Realization of Logical Circuits with Majority Logical Function as Symmetrical Function

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The paper deals with the "production" and design of symmetrical functions, particularly aimed at the design of circuits with majority elements, which lead to interesting solutions of logical structures. The solutions are presented in several examples, which show the applicability of the procedures to the design of FPGA morphology on chips.

Keywords: Shannon extension development, Hamming weight, derivation of Boolean function, symmetrical and majority function.

1 Introduction

Binary logical circuits designed with respect to Boolean symmetrical, particularly majority, output functions are certainly worth attention. The article, therefore, makes an evaluation both by controlling binary one-digit adders and by using functions interpreted by arithmetic polynomials. It also demonstrates how effectively the Shannon decomposition of the output functions can be used in designing a circuit with majority elements.

2 Boolean function

Let a Boolean function $f:\{0,1\}^m \to \{0,1\} : \langle x_1, x_2, \dots, x_m \rangle$ and y be given. If we denote the set $\{x_i\}_{i=1}^{m}$ of arguments x_1 by the symbol X, we can briefly write $f(X)$ instead of $f(x_1, x_2, \dots, x_m)$. In addition, instead of $f(x_1, x_2, \dots, x_{i-1}, \sigma_i, x_{i+1}, \dots, x_m)$, in which $\sigma_i \in \{0,1\}$, let us simply write $f(x{=}_{i\,s\,i})$. Let $x^{\sigma} = x\sigma \vee \bar{x}\bar{\sigma}$ ($\sigma \in \{0,1\}$); any Boolean function $f(x_1, x_2, \dots, x_m)$ can be expressed, without loss of generality, by the **Shannon extension development**

$$f(X) =$$

$$= \bigvee_{\langle \sigma_1, \sigma_2, \dots, \sigma_n \rangle} x_1^{\sigma_1} x_2^{\sigma_2} \dots x_n^{\sigma_n} \; f(\sigma_1, \sigma_2, \dots, \sigma_n, x_{n+1}, \dots, x_m)$$

where $n \le m$ esp.

$$f(X) = \bigvee_{\sigma_i} x_i^{\sigma_1} f(x_i = \sigma_1) = \bar{x}_i f(x_i = 0) \vee x_i f(x_i = 1)$$

the functions $f(\sigma_1, \sigma_2, \dots, \sigma_n, x_{n+1}, \dots, x_m)$ will be called **remainder functions**.

By the **Hamming weight $w_H\ f(X)$ *of the function* $f(X)$** we understand the value of the arithmetic formula

$$w_H\, f(X) = \sum_{\langle \sigma_1, \sigma_2, \dots, \sigma_m \rangle} f(\sigma_1, \sigma_2, \dots, \sigma_m)$$

The **partial derivation** $\frac{\partial f(X)}{\partial x_i}$ [1] of function $f(X)$ by the argument x will be termed the Boolean function

$$\frac{\partial f(X)}{\partial x_i} = f(x_1, x_2, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_m) \oplus$$

$$\oplus\, f(x_1, x_2, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_m)$$

defining the conditions under which $f(X)$ changes its value while the value of the argument x is changed.

For example, for $y = \overline{x_2}\,\overline{x_3} \vee x_1 x_2 x_3$, when

$$w_H \frac{\partial y}{\partial x_i} = w_H\left(\overline{x_2}\,\overline{x_3} \vee (x_2 \equiv x_3)\right) = w_H(x_2 x_3) = 1$$

the function y changes its value while the value of the argument x_1 is changed under one condition for $w_H \frac{\partial y}{\partial x_i} = 1$, which is: $x_2 = x_3 = 1$.

3 Boolean formulae and arithmetic polynomials

Let us have

$$f(x_1, x_2, \dots, x_m) = \bigvee_{i,k} x_{i1}^{\sigma_{i1}} x_{i2}^{\sigma_{i2}} \dots x_{ik}^{\sigma_{ik}}$$

where $\boldsymbol{k} = 1, 2, \dots, m$ and $\boldsymbol{i} = 0, 1, \dots, 2^m - 1$, the function $f(X)$ is represented by a normal disjunctive formula ($ndf\, f(X)$). If there holds

$$\left(x_{i1}^{\sigma_{i1}} x_{i2}^{\sigma_{i2}} \dots x_{ik}^{\sigma_{ik}} \right)\left(x_{j1}^{\sigma_{j1}} x_{j2}^{\sigma_{j2}} \dots x_{jl}^{\sigma_{jl}} \right) = 0,$$

where $\boldsymbol{k} = 1, 2, \dots, \boldsymbol{m}$ and $j = 0, 1, \dots, 2^m - 1$, the conjuncts presented are termed **orthogonal**, i.e., all conjuncts of a complete normal disjunctive formula of the symmetrical Boolean function (see Paragraph 4) are mutually orthogonal. If all conjuncts $ndf\, f(X)$ are mutually orthogonal, we can also write

$$f(x_1, x_2, \dots, x_m) = \bigoplus_{i,k} x_{i1}^{\sigma_{i1}} x_{i2}^{\sigma_{i2}} \dots x_{ik}^{\sigma_{ik}}.$$

Note that the function $f(X)$ can also be conveniently expressed by the Boolean (Zegalkin) polynomial [4].

