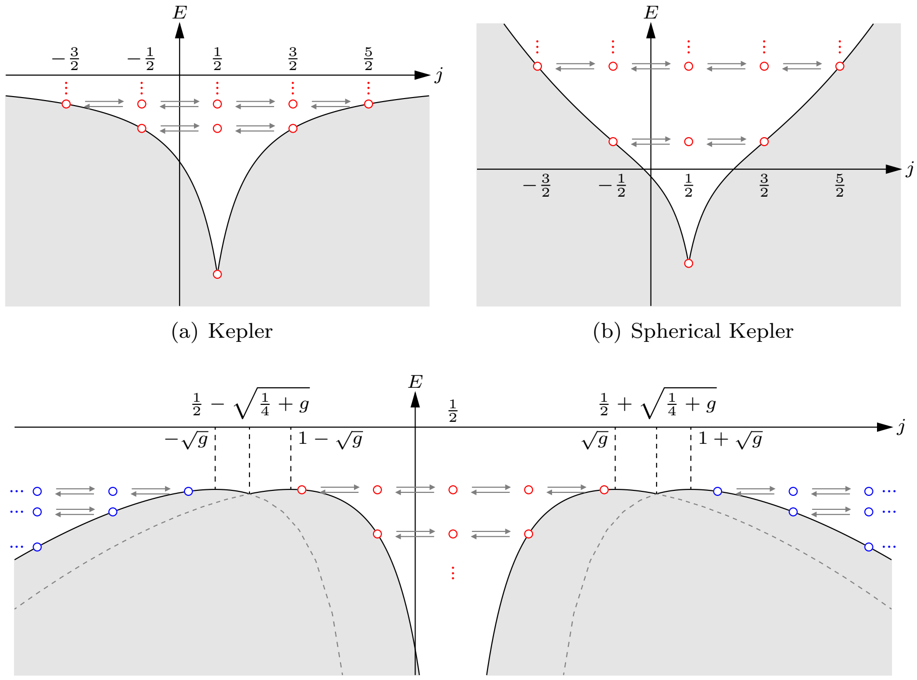
$$\|J_+|E, j\rangle\|^2 = E + \frac{g^2}{j^2} \geq 0, \qquad \|J_-|E, j\rangle\|^2 = E + \frac{g^2}{(j - 1)^2} \geq 0. \tag{2.8}$$

These equations not only fix the proportional coefficients in (2.7) but also provide nontrivial constraints on E and j. In fact, together with the ladder equations (2.7), the conditions (2.8) completely fix the possible values of E and j. To see this, let us consider a negative-energy state $|E, j\rangle$ that corresponds to an arbitrary point in the lower half of the (E, j)-plane. By applying the ladder operators $J_\pm$ to the state $|E, j\rangle$ one can easily see that such an arbitrary point eventually falls into the region in which the squared norms become negative. See the figure below:



The only way to avoid this is to terminate the sequence $\{\cdots, |E, j-1\rangle, |E, j\rangle, |E, j+1\rangle, \cdots\}$ from both above and below. This is possible if and only if there exist both the highest and lowest weight states $|E, j_{\max}\rangle$ and $|E, j_{\min}\rangle$ in the sequence such that $J_+|E, j_{\max}\rangle = 0 = J_-|E, j_{\min}\rangle$, $-g^2/j_{\max}^2 = -g^2/(j_{\min} - 1)^2$, $j_{\max} - j_{\min} \in \mathbb{Z}_{\geq 0}$, and $j_{\max} \geq 1/2$ and $j_{\min} \leq 1/2$. It is not difficult to see that these conditions are fulfilled if and only if the eigenvalue of the invariant operator takes the value $E = -g^2/\nu^2$, $\nu \in \{\frac{1}{2}, 1, \frac{3}{2}, 2, \cdots\}$. With this ν the eigenvalues of J_3 take the values $\{j_{\max} = \nu, \nu - 1, \cdots, 2 - \nu, j_{\min} = 1 - \nu\}$. Note, however, that if ν is an integer, the spectrum of J_3 contains $j = 0$ which makes the operator $1/J_3$ ill-defined. Thus we should disregard this case. To summarize,

