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GENERALIZED THREE-BODY HARMONIC OSCILLATOR SYSTEM: GROUND STATE

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Abstract.

In this work we report on a 3-body system in a $d-$dimensional space $\mathbb{R}^d$ with a quadratic harmonic potential in the relative *distances* $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$ between particles. Our study considers unequal masses, different spring constants and it is defined in the three-dimensional (sub)space of solutions characterized (globally) by zero total angular momentum. This system is exactly-solvable with hidden algebra $s\ell_4(\mathbb{R})$. It is shown that in some particular cases the system becomes maximally (minimally) superintegrable. We pay special attention to a physically relevant generalization of the model where eventually the integrability is lost. In particular, the ground state and the first excited state are determined within a perturbative framework.

Keywords: Three-body system, exact-solvability, hidden algebra, integrability.

1. Introduction

The two-body harmonic oscillator, i.e. two particles with masses m_1 and m_2 interacting via the translational invariant potential $V \propto |\mathbf{r}_i - \mathbf{r}_j|^2$, appears in all textbook in Classical Mechanics. In an arbitrary $d-$dimensional Euclidean space $\mathbb{R}^d$ this system admits separation of variables in the center-of-mass and relative coordinates as well as exact solvability. The relevance of such a system is obvious: any scalar potential $U = U(|\mathbf{r}_i - \mathbf{r}_j|)$ can be approximated by the two-body harmonic oscillator. In this case, the center-of-mass and relative coordinates are nothing but the normal coordinates. Therefore, in the n-body case of $n > 2$ particles interacting by a quadratic pairwise potential it is natural to ask the question about the existence of normal coordinates and the corresponding explicit exact solutions. Interestingly, even for the three-body case $n = 3$ a complete separation of variables can not be achieved in full generality.

Starting in 1935, the quantum $n-$body problem in $\mathbb{R}^3$ was studied by Zernike and Brinkman [1] using the so-called hyperspherical-harmonic expansion. Two decades later, this method possessing an underlying group-theoretical nature was then reacquainted and refined in the papers by Delves [2] and Smith [3]. Nevertheless, in practice the success of the method is limited to the case of highly symmetric systems, namely identical particles with equal masses and equal spring constants.

In a previous work [4], the most general quantum system of a three-body chain of harmonic oscillators, in $\mathbb{R}^d$, was explored exhaustively. For arbitrary masses and spring constants this problem possesses spherical symmetry. It implies that the total angular momentum is a well-defined Observable which allows to reduce effectively the number of degrees of freedom in the corresponding Schrödinger equation governing the states with zero angular momentum. In the sector of vanishing angular momentum, it turns out that this three-body quantum system is exactly solvable. The hidden algebra $s\ell(4,\mathbb{R})$ responsible of the exact solvability was exhibited in [4] using the ρ-representation. In the present work we consider a physically relevant generalization of the model where eventually the integrability properties are lost. Again, in our analysis we assume a system of arbitrary masses and spring constants with the total angular momentum identically zero.

In the current study we revisited the algebraic structure and solvability of the quantum 3-body quantum oscillator system in the special set of coordinates appearing in [5], [6]. Afterwards, a physically motivated generalization of the model is considered. The goal of the paper is two-fold. Firstly, in the (sub)-space of zero total angular momentum we will describe the reduced Hamiltonian operator which admits a hidden $s\ell(4;\mathbb{R})$ algebraic structure, hence, allowing exact-analytical eigenfunctions. Especially, at any $d \geq 1$ it is demonstrated the existence of an exactly-solvable model that solely depends on the moment of inertia of the system. This model, admits a quasi-exactly-solvable extension as well.

Secondly, we explore a physically relevant generalization of the model. Approximate solutions of the problem are presented just for the case of equal masses in the framework of standard perturbation theory and complemented by the variational method. The first excited state, thus the energy gap of the system, is briefly discussed.