Since, as can easily be confirmed, the following equality holds:

$$x \vee y = x + y - xy$$

(probability addition)

$$xy = xy$$

$$\bar{x} = 1 - x$$

and also (x and $\bar{x}y$ are orthogonal)

$$x \vee y = x \vee \bar{x}y = x + (1 - x)\,y,$$

each $ndf\, f(X)$ can be expressed by an arithmetic polynomial

$$f(x_1, x_2, \ldots, x_m) = a_0 + a_1 x_1 + a_2 x_2 + \ldots + a_m x_m + a_{m+1} x_1 x_2 + a_{m+2} x_1 x_2 + \ldots + a_{2m-1} x_1 x_m + \ldots + a_{2^m-1} x_1 x_2 \ldots x_m = A(x_1, x_2, \ldots, x_m)$$

where $a_i \in N (i = 0, 1, \ldots, 2^m - 1)$, this can be done either by applying the equality $x \vee y = x + y - xy$, or orthogonalizing all conjuncts $ndf\, f(X)$ and applying the absorption $x \vee \bar{x} y = x + (1 - x) y$. Note that if the Boolean function $f(X)$ is expressed by the arithmetic polynomial $A(X)$, then

$$f(X) = sign\ A(X).$$

Example 1.: Let $ndf\ f(x_1, x_2, x_3) = (x_1 \vee x_2) . x_3 \vee x_1 x_2$ be given. Express the given formula by means of the arithmetic polynomial $A(x_1, x_2, x_3)$, which is $xx = x^2 = x$:

- $$(x_1 \vee x_2) x_3 \vee x_1 x_2 = x_1 x_2 + (x_1 \vee x_2)(x_3 - x_1 x_2 x_3) = = x_1 x_2 + (x_1 + x_2 - x_1 x_2)(x_3 - x_1 x_2 x_3) = = x_1 x_2 + x_1 x_3 + x_2 x_3 - 2 x_1 x_2 x_3$$
- $$(x_1 \vee x_2) x_3 \vee x_1 x_2 = x_1 x_2 \vee \overline{x_1 x_2}(x_1 \vee \bar{x}_1 x_2) x_3 = = x_1 x_2 \vee (\bar{x}_1 \vee \bar{x}_2)(x_1 \vee \bar{x}_1 x_2) x_3 = x_1 x_2 \vee (x_1 \bar{x}_2 \vee \bar{x}_1 x_2) x_3 = = x_1 x_2 + (x_1(1 - x_2) + (1 - x_1) x_2) x_3 = = x_1 x_2 + x_1 x_3 + x_2 x_3 - 2 x_1 x_2 x_3$$

4 Symmetrical Boolean function

Let the bijection $X \leftrightarrow X$: $\langle x_1, x_2, \ldots, x_m \rangle \leftrightarrow \langle x_{i1}, x_{i2}, \ldots, x_{im} \rangle$ be a set of all permutations of arguments from X; the function $f(X)$ is called **symmetrical** if $f(x_1, x_2, \ldots, x_m) = f(x_{i1}, x_{i2}, \ldots, x_{im})$; i.e. 0, $x \oplus y$, $xy \vee xz \vee yz$ is a symmetrical function. Let $\{P_j\}_{j=1}^k$ be a set of integers P_j (called operational or characteristic numbers) such that $0 \le P_j \le m$. It can be demonstrated [2] that $f(X)$ is symmetrical just if $f(\sigma_1, \sigma_2, \ldots, \sigma_m) = 1$ for $w_H(\sigma_1, \sigma_2, \ldots, \sigma_m) = P_j$. The symmetrical function with characteristic numbers P_j will be denoted $S^m_{\{P_j\}_{j=1}^k}$;

obviously, $S^m_{\varnothing} = 0$ and $S^m_{\{0, 1, \ldots, m\}} = 1$.

For example: $S^2_{\{1\}} = x \oplus y$, $S^3_{\{2,3\}} = xy \vee xz \vee yz$.

The symmetrical function $S^m_{\{P\}}$ is **elementary**; for the length $\left|cndf\ S^m_{\{P\}}\right|$ of the completely normal disjunctive formula $S^m_{\{P\}}$ – there holds $\left|Cndf\ S^m_{\{P\}}\right| = m\binom{m}{P}$, since

$$Cndf\ S^m_{\{P\}} = \bigvee_{\substack{\langle \sigma_1, \sigma_2, \ldots, \sigma_m \rangle \\ w_H(\sigma_1 \sigma_2 \ldots \sigma_n) = P}} x_1^{\sigma_1} x_2^{\sigma_2} \ldots x_m^{\sigma_m}.$$

There also holds [2]

$$\left| \bigcup_{\{Q\} \in 2^{\{0, 1, \ldots, m\}}} \left\{ S^m_{\{Q\}} \right\} \right| = \sum_{i=0}^{m+1} \binom{m+1}{i} = 2^{m+1}$$ as well as

$S^m_{\{P_i\}} \cdot S^m_{\{P_j\}} = 0$ for $P_i \ne P_j$. Any symmetrical function $S^m_{\{P_j\}_{j=1}^k}$ can be written in the form of $cndf\ S^m_{\{P_j\}_{j=1}^k}$:

$$S^m_{\{P_j\}_{j=1}^k} = \bigvee_{\substack{\langle \sigma_1, \sigma_2, \ldots, \sigma_m \rangle \\ w_H(\sigma_1 \sigma_2 \ldots \sigma_m) = P_1, P_2, \ldots, P_k}} x_1^{\sigma_1} x_2^{\sigma_2} \ldots x_m^{\sigma_m}$$

since [2] $S^m_{\{P_j\}_{j=1}^k} = \bigvee_{j=1}^k S^m_{\{P_j\}}$. We can also write

$$S^m_{\{P_j\}_{j=1}^k} = \bigvee_{j=0}^m \tau_i S^m_{\{i\}}$$

where $\tau_i = \begin{cases} 1 & \text{for} \quad i = P_j \\ 0 & \text{for} \quad i \ne P_j \end{cases}$.