[4] The commutation relation $[H, J_3] = 0$ is trivial. In order to prove $[H, J_\pm] = 0$, one should first note that $HJ_+ - J_+H = g^2(J_+\frac{1}{J_3^2} - \frac{1}{(J_3-1)^2}J_+)$ and $HJ_- - J_-H = g^2(J_-\frac{1}{(J_3-1)^2} - \frac{1}{J_3^2}J_-)$, which follow from (2.5). Then by using (2.3) and $\mathrm{e}^{-i\theta}\frac{1}{(J_3-1)^2}\mathrm{e}^{i\theta} = \frac{1}{J_3^2}$, one arrives at $[H, J_\pm] = 0$.



(a) Kepler

(b) Spherical Kepler

(c) Hyperbolic Kepler & Rosen–Morse: Case $g > 1/4$

FIGURE 2. Representations of the potential algebras. Gray shaded regions are the domains in which the squared norms $\|J_{\pm}|E, j\rangle\|^2$ become negative. Red circles represent the finite-dimensional representations, whereas blue circles represent the infinite-dimensional representations. Right and left arrows indicate the actions of ladder operators J_+ and J_-, respectively.

the representation of the potential algebra is specified by a half-integer $\nu \in \{\frac{1}{2}, \frac{3}{2}, \cdots\}$ and the representation space is spanned by the following 2ν vectors:

$$\Big\{|E, j\rangle : E = -\frac{g^2}{\nu^2} \text{ and } j \in \{\nu, \nu - 1, \cdots, 1 - \nu\}\Big\}. \tag{2.9}$$

These 2ν-dimensional representations are schematically depicted in Figure 2(a).

Now it is straightforward to solve the original spectral problem of the Kepler Hamiltonian H_{Kepler}. To this end, let $j \in \{\pm\frac{1}{2}, \pm\frac{3}{2}, \cdots\}$ be fixed. Since the Hamiltonian is invariant under $j \to 1 - j$, without any loss of generality we can focus on the case $j \in \{\frac{1}{2}, \frac{3}{2}, \cdots\}$. Then the discrete energy eigenvalues read

$$E_n = -\frac{g^2}{(j+n)^2}, \quad n \in \{0, 1, \cdots\}. \tag{2.10}$$

The energy eigenfunction $\psi_{E_n,j}(x)$ that satisfies the Schrödinger equation $H_{\text{Kepler}}\psi_{E_n,j} = E_n\psi_{E_n,j}$ can be determined by the formula $|E_n, j\rangle \propto (J_-)^n|E_n, j+n\rangle$. Noting that $|E, j\rangle$ corresponds to the function $\psi_{E,j}(x)\mathrm{e}^{ij\theta}$ and J_- is given by $J_- = A_-(J_3)\mathrm{e}^{-i\theta}$, we get the following Rodrigues-like formula:

$$\psi_{E_n,j}(x) \propto A_-(j)A_-(j+1)\cdots A_-(j+n-1)\psi_{E_n,j+n}(x), \tag{2.11}$$

where $\psi_{E_n,j+n}(x)$ is a solution to the first-order differential equation $A_+(j+n)\psi_{E_n,j+n}(x) = 0$ and given by $\psi_{E_n,j+n}(x) \propto x^{j+n}\exp(-\frac{g}{j+n}x)$. All of these exactly coincide with the well-known results.

In the rest of the note we would like to apply the same idea to the spectral problem for the spherical Kepler, hyperbolic Kepler, and Rosen–Morse Hamiltonians. We shall first introduce the potential algebras, and then classify their representations, and then solve the bound-state problems. As we will see below, the spherical Kepler problem is rather straightforward but the hyperbolic Kepler and Rosen–Morse potential problems are more intriguing and require careful analysis.