2. Generalities

The quantum Hamiltonian in $\mathbb{R}^d$ ($d>1$) for three non-relativistic spinless particles with masses m_1, m_2, m_3 and translationally invariant potential is given by

$$\mathcal{H} \ = \ -\sum_{i=1}^{3} \frac{1}{2\,m_i}\Delta_i^{(d)} \ + \ V(r_{12},\, r_{13},\, r_{23})\ , \qquad (1)$$

($\hbar = 1$) see e.g. [4, 5], where $\Delta_i^{(d)}$ stands for the individual Laplace operator of the ith mass with $d-$dimensional position vector $\mathbf{r}_i$, and

$$r_{ij} \ = \ |\mathbf{r}_i - \mathbf{r}_j|\ , \qquad (2)$$

($j=1,2,3$) is the relative mutual distance between the bodies i and j. The eigenfunctions of (1) which solely depend on the ρ-variables, $\rho_{ij}=r_{ij}^2$, are governed by a three-dimensional reduced Hamiltonian [4]

$$\mathcal{H}_{\rm rad} \ \equiv \ -\Delta_{\rm rad} \ + \ V(\rho)\ , \qquad (3)$$

where

$$\begin{aligned}\Delta_{\rm rad} \ = \ & \frac{2}{\mu_{12}}\rho_{12}\,\partial^2_{\rho_{12}} \ + \ \frac{2}{\mu_{13}}\rho_{13}\,\partial^2_{\rho_{13}} \\ & + \frac{2}{\mu_{23}}\rho_{23}\,\partial^2_{\rho_{23}} \ + \ \frac{2(\rho_{13}+\rho_{12}-\rho_{23})}{m_1}\partial_{\rho_{13},\,\rho_{12}} \\ & + \frac{2(\rho_{13}+\rho_{23}-\rho_{12})}{m_3}\partial_{\rho_{13},\,\rho_{23}} + \frac{2(\rho_{23}+\rho_{12}-\rho_{13})}{m_2}\partial_{\rho_{23},\,\rho_{12}} \\ & + \frac{d}{\mu_{12}}\partial_{\rho_{12}} + \frac{d}{\mu_{13}}\partial_{\rho_{13}} + \frac{d}{\mu_{23}}\partial_{\rho_{23}}\ , \qquad (4)\end{aligned}$$

c.f. [5], and

$$\mu_{ij} \ = \ \frac{m_i\,m_j}{m_i+m_j}\ ,$$

denotes a reduced mass. The operator (3) describes three-dimensional (radial) dynamics in variables $\rho_{12}, \rho_{13}, \rho_{23}$. This operator $\mathcal{H}_{\rm rad}$ is, in fact, equivalent to a Schrödinger operator, see [4]. We call it three-dimensional (radial) Hamiltonian. All the $d-$dependence in (3) occurs in the coefficients in front of the first derivatives.

2.1. Case of identical particles: τ-representation

Now, let us consider the case of identical masses

$$m_1 \ = \ 1\ ;\ m_2 \ = \ 1\ ;\ m_3 \ = \ 1\ ,$$

thus, $\mu_{ij} = \frac{1}{2}$, and the operator (4) is S_3 permutationally-invariant in the ρ-variables. It suggests the change of variables $\rho \leftrightarrow \tau$ where

$$\begin{aligned}\tau_1 \ &= \ \rho_{12}+\rho_{13}+\rho_{23}\ , \\ \tau_2 \ &= \ \rho_{12}\,\rho_{13}+\rho_{12}\,\rho_{23}+\rho_{13}\,\rho_{23}\ , \qquad (5) \\ \tau_3 \ &= \ \rho_{12}\,\rho_{13}\,\rho_{23}\ ,\end{aligned}$$

are nothing but the lowest elementary symmetric polynomials in ρ-coordinates.

In these variables (5), the coefficients of the operator $\Delta_{\rm rad}$ are also polynomials, hence, this operator is algebraic in both representations. Explicitly,

$$\begin{aligned}\Delta_{\rm rad} \ = \ & 6\,\tau_1\partial_1^2 \ + \ 2\tau_1(7\tau_2-\tau_1^2)\partial_2^2 \ + \ 2\tau_3(6\tau_2-\tau_1^2)\partial_3^2 \\ & + \ 24\,\tau_2\partial^2_{1,2} \ + \ 36\tau_3\partial^2_{1,3} \ + \ 2\,[9\tau_3\tau_1+4\tau_2(\tau_2-\tau_1^2)]\partial^2_{2,3} \\ & + \ 6\,d\,\partial_1 \ + \ 2\,(2d+1)\tau_1\,\partial_2 \ + \ 2\,[(d+4)\tau_2-\tau_1^2]\,\partial_3 \qquad (6) \\ & \partial_i \ \equiv \ \partial_{\tau_i},\ i=1,2,3.\end{aligned}$$