Denote the elementary symmetrical Boolean functions the representation of which in the form of normal disjunctive formulae (*ndf*) does not contain negated variables with the symbol $\mathsf{S}^m_{\{n\}}$ $(n = 0, 1, \ldots, m)$.

For example

$\mathsf{S}^m_{\{0\}} = 1$,

$\mathsf{S}^m_{\{1\}} = x_1 x_2 \ldots x_m, \ldots,$

$\mathsf{S}^m_{\{m-1\}} = x_1 x_2 \vee x_1 x_3 \vee \ldots \vee x_{m-1} x_m$,

$\mathsf{S}^m_{\{m\}} = x_1 \vee x_2 \vee \ldots \vee x_m$.

Every function $\mathsf{S}^m_{\{n\}}$ in which $n \ne m$ ($S^m_{\{n\}} = x_1 x_2 \ldots x_m$), can be expressed by the composition

$$S^m_{\{n\}} = \mathsf{S}^m_{\{n\}} \overline{\mathsf{S}^m_{\{m-n\}}}.$$

For example:

$S^3_{\{2\}} = \mathsf{S}^3_{\{2\}} \cdot \bar{\mathsf{S}}^3_{\{1\}} = (x_1 x_2 \vee x_1 x_3 \vee x_2 x_3)\overline{x_1 x_2 x_3} = = x_1 x_2 \bar{x}_3 \vee x_1 \bar{x}_2 x_3 \vee \bar{x}_1 x_2 x_3$, resp.

$S^3_{\{2\}} = \mathsf{S}^3_{\{1\}} \cdot \bar{\mathsf{S}}^3_{\{2\}} = x_1 x_2 x_3\ \overline{x_1 x_2 \vee x_1 x_3 \vee x_2 x_3} = = x_1 x_2 x_3 \left(\overline{x_1 x_2} \vee \overline{x_1 x_3} \vee \overline{x_2 x_3} \right) = x_1 \bar{x}_2 \bar{x}_3 \vee \bar{x}_1 x_2 \bar{x}_3 \vee \bar{x}_1 \bar{x}_2 x_3$.

Symmetrical functions are discussed in greater detail, e.g., in [2, 3, 4].The majority function $Maj^m_{\{M\}}$ refers to the symmetrical function

$$\bigvee_{i=0}^{m} S^m_{\{M+i\}} = S^m_{\{M, M+1, \ldots, m\}}.$$

For the three-variable majority function

$$Maj^3_{\{2\}} = x_1 x_2 \vee x_1 x_3 \vee x_2 x_3$$

an infix notation $x_1 \# x_1 \# \# x_3$ can be used. There obviously holds $x_1 \# x_1 \# x_3 = x_1$ and $x_1 \# \bar{x}_1 \# x_3 = x_3$.

5 Numerical representation of symmetrical functions

We might construct a minimal normal disjunctive formula to a given symmetrical function [4] or decompose the given

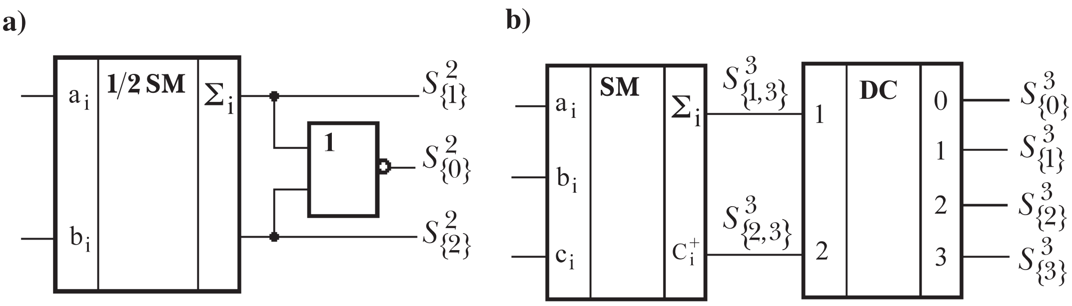

Fig. 1: a) Half-adder , b) Adder

function ($S^m_{\{n\}} = S^m_{\{n\}} \overline{S^m_{\{m-n\}}}$) and design, according to the constructed formulae, a structural model of the given function in one of the structurally complete systems of statistical elements [5].

Further, consider a one-digit binary half-adder or an adder (Fig. 1) with which a_i, b_i are binary augmenters, Σ_i is the sum in i position, and C_i^- and C_i^+ denote the transfer from the position $i-1$ to the position $i+1$, respectively. The half-adder can be modeled by a system of output functions $\Sigma_i a_i \oplus b_i = S^2_{\{1\}}$ and $C_i^+ = a_i b_i = S^2_{\{2\}}$; by analogy for the adder we obtain

$\Sigma_i = a_i \oplus b_i \oplus C_i^- = S^3_{\{1,3\}}$ and

$$C_i^+ = \overline{a_i}\, b_i\, C_i^- \vee a_i\, \overline{b_i}\, C_i^- \vee a_i\, b_i\, \overline{C_i^-} \vee a_i\, b_i\, C_i^- = S^3_{\{2,3\}}.$$

