Non-Standard Numeration Systems

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We study some properties of non-standard numeration systems with an irrational base $\beta > 1$, based on the so-called beta-expansions of real numbers [1]. We discuss two important properties of these systems, namely the Finiteness property, stating whether the set of finite expansions in a given system forms a ring, and then the problem of fractional digits arising under arithmetic operations with integers in a given system. Then we introduce another way of irrational representation of numbers, slightly different from classical beta-expansions. Here we restrict ourselves to one irrational base – the golden mean τ – and we study the Finiteness property again.

1 Introduction

Numeration systems with an irrational base β give us an opportunity to perform exact arithmetic operations with irrational numbers – a tool necessary for the new methods of building aperiodic random number generators, for new cryptographic methods and also for the mathematical modelling of recently discovered materials with a long range order – so called quasicrystals.

The whole field of irrational numeration originates from the article of A. Rényi [1], in which he proved that for each real base $\beta > 1$ and for every positive real number x, there exists a unique representation of the number x in the numeration system with the base β. However, contrary to the usual numeration system with an integer base (such as binary or decimal numeration systems), there are several strange and unsteady (i.e. depending on the nature of the base β) phenomena, which have to be examined and described before we are able to employ these systems.

The work is organized as follows. In the first part, there is a survey of the already classical β-expansions of real numbers. We recall basic definitions and fundamentals; then we discuss two key problems of these numeration systems – the so-called Finiteness property and the problem of fractional parts arising under addition and multiplication of integers.

In the second part, we study a slightly different approach to irrational representations of real numbers, which is called τ-adic expansions. Since this concept has been much less studied in the past, we focus on one particular irrationality (namely the golden mean τ), rather than trying to explain everything in general. After giving necessary definitions we discuss the relations of these τ-adic expansions with integers and rationals, and we also study an analogue to the finiteness problem for β-expansions.

2 Beta-numeration systems

Let $\beta > 1$ be a fixed real number. A representation in base β (or simply β-*representation*) of a real number $x \in \mathbf{R}_+$ is an infinite sequence $(x_i)_{n \geq i > -\infty}$, $x_i \in \mathbf{Z}$ such that

$$x = x_n \beta^n + x_{n-1} \beta^{n-1} + \ldots + x_0 \beta^0 + x_{-1} \beta^{-1} + \ldots$$

for some $n \in \mathbf{Z}$. A particular β-representation with

$$\sum_{i=-\infty}^{N} x_i \beta^i < \beta^{N+1} \text{ for all } -\infty < N < n$$

is called β-*expansion*, see [1]. The greedy algorithm computes the digits of the β-expansions of a real number x: Let $[x]$ and $\{x\}$ denote the integer and the fractional part of x, respectively. Find $n \in \mathbf{Z}$ such that $\beta^n \leq x < \beta^{n+1}$. Put $x_n = [x/\beta^n]$ and $r_n = [x/\beta^n]$. For $i = n-1, n-2, \ldots$ put $x_i = [\beta r_{i+1}]$ and $r_i = [\beta r_{i+1}]$.

The β-expansion can be seen as an analogue to the ordinary expansion in a system with an integer base (such as a decimal or binary system), we usually use the natural notation

$$\langle x \rangle_\beta = x_n x_{n-1} \ldots x_0 \bullet x_{-1} \ldots$$

and we say that the coefficients $x_n \ldots x_0$ form the integer part, whereas the coefficients $x_{-1} x_{-2} \ldots$ form the fractional part of the β-expansion of a number x.

A sequence of integer coefficients that corresponds to some β-expansion is sometimes called *admissible* in the β-numeration system. A sequence (or string) of integer coefficients that is not admissible is called *forbidden*. For the characterization of an admissible sequence one needs to introduce the so-called *Rényi expansion of 1*, $d_\beta(1) := t_1 t_2 t_3 \ldots$, where $t_1 = [\beta]$ and $\sum_{m=2}^{\infty} t_n / \beta^n$ is the β-expansion of $1 - t_1/\beta$. The β-expansions (or β-admissible sequences) are then characterized by the *Parry condition*.