3. Spherical Kepler

Let us next move on to the spherical Kepler problem [1–3], whose Hamiltonian is factorized as follows:

$$H_{\text{spherical Kepler}} = A_-(j)A_+(j) + j^2 - \frac{g^2}{j^2}, \tag{3.1}$$

where

$$A_\pm(j) = \pm\frac{d}{dx} - j\cot x + \frac{g}{j}. \tag{3.2}$$

Just as in the previous section, let us next introduce the following operators:

$$J_3 = -i\partial_\theta, \qquad J_+ = \mathrm{e}^{i\theta}\Big(\partial_x - \cot x J_3 + \frac{g}{J_3}\Big), \qquad J_- = \Big(-\partial_x - \cot x J_3 + \frac{g}{J_3}\Big)\mathrm{e}^{-i\theta}, \tag{3.3}$$

which satisfy the following commutation relations:

$$[J_3, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = J_3^2 - \frac{g^2}{J_3^2} - (J_3-1)^2 + \frac{g^2}{(J_3-1)^2}. \tag{3.4}$$

The invariant operator that commutes with J_3 and $J_\pm$ is given by

$$H = J_-J_+ + J_3^2 - \frac{g^2}{J_3^2} = J_+J_- + (J_3-1)^2 - \frac{g^2}{(J_3-1)^2} = -\partial_x^2 + \frac{J_3(J_3-1)}{\sin^2 x} - 2g\cot x. \tag{3.5}$$

It should be noted that, if $g = 0$, (3.4) reduces to the standard commutation relations for the Lie algebra $\mathfrak{so}(3)$ under the appropriate shift $J_3 \to J_3 + 1/2$. In this case the invariant operator H is nothing but the Casimir operator of $\mathfrak{so}(3)$ and provides a well-known example of interplay between shape invariance and Lie algebra; see, e.g., the review [12].

Now, let $|E, j\rangle$ be a simultaneous eigenstate of H and J_3 that satisfies the eigenvalue equations

$$H|E, j\rangle = E|E, j\rangle, \qquad J_3|E, j\rangle = j|E, j\rangle, \tag{3.6}$$

as well as the normalization condition $\||E, j\rangle\| = 1$. Then we have the following conditions:

$$\|J_+|E, j\rangle\|^2 = E - j^2 + \frac{g^2}{j^2} \geq 0, \qquad \|J_-|E, j\rangle\|^2 = E - (j-1)^2 + \frac{g^2}{(j-1)^2} \geq 0, \tag{3.7}$$

which, together with the ladder equations $J_\pm|E, j\rangle \propto |E, j \pm 1\rangle$, restrict the possible values of E and j. As discussed in the previous section, these conditions are compatible with each other if and only if the eigenvalue of the invariant operator takes the value $E = \nu^2 - g^2/\nu^2$, $\nu \in \{\frac{1}{2}, \frac{3}{2}, \cdots\}$. Now let $\nu \in \{\frac{1}{2}, \frac{3}{2}, \cdots\}$ be fixed. Then the representation space is spanned by the following 2ν vectors:

$$\Big\{|E, j\rangle : E = \nu^2 - \frac{g^2}{\nu^2} \text{ and } j \in \{\nu, \nu-1, \cdots, 1-\nu\}\Big\}. \tag{3.8}$$

These 2ν-dimensional representations are schematically depicted in Figure 2(b).

Now it is easy to find the spectrum of the original Hamiltonian $H_{\text{spherical Kepler}}$. For fixed $j \in \{\frac{1}{2}, \frac{3}{2}, \cdots\}$ the energy eigenvalues and eigenfunctions read

$$E_n = (j+n)^2 - \frac{g^2}{(j+n)^2}, \quad n \in \{0, 1, \cdots\}, \tag{3.9}$$

and

$$\psi_{E_n,j}(x) \propto A_-(j)A_-(j+1)\cdots A_-(j+n-1)\psi_{E_n,j+n}(x), \tag{3.10}$$

where $\psi_{E_n,j+n}(x) \propto (\sin x)^{j+n}\exp(-\frac{g}{j+n}x)$. We note that (3.9) and (3.10) are consistent with the known results [1–3].

4. Hyperbolic Kepler & Rosen–Morse

Let us finally move on to the study of potential algebras for the hyperbolic Kepler and Rosen–Morse Hamiltonians. We shall see that the bound-state spectra of these problems correspond to two distinct representations of a single algebraic system.