It is therefore sufficient to provide the half-adder with an inverse disjunctor (Fig.1a) and the adder with a decoder (Fig. 1b) and we obtain the products $S^2_{\{0\}}, S^2_{\{1\}}, S^2_{\{2\}}$, since

$$\overline{S^2_{\{1\}}}\,\overline{S^2_{\{2\}}} = S^2_{\{0,2\}} S^2_{\{0,1\}} = S^2_{\{0\}}$$

and $S^3_{\{0\}}, S^3_{\{1\}}, S^3_{\{2\}}, S^3_{\{3\}}$, since

$$\overline{S^3_{\{1,3\}}} \cdot \overline{S^3_{\{2,3\}}} = S^3_{\{0,2\}} \cdot S^3_{\{0,1\}} = S^3_{\{0\}},$$

$$S^3_{\{1,3\}} \cdot \overline{S^3_{\{2,3\}}} = S^3_{\{1,3\}} \cdot S^3_{\{0,1\}} = S^3_{\{1\}},$$

$$\overline{S^3_{\{1,3\}}} \cdot S^3_{\{2,3\}} = S^3_{\{0,2\}} \cdot S^3_{\{2,3\}} = S^3_{\{2\}},$$

$$S^3_{\{1,3\}} \cdot S^3_{\{2,3\}} = S^3_{\{3\}}.$$

Example 2.: Design a structural model with adders or half-adders modeled with a system of output symmetrical functions S – Fig. 2. Indeed,

$$\Sigma_1 = x_1 \oplus x_2 \oplus x_3 = (x_1 \oplus x_2)\overline{x_3} \vee (x_1 \equiv x_2) x_3 = S^3_{\{1,3\}},$$

$$C_1 = S^3_{\{2,3\}},$$

$$\Sigma_2 = x_4 \oplus S^3_{\{1,3\}} = \overline{x_4} S^3_{\{1,3\}} \vee x_4 \overline{S^3_{\{1,3\}}} =$$

$$= S^4_{\{1,3\}} \vee x_4 S^3_{\{0,2\}} = S^4_{\{1,3\}} \vee S^4_{\{1,3\}} = S^4_{\{1,3\}},$$

$$C_2 = x_4\, S^3_{\{1,3\}} = S^4_{\{2,4\}},$$

$$\Sigma_3 =$$

$$= S^4_{\{2,4\}} \oplus S^3_{\{2,3\}} (\overline{x_4} \oplus x_4) =$$

$$= \left(S^4_{\{0,1,3\}} S^3_{\{1,3\}} \vee S^4_{\{2,4\}} S^3_{\{0,1\}} \right) (\overline{x_4} \vee x_4) =$$

$$= S^4_{\{0,1,3\}} S^4_{\{2,3\}} \vee S^4_{\{2,4\}} S^4_{\{0,1\}} \vee S^4_{\{0,1,3\}} S^4_{\{3,4\}} \vee S^4_{\{2,4\}} S^4_{\{1,2\}} =$$

$$= S^4_{\{3\}} \vee S^4_{\{2\}} = S^4_{\{2,3\}},$$

$$C_3 = S^4_{\{2,4\}} S^3_{\{2,3\}} (\overline{x_4} \vee x_4) = S^4_{\{2,4\}} S^4_{\{2,3\}} \vee S^4_{\{2,4\}} S^4_{\{3,4\}} =$$

$$= S^4_{\{2\}} \vee S^4_{\{4\}} = S^4_{\{2,4\}}.$$

It is easy to obtain

$S^4_{\{1,3\}} = S^4_{\{0,2,4\}}$, $\overline{S^4_{\{2,3\}}} = S^4_{\{0,1,4\}}$ and $\overline{S^4_{\{2,4\}}} = S^4_{\{0,1,3\}}$; hence for the decoder:

$$S^4_{\{0\}} = S^4_{\{1,3\}} S^4_{\{2,4\}},\ S^4_{\{2\}} = S^4_{\{2,3\}} S^4_{\{2,4\}},\ S^4_{\{3\}} = S^4_{\{1,3\}} S^4_{\{2,3\}},$$

$$S^4_{\{1\}} = S^4_{\{0,1,3\}} S^4_{\{0,1,4\}} S^4_{\{1,3\}} = \overline{S^4_{\{2,4\}}}\, \overline{S^4_{\{2,3\}}}\, S^4_{\{1,3\}},$$

$$S^4_{\{4\}} = S^4_{\{0,1,4\}} S^4_{\{0,2,4\}} S^4_{\{2,4\}} = \overline{S^4_{\{2,3\}}}\, \overline{S^4_{\{1,3\}}}\, S^4_{\{2,4\}}.$$

If the Boolean function $f(X)$ is symmetrical, it can be suitably expressed by an arithmetic polynomial in the form

$$f(x_1, x_2, \ldots, x_m) = b_0 + b_1(x_1 + x_2 + \ldots + x_m) +$$
$$+ b_2(x_1 x_2 + x_2 x_3 + \ldots + x_{m-1} x_m) + b_3(x_1 x_2 x_3 + x_1 x_2 x_4 +$$
$$+ \ldots + x_{m-2} x_{m-1} x_m) + \ldots + b_{m+1} x_1 x_2 + \ldots + x_m =$$
$$= b_0 + b_1 X^1 + b_2 X^2 + \ldots + b_m X^m = \sum_{i=0}^{m} b_i X^i \quad [6];$$

for example for

$$S^3_{\{1,2\}} = \overline{x_1} x_2 \vee x_1 \overline{x_2} \vee x_1 x_2 = (1 - x_1) x_2 + x_1 x_2 = x_1 + x_2 - x_1 x_2$$

we obtain $b_0 = 0$, $b_1 = 1$, $b_2 = -1$ and
$X^0 = 1$, $X^1 = x_1 + x_2$, $X^2 = x_1 x_2$.
The parametric notation *cndf* $S^m_{\{P_j\}_{j=1}^{k}}$ being

$$S^m_{\{P_j\}_{j=1}^{k}} = \bigvee_{P_j} X^{P_j} \overline{X}^{(m-P_j)} = \sum_{P_j} \left[m \Big/ \binom{m}{P_j} \right] X^{P_j} (1 - X)^{m-P_j}$$