Theorem 1.1: (Parry [2]). Let $\beta > 1$ be a real number. Let $d_\beta(1) := t_1 t_2 t_3 \ldots$ be the Rényi expansion of one. The sequence $(x_i)_{n \geq i \geq m}$ with $x_i \in \{0, 1, \ldots, [\beta]\}$ is a β-expansion of some $x > 0$ if and only if $x_{n-p} x_{n-p-1} \ldots x_m$ is lexicographically strictly smaller (denoted by the symbol $\prec$) than $d_\beta(1)$ for all $0 \leq p \leq n - m$.

The set of those $x \in \mathbf{R}$ for which the β-expansion of $|x|$ has only a finite number of non-zero fractional coefficients is denoted by $\mathrm{Fin}(\beta)$. If the β-expansion of a real number x is of the form $x = \sum_{i=0}^{n} x_i \beta^i$, i.e. there is no fractional part, the number x is said to be a β-*integer*. The set of all β-integers is denoted by $\mathbf{Z}_\beta$.

2.1 Finiteness condition

In general, the set $\mathrm{Fin}(\beta)$ does not have to be closed under arithmetic operations, and there is no criterion known which would decide whether $\mathrm{Fin}(\beta)$ is a ring.

The β-numeration systems for which $\mathrm{Fin}(\beta)$ is a ring are equally characterized by the so-called Property (F)

$$\text{(F)} \quad \mathbf{Z}[\beta^{-1}] = \mathrm{Fin}(\beta),$$

where $\mathbf{Z}[\beta]$ denotes the ring of polynomials with integer coefficients in β.

It is known [3] that the condition (F) implies that β is a Pisot number (an algebraic integer with all its algebraic conjugates in modulus smaller than one). Several authors have found partial answers to the finiteness question by giving some sufficient conditions on the minimal polynomial of β.

Theorem 2.1: (Frougny, Solomyak [3]). Let β be a positive root of the polynomial $M(x) = x^m - a_1 x^{m-1} - \cdots - a_m$, $a_i \in \mathbf{Z}$ and $a_1 \geq a_2 \geq \cdots \geq a_m > 0$. Then β is a Pisot number and Property (F) holds for β.

Theorem 2.2: (Hollander [4]). Let β be an algebraic integer with the minimal polynomial $M(x) = x^m - a_1 x^{m-1} - \cdots - a_m$, $a_i \in \mathbf{Z}$, $a_i \geq 0$ and $a_1 > a_2 + a_3 + \cdots + a_m$. Then Property (F) holds for β.

Akiyama [5] proved a necessary and sufficient condition, however it is unfortunately quite vague.

Theorem 2.3: (Akiyama [5]). Let β be a Pisot number of the degree m. Then β has Property (F) if and only if every element of

$$\left\{ x \in \mathbf{Z}[\beta] \,\middle|\, 0 < x - x^{(1)} < 1, \, |x^{(j)}| \leq \frac{[\beta]}{1 - |\beta^{(j)}|} \text{ for } j = 2, 3, \ldots, m \right\}$$

has a finite β-expansion in base β. Here $x^{(i)}$ with $i = 1, 2, \ldots, m$ are the conjugates of $x \in \mathbf{Q}(\beta)$ in the algebraic field.

Ambrož et al. [6] gave a partial characterization of numbers β fulfilling Property (F) in terms of *minimal forbidden strings*.

Definition 2.4: Let $\beta > 1$. A forbidden string $u_k u_{k-1} \ldots u_0$ of non-negative integers is called *minimal*, if

- $u_{k-1} \ldots u_0$ and $u_k \ldots u_1$ are admissible,
- $u_i \geq 1$ implies $u_k \ldots u_{i+1}(u_i - 1)u_{i-1} \ldots u_0$ is admissible, for all $i = 0, 1, \ldots, k$.