4.1. Hyperbolic Kepler

The Hamiltonian for the hyperbolic Kepler problem [4, 5] can be factorized as follows:

$$H_{\text{hyperbolic Kepler}} = A_-(j)A_+(j) - j^2 - \frac{g^2}{j^2}, \tag{4.1}$$

where

$$A_\pm(j) = \pm\frac{d}{dx} - j\coth x + \frac{g}{j}. \tag{4.2}$$

We then introduce the following operators:

$$J_3 = -i\partial_\theta, \qquad J_+ = \mathrm{e}^{i\theta}\Big(\partial_x - \coth x J_3 + \frac{g}{J_3}\Big), \qquad J_- = \Big(-\partial_x - \coth x J_3 + \frac{g}{J_3}\Big)\mathrm{e}^{-i\theta}, \tag{4.3}$$

which satisfy the following commutation relations:

$$[J_3, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = -J_3^2 - \frac{g^2}{J_3^2} + (J_3-1)^2 + \frac{g^2}{(J_3-1)^2}. \tag{4.4}$$

The invariant operator is given by

$$H = J_-J_+ - J_3^2 - \frac{g^2}{J_3^2} = J_+J_- - (J_3-1)^2 - \frac{g^2}{(J_3-1)^2} = -\partial_x^2 + \frac{J_3(J_3-1)}{\sinh^2 x} - 2g\coth x. \tag{4.5}$$

We note that, if $g=0$, (4.4) reduce to the standard commutation relations for the Lie algebra $\mathfrak{so}(2,1)$ under the shift $J_3 \to J_3 + 1/2$. In other words, the operators (4.3) provide one of differential realizations of $\mathfrak{so}(2,1)$ if $g=0$ and $J_3 \to J_3 + 1/2$. Unfortunately, however, this Lie-algebraic structure is less useful in the present problem because the invariant operator (4.5) does not contain discrete eigenvalues if $g=0$ and J_3 has real eigenvalues. As we will see shortly, however, this situation gets changed if g is non-vanishing.

Now, let $|E,j\rangle$ be a simultaneous eigenstate of H and J_3:

$$H|E,j\rangle = E|E,j\rangle, \qquad J_3|E,j\rangle = j|E,j\rangle. \tag{4.6}$$

Then, under the normalization condition $\||E,j\rangle\| = 1$, the squared norms $\|J_\pm|E,j\rangle\|^2$ are evaluated as follows:

$$\|J_+|E,j\rangle\|^2 = E + j^2 + \frac{g^2}{j^2} \geq 0, \qquad \|J_-|E,j\rangle\|^2 = E + (j-1)^2 + \frac{g^2}{(j-1)^2} \geq 0. \tag{4.7}$$

These conditions are enough to classify representations. In contrast to the previous two examples, there are several nontrivial representations depending on the range of j. For $g > 1/4$, we have the following three distinct representations (see Figure 2(c)):

- **Case $j \in (-\infty, -\sqrt{g})$: Infinite-dimensional representation.** Let $\nu \in (-\infty, -\sqrt{g})$ be fixed. Then the representation space is spanned by the following infinitely many vectors:

$$\Big\{|E,j\rangle : E = -\nu^2 - \frac{g^2}{\nu^2} \text{ and } j \in \{\nu, \nu-1, \cdots\}\Big\}. \tag{4.8}$$

  We emphasize that in this case the parameter $\nu \in (-\infty, -\sqrt{g})$ is not necessarily restricted to an integer or half-integer. This is a one-parameter family of infinite-dimensional representation of the algebraic system $\{J_3, J_+, J_-\}$.

- **Case $j \in (1-\sqrt{g}, \sqrt{g})$: Finite-dimensional representation.** Let $\nu \in \{\frac{1}{2}, \frac{3}{2}, \cdots, \nu_{\max}\}$ be fixed, where $\nu_{\max}$ is the maximal half-integer smaller than $\sqrt{g}$; i.e., $\nu_{\max} = \max\{\nu \in \frac{1}{2}\mathbb{N} : \nu < \sqrt{g}\}$. Then the representation space is spanned by the following 2ν vectors:

$$\Big\{|E,j\rangle : E = -\nu^2 - \frac{g^2}{\nu^2} \text{ and } j \in \{\nu, \nu-1, \cdots, 1-\nu\}\Big\}. \tag{4.9}$$

  This is a 2ν-dimensional representation of the algebraic system $\{J_3, J_+, J_-\}$.