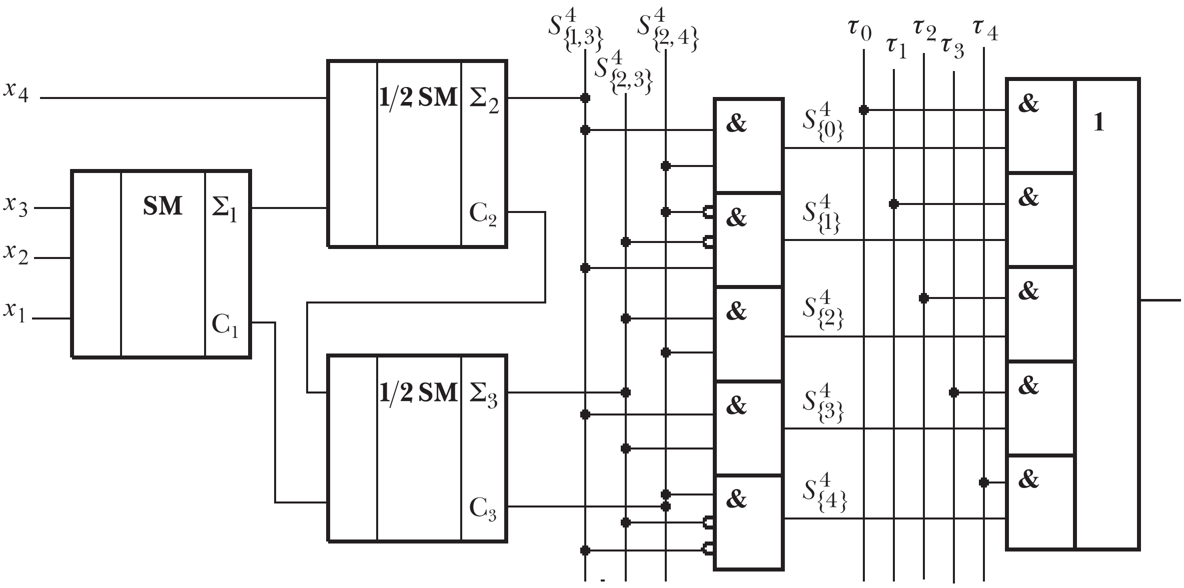

Fig. 2: Production of symmetrical functions $S^4_{\{i\}} (i = 0, 1, 2, 3, 4)$ from Example 2

Hence

$$\sum_{i=0}^{m} b_i X^i = \sum_{P_j} \left[m \Big/ \binom{m}{P_j} \right] \mathrm{X}^{\mathrm{P_j}} (1-X)^{m-P_j}$$

we can easily (!) determine the values of coefficients b, provided the structure of polynomials X^i is known.

Example 3.: Construct an arithmetic polynomial of the symmetrical function $S^3_{\{1,2\}}$. Since

$$S^3_{\{1,2\}} = \bigvee_{\substack{\langle \sigma_1, \sigma_2, \sigma_3 \rangle \\ w_H(\sigma_1 \sigma_2 \sigma_3)}} x_1^{\sigma_1} x_2^{\sigma_2} x_3^{\sigma_3} = \overline{x_1}\,\overline{x_2}\,x_3 \vee \overline{x_1}\,x_2\,\overline{x_3} \vee$$

$$\vee\, x_1 \overline{x_2}\,\overline{x_3} \vee \overline{x_1}\,x_2\,x_3 \vee x_1\,\overline{x_2}\,x_3 \vee x_1\,x_2\,\overline{x_3} =$$

$$= \sum_{j=1,2} \left[3 \Big/ \binom{3}{j} \right] \mathrm{X}^{j} (1-X)^{3-j} =$$

$$= \left[3 \Big/ \binom{3}{1} \right] X(1-X)^2 + \left[3 \Big/ \binom{3}{2} \right] X^2(1-X) =$$

$$= X - X^2 = b_0 + b_1 X^1 + b_2 X^2 + b_3 X^3.$$

Hence $b_0 = b_3 = 0, b_2 = 1, b_3 = -1$ provided that
$X^1 = x_1 + x_2 + x_3$ and
$X^2 = x_1x_2 + x_1x_3 + x_2x_3$;
because

$$S^3_{\{1,2\}} = (1-x_1)x_2x_3 + x_1(1-x_2)x_3 + x_1x_2(1-x_3) +$$
$$+(1-x_1)(1-x_2)x_3 + (1-x_1)x_2(1-x_3) + x_1(1-x_2)(1-x_3) =$$
$$= (x_1 + x_2 + x_3) - 2(x_1x_2 + x_1x_3 + x_2x_3) + 3x_1x_2x_3 +$$
$$+(x_1 + x_2 + x_3) + (x_1x_2 + x_1x_3 + x_2x_3) - 3x_1x_2x_3 .$$

5 Circuits with majority elements

Let us limit ourselves to majority elements modeled with the function $Maj^3_{\{2\}}$.

Since, as can be easily confirmed, there holds

$$x \vee y = x \# y \# 1 ,$$

$$xy = x \# y \# 0 ,$$

the *ndf* of the given function $f(X)$ can be rewritten according to the above quoted equities and we can design the respective static structural model.