The conditions based on the above-defined notion of minimal forbidden strings were given by the following two propositions.

Proposition 2.5: (Property (T)). If Fin(β) is closed under addition of two positive numbers, then β must satisfy the following property: For every minimal forbidden string $u_k u_{k-1} \ldots u_0$ there exists a finite sequence $v_n v_{n-1} \ldots v_\ell$ of non-negative integers, such that

- $k, \ell \leq n$,
- $v_n \beta^n + \cdots + v_\ell = u_k \beta^k + \cdots + u_1 \beta + u_0$,
- $v_n v_{n-1} \ldots v_\ell \succ \underbrace{00 \ldots 0}_{(n-k) \text{ times}} u_k \ldots u_0$.

Theorem 2.6: (Ambrož et al. [6]). Let $\beta > 1$ satisfy Property (T), and suppose that for every minimal forbidden string $u_k u_{k-1} \ldots u_0$ we have the following condition: If $v_n v_{n-1} \ldots v_\ell$ is the lexicographically greater string of (T) corresponding to $u_k u_{k-1} \ldots u_0$ then

$$v_n + v_{n-1} + \cdots + v_\ell \leq u_k + u_{k-1} + \cdots + u_0.$$

Then Fin(β) is closed under addition of positive elements. Moreover, for every positive $x, y \in$ Fin(β), the β-expansion of

$x + y$ can be obtained from any β-representation of $x + y$ using finitely many transcriptions.

Finally, Akiyama gave an algebraic characterization of cubic Pisot numbers satisfying Property (F).

Theorem 2.7: (Akiyama [7]) Let β be a cubic Pisot number. β has Property (F) if and only if β is a root of the following polynomial with integer coefficients

$$x^3 - ax^2 - bx - 1 \quad \text{where} \quad a \geq 0 \quad \text{and} \quad -1 \leq b \leq a + 1.$$

2.2 Number of fractional digits

The second question concerning arithmetics in β-numeration systems is connected to the fact that for non-integer β the set $\mathbf{Z}_\beta$ is not closed under arithmetic operations. However, sometimes it is true that the result of addition or multiplication of two β-integers has only a finite fractional part. Then it is interesting to try to estimate the maximal length of such fractional part. More precisely, the task is as follows. For given β find the value, or at least some good estimate, of the quantities

$$L_\oplus(\beta) := \min\{n \in \mathbf{N} | \, \forall x, y \in \mathbf{Z}_\beta, \, x + y \in \text{Fin}(\beta) \Rightarrow x + y \in \beta^{-n} \mathbf{Z}_\beta\},$$

$$L_\otimes(\beta) := \min\{n \in \mathbf{N} | \, \forall x, y \in \mathbf{Z}_\beta, \, xy \in \text{Fin}(\beta) \Rightarrow xy \in \beta^{-n} \mathbf{Z}_\beta\}.$$

Two methods for estimation of $L_\oplus(\beta)$, $L_\otimes(\beta)$ are known. The first of them uses the so-called Meyer property of the set of β-integers for β Pisot number, namely that

$\mathbf{Z}_\beta + \mathbf{Z}_\beta \subset \mathbf{Z}_\beta + F$ where F is a finite set,

and

$\mathbf{Z}_\beta \mathbf{Z}_\beta \subset \mathbf{Z}_\beta + G$ where G is a finite set.

This method is used in [6] to find values of $L_\oplus(\beta)$, $L_\otimes(\beta)$ for β the Pisot number, solution of the equation $x^3 = 25x^2 + 15x + 2$.

The second, much more widely used method for estimation of $L_\oplus(\beta)$, $L_\otimes(\beta)$ is based on the following theorem. Several version of this method are employed in [6, 8, 9, 10, 11, 12].