- **Case $j \in (1+\sqrt{g}, \infty)$: Infinite-dimensional representation.** Let $\nu \in (1+\sqrt{g}, \infty)$ be fixed. Then the representation space is spanned by the following infinitely many vectors:

$$\Big\{|E,j\rangle : E = -(\nu-1)^2 - \frac{g^2}{(\nu-1)^2} \text{ and } j \in \{\nu, \nu+1, \cdots\}\Big\}. \tag{4.10}$$

  Note that $\nu \in (1+\sqrt{g}, \infty)$ is a continuous parameter and is not necessarily be an integer or half-integer. This is another one-parameter family of infinite-dimensional representation of the algebraic system $\{J_3, J_+, J_-\}$.

One may notice that the region $[-\sqrt{g}, 1-\sqrt{g}] \cup [\sqrt{g}, 1+\sqrt{g}]$ is excluded in the above classification. This is because there is no bound state in this region for both the hyperbolic Kepler and Rosen–Morse potential problems. We note that the finite-dimensional representation (4.9) disappears for $g \leq 1/4$, whereas the infinite-dimensional representations (4.8) and (4.10) remain present for $g \leq 1/4$.

Now we have classified the representations of the potential algebra. The next task we have to do is to understand which representations are realized in the hyperbolic Kepler problem. To see this, let us consider the potential $V(x) = j(j-1)/\sinh^2 x - 2g\coth x$. In order to have a bound state, it is necessary that $V(x)$ has a minimum on the half line.[5] This is achieved if and only if j is in the range $(\frac{1}{2} - \sqrt{g+\frac{1}{4}}, \frac{1}{2} + \sqrt{g+\frac{1}{4}})$, which includes $(1-\sqrt{g}, \sqrt{g})$; see Figure 2(c). Hence the bound state spectrum should be related to the finite-dimensional representation (4.9).

Now it is easy to solve the original eigenvalue problem $H_{\text{hyperbolic Kepler}}\psi_{E_n,j} = E_n\psi_{E_n,j}$ for the hyperbolic Kepler Hamiltonian. For fixed $j \in \{\frac{1}{2}, \frac{3}{2}, \cdots, \nu_{\max}\}$, the energy eigenvalues and eigenfunctions are given by

$$E_n = -(j+n)^2 - \frac{g^2}{(j+n)^2}, \quad n \in \{0, 1, \cdots, N\}, \tag{4.11}$$

and

$$\psi_{E_n,j}(x) \propto A_-(j)A_-(j+1)\cdots A_-(j+n-1)\psi_{E_n,j+n}(x), \tag{4.12}$$

where $N = \max\{n \in \mathbb{Z}_{\geq 0} : j+n < \sqrt{g}\} = \nu_{\max} - j$ and $\psi_{E_n,j+n}(x) \propto (\sinh x)^{j+n}\exp(-\frac{g}{j+n}x)$. Notice that these results are consistent with the known results [5].

Before closing this subsection it is worthwhile to comment on the case $g \leq 1/4$. As mentioned before, the finite-dimensional representation (4.9) disappears for $g \leq 1/4$. However, new finite-dimensional representations appear in this case. The relevant one is the following one-dimensional representation spanned by a single vector:

$$\Big\{|E, j\rangle : E = -j^2 - \frac{g^2}{j^2} \text{ and } j = \frac{1}{2} - \sqrt{\frac{1}{4} - g}\Big\}, \tag{4.13}$$

where $g \in (0, 1/4)$. Notice that this j is one of the solutions to the condition $-j^2 - g^2/j^2 = -(j-1)^2 - g^2/(j-1)^2$. Now one can easily check that this state vector satisfies $J_\pm|E, j\rangle = 0$. It is also easy to see that, for $g \in (0, 1/4)$, $j = 1/2 - \sqrt{1/4 - g}$ satisfies the condition $j < \sqrt{g}$, which is the necessary condition for the ground-state wavefunction to be normalizable. The point is that, just as in the case $g > 1/4$, j must be quantized in a particular manner in this representation theoretic approach.