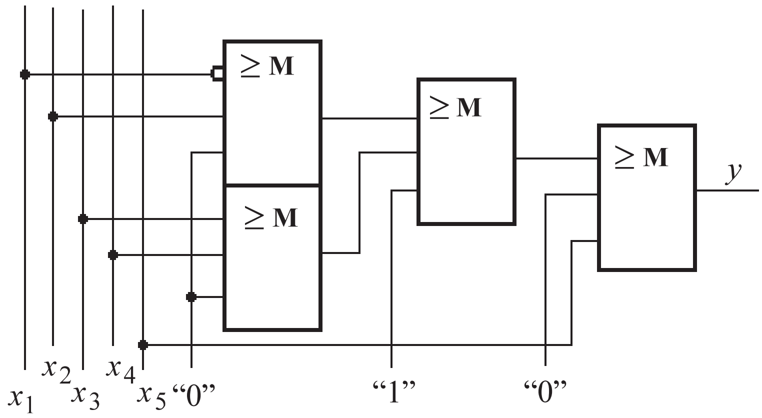

Fig. 3: Structural model from Example 4

Example 4.: Construct a structural model given by the output function

$$y = (\overline{x_1}\, x_2 \vee x_3\, x_4)\, x_5$$

hence

$$y = \Big(\big((\overline{x_1} \# x_2 \# 0) \# (x_3 \# x_4 \# 0)\big) \# 1\Big) \# x_5 \# 0,$$

and thus Fig. 3.

It is also helpful to use the Shannon extension development of the given Boolean output function $y = f(X)$

$$y = \overline{x_i} f(x_i = 0) \vee x_i f(x_i = 1) =$$

$$= \big(\overline{x_i} \# f(x_i = 0) \# 0\big) \# \big(x_i \# f(x_i = 1) \# 0\big) \# 1,$$

and it remains only to decide according to which argument to start and according to which arguments to continue the repeated application of the development. We, therefore, heuristically develop $f(X)$, first according to the arguments whose change of values leads to the change of values $f(X)$ under the highest number of conditions, i.e., at the highest.

Hamming weights pertaining to the derivation of the function $f(X)$ according to the respective arguments.

Example 5.: Design a structural model given by the output function

$$y = x_1 \overline{x_2} x_3 \vee \overline{x_1}\, \overline{x_3} x_4 \vee x_1 x_3 \overline{x_5} \vee x_1 x_2 x_4 \vee \overline{x_2} x_3 x_5 \vee \overline{x_3}\, \overline{x_4} x_5$$

with majority elements. According to the map of the given output function, the *ndf* of the remainder functions of its Shannon extension development can easily be constructed (Fig. 4.).

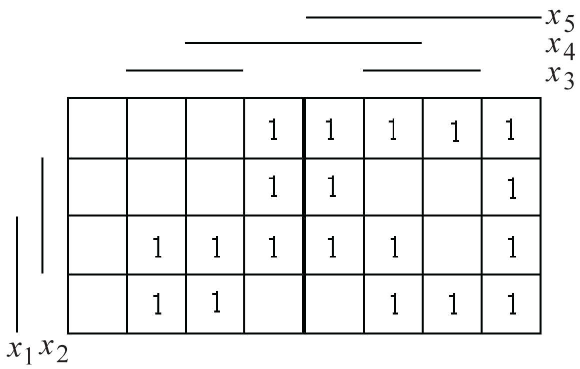

Fig. 4: The Karnaugh map of the output function from Example 5

Hence

$$w_H \frac{\partial y}{\partial x_1} = w_H\big(y(x_1 = 0) \oplus y(x_1 = 1)\big) =$$

$$= w_H\Big((\overline{x_2} x_5 \vee \overline{x_3} x_4 \vee \overline{x_3} x_5) \oplus$$

$$\oplus (\overline{x_2} x_3 x_5 \vee x_2 x_4 \vee \overline{x_3}\, \overline{x_4} x_5 \vee x_3 \overline{x_5})\Big) = 7$$

When stating the formula which expresses the derivation of the function we will preferably use a map, to each field of which we will write the value in the form of a fraction:

$$y(x_i = 0)/y(x_i = 1);$$

the resulting value of the remainder function formula as well as the weight of its derivation is evident (Fig. 5). There also holds

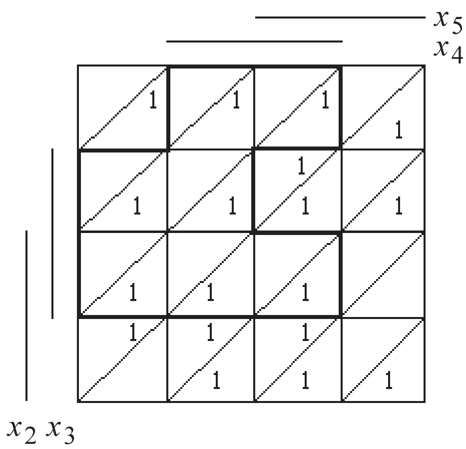

Fig. 5: Recording of remainder functions $\frac{\partial y}{\partial x_1}$ from Example 5

$$w_H \frac{\partial y}{\partial x_2} = w_H\big(y(x_2 = 0) \oplus y(x_2 = 1)\big) =$$

$$= w_H\Big((\overline{x_1}\, \overline{x_3} x_4 \vee x_1 x_3 \vee x_3 x_5 \vee \overline{x_4} x_5) \oplus$$

$$\oplus (x_1 x_4 \vee x_1 x_3 \overline{x_5} \vee \overline{x_3} x_4 \vee \overline{x_3} x_5)\Big) = 5,$$