Theorem 2.8: (Guimond et al. [10]). Let $\beta > 1$ be an algebraic number, and let β' be its algebraic conjugate. For $z \in \mathbf{Q}(\beta)$ we denote by z' the image of z under the field isomorphism $' : \mathbf{Q}(\beta) \mapsto \mathbf{Q}(\beta')$. If

$$H := \sup\{|z'| \, | \, z \in Z_\beta\} < +\infty,$$

$$K := \inf\{|z'| \, | \, z \in Z_\beta \setminus \beta Z_\beta\} > 0,$$

then

$$\left(\frac{1}{|\beta'|}\right)^{L_\oplus} < \frac{2H}{K} \quad \text{and} \quad \left(\frac{1}{|\beta'|}\right)^{L_\otimes} < \frac{H^2}{K}.$$

Some known results on the values of $L_\oplus(\beta)$, $L_\otimes(\beta)$

- Knuth [13]: $L_\otimes(\beta) = 2$, β root of $x^2 = x + 1$.
- Burdík et al. [14]

  $L_\oplus(\beta) = L_\otimes(\beta) = 1$ for β root of equation $x^2 = mx - 1$,

  $L_\oplus(\beta) = L_\otimes(\beta) = 2$ for β root of equation $x^2 = mx + 1$.

- Guimond et al. [10] for β root the equation $x^2 = mx + n$, $m \leq n$

$$2\left\lfloor \frac{m+1}{m-n+1} \right\rfloor \le L_\oplus(\beta) \le 2\left\lceil \frac{m}{m-n+1} \right\rceil,$$

$$L_\otimes(\beta) \le 4 L_\oplus(\beta) \log_2(m+2).$$

- Messaoudi [15]: $L_\otimes(\beta) \le 6$ for β root of the equation $x^3 = x^2 + x + 1$, that is the so-called Tribonacci number.

- Ambrož et al. [6] for β Tribonacci number $5 \le L_\oplus(\beta) \le 6$ and $4 \le L_\otimes(\beta) \le 6$ and $L_\oplus(\beta) = 5$, $L_\otimes(\beta) = 7$ for β root of the equation $x^3 = 25x^2 + 15x + 2$.

- Bernat [9] gave an exact value for the Tribonacci addition, $L_\oplus(\beta) = 5$.

- Ambrož et al. [8] for the β root of the equation $x^3 = mx^2 + x + 1$

  $5 \le L_\oplus(\beta) \le 6$ for $m = 2$,

  $4 \le L_\oplus(\beta) \le 5$ for $m \ge 3$,

  $4 \le L_\otimes(\beta) \le 6$ for $m \ge 2$.

3 Tau-adic numeration system

Let τ be the golden mean, i.e. the Pisot number, root of the equation $x^2 = x + 1$ and τ' its algebraic conjugate. A *τ-adic representation* of a real number $x \in \mathbf{R}$ is a left-infinite sequence $(d_i)_{-n \le i < \infty}$, $d_i \in \mathbf{Z}$, $n \in \mathbf{Z}_+$, such that $x = \sum_{i=-n}^{\infty} d_i(\tau')^i$. It is denoted $_{\tau'}\langle x \rangle := \ldots d_1 d_0 \bullet d_{-1} \ldots d_{-n}$. The *value of a τ-adic representation* is obtained by the function $\pi: \mathbf{Z}^* \mapsto \mathbf{R}$ given by

$$\pi\left((d_i)_{-n \le i < \infty}\right) := \sum_{i=-n}^{\infty} d_i(\tau')^i.$$

If all finite factors of the sequence $(d_i)_{-n \le i < \infty}$ are admissible in the τ-numeration system, the sequence $(d_i)_{-n \le i < \infty}$ is said to be the *τ-adic expansion* of the number x, and it is denoted $_{\tau'}\langle x \rangle$.