4.2. Rosen–Morse

Let us finally move on to the bound-state problem of the Rosen–Morse Hamiltonian [6, 7]. First, the Hamiltonian $H_{\text{Rosen–Morse}}$ in (1.1) is factorized as follows:

$$H_{\text{Rosen–Morse}} = A_-(j)A_+(j) - j^2 - \frac{g^2}{j^2}, \tag{4.14}$$

where

$$A_\pm(j) = \pm\frac{d}{dx} - j\tanh x + \frac{g}{j}. \tag{4.15}$$

Let us then introduce the following operators:

$$J_3 = -i\partial_\theta, \qquad J_+ = \mathrm{e}^{i\theta}\Big(\partial_x - \tanh x J_3 + \frac{g}{J_3}\Big), \qquad J_- = \Big(-\partial_x - \tanh x J_3 + \frac{g}{J_3}\Big)\mathrm{e}^{-i\theta}, \tag{4.16}$$

which satisfy the commutation relations:

$$[J_3, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = -J_3^2 - \frac{g^2}{J_3^2} + (J_3-1)^2 + \frac{g^2}{(J_3-1)^2}. \tag{4.17}$$

The invariant operator is

$$H = J_-J_+ - J_3^2 - \frac{g^2}{J_3^2} = J_+J_- - (J_3-1)^2 - \frac{g^2}{(J_3-1)^2} = -\partial_x^2 - \frac{J_3(J_3-1)}{\cosh^2 x} - 2g\tanh x. \tag{4.18}$$

Note that the commutation relations (4.17) are exactly the same as those for the hyperbolic Kepler problem. Hence the bound-state spectrum should be related to the representations classified in the previous subsection.

[5]This is, of course, not sufficient condition.

To understand which representations are realized, let us study the minimum of the potential $V(x) = -j(j-1)/\cosh^2 x - 2g\tanh x$. Thanks to the symmetry $j \to 1-j$, without any loss of generality we can focus on the case $j \geq 1/2$. It is then easy to see that the potential has a minimum if j is in the range $(\frac{1}{2} + \sqrt{g + \frac{1}{4}}, \infty)$, which contains the region $(1+\sqrt{g}, \infty)$; see Figure 2(c). Hence, in contrast to the previous case, the bound-state problem for the Rosen–Morse Hamiltonian should be related to the infinite-dimensional representation (4.10).

Now it is easy to find the energy eigenvalue of the original Hamiltonian. Let $j \in (1+\sqrt{g}, \infty)$ be fixed. Then the energy eigenvalues and eigenfunctions read

$$E_n = -(j-n-1)^2 - \frac{g^2}{(j-n-1)^2}, \quad n \in \{0, 1, \cdots, N\}, \tag{4.19}$$

and

$$\psi_{E_n,j}(x) \propto A_+(j-1)A_+(j-2)\cdots A_+(j-n)\psi_{E_n,j-n}(x), \tag{4.20}$$

where $N = \max\{n \in \mathbb{Z}_{\geq 0} : 1+\sqrt{g} < j-n\}$ and $\psi_{E_n,j-n}(x) \propto (\cosh x)^{-j+n+1}\exp(\frac{g}{j-n-1}x)$. Notice that (4.19) and (4.20) are consistent with the known results [7].

5. Conclusions

In this note we have revisited the bound-state problems for the Kepler, spherical Kepler, hyperbolic Kepler, and Rosen–Morse Hamiltonians, all of which have not been solved before in terms of potential algebra. We have introduced three nonlinear algebraic systems and solved the problems by means of representation theory. We have seen that the discrete energy spectra can be obtained just from the four conditions: $J_\pm|E,j\rangle \propto |E, j\pm 1\rangle$ and $\|J_\pm|E,j\rangle\|^2 \geq 0$. These conditions correctly reproduce the known results in a purely algebraic fashion. The price to pay, however, is that in this approach j must be a half-integer (except for the Rosen–Morse potential problem and the hyperbolic Kepler problem in the domain $g \in (0, 1/4)$), otherwise there arise inconsistencies. This is a weakness of this representation theoretic approach.

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