$$w_H \frac{\partial y}{\partial x_3} = w_H\big(y(x_3 = 0) \oplus y(x_3 = 1)\big) =$$

$$= w_H\Big((\overline{x_1} x_4 \vee x_2 x_4 \vee \overline{x_4} x_5) \oplus$$

$$\oplus (x_1 \overline{x_5} \vee x_1 x_4 x_5 \vee \overline{x_2} x_5)\Big) = 8,$$

$$w_H \frac{\partial y}{\partial x_4} = w_H\big(y(x_4 = 0) \oplus y(x_4 = 1)\big) =$$

$$= w_H\Big((x_1 \overline{x_2} x_5 \vee x_1 x_3 \overline{x_5} \vee x_1 \overline{x_4} x_5) \oplus$$

$$\oplus (\overline{x_1}\, \overline{x_3} \vee x_1 x_3 \vee x_2 \overline{x_3})\Big) = 5,$$

$$w_H \frac{\partial y}{\partial x_5} = w_H\big(y(x_5 = 0) \oplus y(x_5 = 1)\big) =$$

$$= w_H\Big((x_1 x_3 \vee \overline{x_1}\, \overline{x_3} x_4 \vee x_2 \overline{x_3} x_4) \oplus$$

$$\oplus (\overline{x_1}\, \overline{x_3} \vee \overline{x_2} x_3 \vee x_2 \overline{x_3} \vee \overline{x_3}\, \overline{x_4} \vee x_1 x_3 x_4)\Big) = 7.$$

Since

$$\max_i \left\{ w_H \frac{\partial y}{\partial x_i} \right\} = w_H \frac{\partial y}{\partial x_3} = 8,$$

we write

$$y = x_3\, y(x_3 = 0) \vee x_3\, y(x_3 = 1),$$

where

$$y(x_3 = 0) = \overline{x_1} x_4 \vee x_2 x_4 \vee \overline{x_4} x_5$$

and

$$y(x_3 = 1) = x_1 x_4 \vee x_1 \overline{x_5} \vee \overline{x_2} x_5 \text{ (Fig. 6).}$$

And, further, there is

$$w_H \frac{\partial y(x_3 = 0)}{\partial x_1} = w_H\big(y(x_1 = 0, x_3 = 0) \oplus y(x_1 = 1, x_3 = 0)\big) =$$

$$= w_H\big((x_4 \vee x_5) \oplus (x_2 x_4 \vee \overline{x_4} x_5)\big) = 2,$$

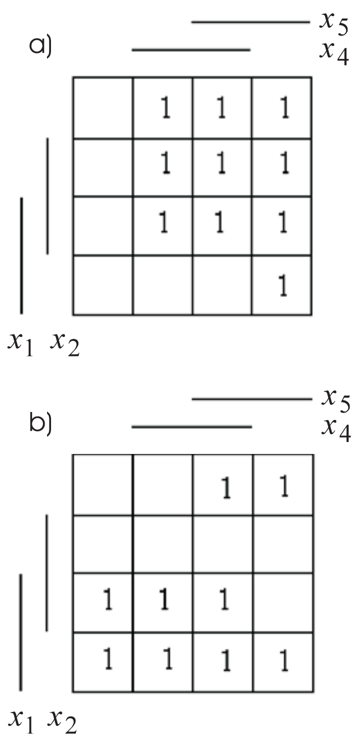

Fig. 6: Map entries of a) $y(x_3=0)$, b) $y(x_3=1)$ from Example 5

$$w_H \frac{\partial y(x_3=0)}{\partial x_2} = w_H\big(y(x_2=0, x_3=0) \oplus y(x_2=1, x_3=0)\big) =$$
$$= w_H\big((\bar{x}_1 x_4 \vee \bar{x}_4 x_5) \oplus (x_4 \vee x_5)\big) = 2,$$
$$w_H \frac{\partial y(x_3=0)}{\partial x_4} = w_H\big(y(x_3=0, x_4=0) \oplus y(x_3=0, x_4=1)\big) =$$
$$= w_H\big(x_5 \oplus (\bar{x}_1 \vee x_2)\big) = 4,$$
$$w_H \frac{\partial y(x_3=0)}{\partial x_5} = w_H\big(y(x_3=0, x_5=0) \oplus y(x_3=0, x_5=1)\big) =$$
$$= w_H\big((\bar{x}_1 x_4 \vee x_2 x_4) \oplus (\bar{x}_1 \vee x_2 \vee \bar{x}_4)\big) = 4,$$
$$w_H \frac{\partial y(x_3=1)}{\partial x_1} = w_H\big(y(x_1=0, x_3=1) \oplus y(x_1=1, x_3=1)\big) =$$
$$= w_H\big(\bar{x}_2 x_5 \oplus (\bar{x}_2 \vee x_4 \vee \bar{x}_5)\big) = 5,$$
$$w_H \frac{\partial y(x_3=1)}{\partial x_2} = w_H\big(y(x_2=0, x_3=1) \oplus y(x_2=1, x_3=1)\big) =$$
$$= w_H\big((x_1 \vee x_5) \oplus (x_1 x_4 \vee x_1 \bar{x}_5)\big) = 3,$$
$$w_H \frac{\partial y(x_3=1)}{\partial x_4} = w_H\big(y(x_3=1, x_4=0) \oplus y(x_3=1, x_4=1)\big) =$$
$$= w_H\big((x_1 \bar{x}_5 \vee \bar{x}_2 x_5) \oplus (x_1 \vee \bar{x}_2 x_5)\big) = 1,$$
$$w_H \frac{\partial y(x_3=1)}{\partial x_5} = w_H\big(y(x_3=1, x_5=0) \oplus y(x_3=1, x_5=1)\big) =$$
$$= w_H\big(x_1 \oplus (\bar{x}_2 \vee x_1 x_4)\big) = 3.$$