3.1 Basic properties of τ-adic expansions

We know [3] that $\mathrm{Fin}(\tau) = \mathbf{Z}[\tau] = \{a + b\tau \,|\, a, b \in \mathbf{Z}\}$. Hence for any $z \in \mathbf{Z}_+$ there exist $m, n \in \mathbf{Z}$, $m \le n$ such that $z = \sum_{i=m}^{n} z_i \tau^i$ thus $z = \sum_{i=m}^{n} z_i(\tau')^i$, i.e. for each $z \in \mathbf{Z}_+$ the τ-adic expansion $_{\tau'}\langle z \rangle = \langle z \rangle_\tau$ is a finite word with some possible fractional part. For negative integers $z \in \mathbf{Z}_-$ we have the following proposition.

Proposition 3.1: (Ambrož [16]). Let $z \in \mathbf{Z}_-$ be a negative integer. Its τ-adic expansion $_{\tau'}\langle z \rangle$ is a left infinite, eventually periodic word of the form $_{\tau'}\langle z \rangle = {}^\omega(10)v$, where $v = \langle x \rangle_\tau$, $x \in \mathrm{Fin}(\tau)$.

Moreover, in [16], an algorithm is given for computing τ-adic expansions of negative integers.

Since most of the τ-adic expansions that we are dealing with are left infinite, eventually periodic, we define the following two sets of numbers.

$$\mathbf{I}_{ep}(\tau') := \{x \in R \,|\, {}_\tau\langle x \rangle = {}^\omega(d_{k+\ell} \ldots d_{k+1})d_k \ldots d_0 \bullet\},$$

$$\mathbf{F}_{ep}(\tau') := \{x \in R \,|\, {}_\tau\langle x \rangle = {}^\omega(d_{k+\ell} \ldots d_{k+1})d_k \ldots d_0 \bullet d_{-1} \ldots d_{-m}\},$$

Indeed, $\mathbf{Z} \subset \mathbf{F}_{ep}(\tau')$.

Concerning rational numbers, there is also an algorithm for computing their τ-adic expansions [16], which was used to prove that also every rational number has its τ-adic expansion, eventually periodic to the left, more precisely we have $\mathbf{Q} \cap (-1, 1] \subset \mathbf{I}_{ep}(\tau')$ and $\mathbf{Q} \subset \mathbf{F}_{ep}(\tau')$.

3.2 Finiteness condition

In accordance with the usual β-expansions, it is interesting to inspect the properties of the set $\mathbf{F}_{ep}(\tau')$, which can be seen as a τ-adic correlate of the set $\mathrm{Fin}(\beta)$. In [17] there is a proof that the set $\mathbf{F}_{ep}(\tau')$ is closed under the addition. The proof is of a constructive type, consisting of building a right to left transducer performing the addition, which preserves the periodicity of its input. When we know that $\mathbf{F}_{ep}(\tau')$ is closed under addition of positive elements, it is quite easy to prove that it is a ring. Finally, there has been shown a relation between eventually periodic τ-adic expansions and the elements of the ring $\mathbf{Q}(\tau)$.

Theorem 3.2: (Ambrož [17]). The field $\mathbf{Q}(\tau')$ (and therefore also the field $\mathbf{Q}(\tau)$) is equal to the set $\mathbf{F}_{ep}(\tau')$ of all real numbers having eventually periodic τ-adic expansion with a finite fractional part.

4 Future perspectives

There are several objectives that can be pursued in the future. Concerning the already classical β-expansions, there is always the challenging problem of the algebraic characterization of numbers β satisfying Property (F). Moreover, there is a lot of work to be done while looking for the number of fractional digits arising under arithmetic operations. Finally, there is an open question [6] of finding arithmetic algorithms working with infinite, but periodic β-expansions. A partial answer to the last question was recently given by the author in [18].

Concerning the second notion of irrational representation – τ-adic expansions – there are also several possible ways of future research. It seems quite obvious that it will be only a question of solving some technical obstacles to broaden the τ-adic expansions onto the β-adic expansions, for some class of β. Indeed, those classes of β fulfilling the Property (F) are suitable candidates. On the other hand, the proof of the finiteness property for the τ-adic expansions is deeply connected with the nature of the system itself and its generalization for some other irrationalities will be difficult or even impossible to perform without changing the approach.

Acknowledgment

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