Since

$$\max_{\substack{i \\ i\neq 3}} \left\{ w_H \frac{\partial y(x_3=0)}{\partial x_i} \right\} = w_H \frac{\partial y(x_3=0)}{\partial x_4} = 4$$

and

$$\max_{\substack{i \\ i\neq 3}} \left\{ w_H \frac{\partial y(x_3=1)}{\partial x_i} \right\} = w_H \frac{\partial y(x_3=1)}{\partial x_1} = 5$$

we write

$$y = \bar{x}_3(\bar{x}_4 y(x_3=0, x_4=0) \vee x_4 y(x_3=0, x_4=1)) \vee x_3(\bar{x}_1 y(x_1=0, x_3=1) \vee x_1 y(x_1=1, x_3=1)),$$

where

$y(x_3=0, x_4=0) = x_5$, $y(x_3=0, x_4=1) = \bar{x}_1 \vee x_2$,

$y(x_1=0, x_3=1) = \bar{x}_2 x_5$ *and*

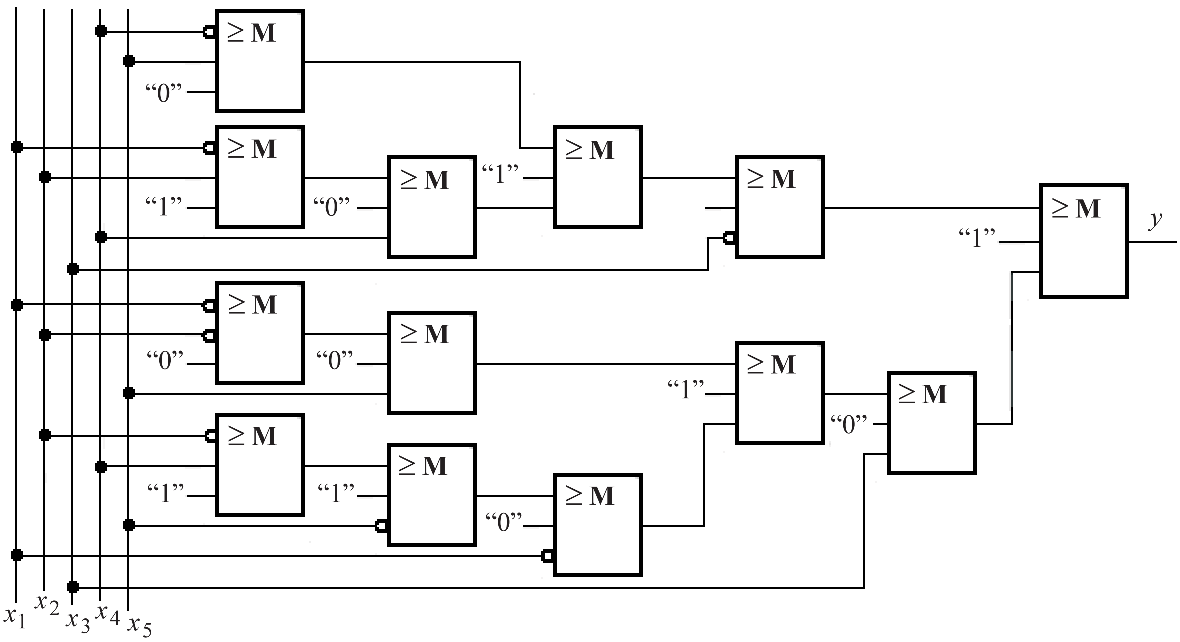

Fig. 7: Structural model with majority elements from Example 5

$y(x_1 = 1, x_3 = 1) = \overline{x_2} \vee x_4 \vee \overline{x_5}$, or

$y = \overline{x_3}(\overline{x_4}\, x_5 \vee x_4(\overline{x_1} \vee x_2)) \vee x_3(\overline{x_1}\,\overline{x_2}\, x_5 \vee x_1(\overline{x_2} \vee x_4 \vee \overline{x_5}))$.

In other words

$$y = \left\{\overline{x_3} \# \left[(\overline{x_4} \# x_5 \# 0) \# (x_4 \# (\overline{x_1} \# x_2 \# 1) \# 0) \# 1\right] \# 0\right\} \#$$
$$\# \left\{x_3 \# \left[((\overline{x_1} \# \overline{x_2} \# 0) \# x_5 \# 0) \# \right.\right.$$
$$\left.\left. \# (x_1 \# ((\overline{x_2} \# x_4 \# 1) \overline{x_5} \# 1) \# 0) \# 1\right] \# 0\right\} \# 1.$$

and hence also the structural model (Fig. 7)

Obviously, there is also

$$x_1 \# x_2 \# x_3 = (x_1 \vee x_2) x_3 \vee x_1 x_2 = (x_1 \oplus x_2) x_3 \oplus x_1 x_2 =$$
$$= S^3_{\{2,3\}}(x_1, x_2, x_3) = x_1 x_2 + x_1 x_3 + x_2 x_3 - 2 x_1 x_2 x_3.$$

7 Conclusion

It appears that it is feasible to produce symmetrical Boolean functions in a sufficiently simple way by a suitable control of one-digit binary adders or by numerical representation of values of the respective arithmetic polynomials, and to design logical circuits with majority elements by applying the Shannon decomposition of the given output function through effective selection of the arguments by which the decomposition is carried.

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