3. Laplace-Beltrami operator

Now, as a result of calculations it is convenient to consider the following gauge factor

$$\Gamma^4 \ = \ \frac{(S^2_\triangle)^{2-d}}{M\,\mathcal{I}}\ , \qquad (7)$$

$M=m_1+m_2+m_3$, where

$$S^2_\triangle = \frac{2\rho_{12}\,\rho_{13}+2\rho_{12}\,\rho_{23}+2\rho_{23}\,\rho_{13}-\rho_{12}^2-\rho_{13}^2-\rho_{23}^2}{16}\ ,$$

and

$$\mathcal{I} \ = \ \frac{m_1m_2\,\rho_{12} \ + \ m_1m_3\,\rho_{13} \ + \ m_2m_3\,\rho_{23}}{M}\ ,$$

possess a geometrical meaning. The term $S^2_\triangle$ is the area (squared) of the triangle formed by the position vectors of the three bodies whilst the term $\mathcal{I}$ is the *moment of inertia* of the system with respect to its center of mass. The radial operator $\mathcal{H}_{\rm rad}$ (3) is gauge-transformed to a truly Schrödinger operator [4],

$$\mathcal{H}_{\rm LB} \equiv \Gamma^{-1}\,\mathcal{H}_{\rm rad}\,\Gamma \ = \ -\Delta_{\rm LB} \ + \ V \ + \ V^{(\rm eff)}\ , \qquad (8)$$

here $\Delta_{\rm LB}$ stands for the Laplace-Beltrami operator

$$\Delta_{LB}(\rho) \ = \ \sqrt{|\,g\,|}\ \partial_\mu\ \frac{1}{\sqrt{|\,g\,|}}\ g^{\mu\nu}\partial_\nu\ ,$$

($\nu,\mu=1,2,3$) and $\partial_1=\frac{\partial}{\partial\rho_{12}}, \partial_2=\frac{\partial}{\partial\rho_{13}}, \partial_3=\frac{\partial}{\partial\rho_{23}}$. The corresponding co-metric in $\Delta_{LB}(\rho)$ reads

$$g^{\mu\nu} \ = \ \begin{pmatrix} \frac{2}{\mu_{12}}\rho_{12} & \frac{(\rho_{13}+\rho_{12}-\rho_{23})}{m_1} & \frac{(\rho_{23}+\rho_{12}-\rho_{13})}{m_2} \\ \frac{(\rho_{13}+\rho_{12}-\rho_{23})}{m_1} & \frac{2}{\mu_{13}}\rho_{13} & \frac{(\rho_{13}+\rho_{23}-\rho_{12})}{m_3} \\ \frac{(\rho_{23}+\rho_{12}-\rho_{13})}{m_2} & \frac{(\rho_{13}+\rho_{23}-\rho_{12})}{m_3} & \frac{2}{\mu_{23}}\rho_{23} \end{pmatrix}.$$

Its determinant

$$|\,g\,| \equiv \ {\rm Det}g^{\mu\nu} \ = \ 32\,\frac{M^2}{m_1^2m_2^2m_3^2}\,\mathcal{I}\,S^2_\triangle\ , \qquad (9)$$

admits factorization and is positive definite. The term $V^{(\rm eff)}$ denotes an *effective potential*

$$V^{(\rm eff)} \ = \ \frac{3}{8}\,\frac{1}{\mathcal{I}} \ + \ \frac{(d-2)(d-4)}{32}\,\frac{M\,\mathcal{I}}{m_1\,m_2\,m_3\,S^2_\triangle}\ ,$$

which depends on the two variables $\mathcal{I}$ and $S^2_\triangle$ alone. Thus, the underlying geometry of the system emerges.

The classical analogue of the quantum Hamiltonian operator (8) describes an effective non-relativistic

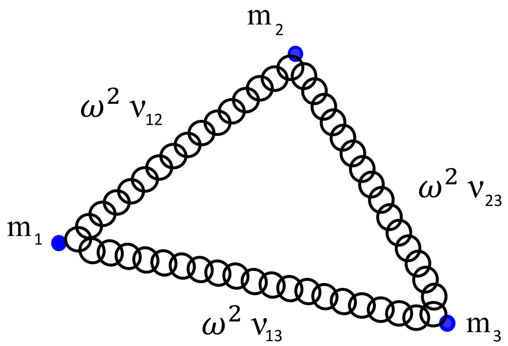

Figure 1. 3-body chain of harmonic oscillators.

classical particle in a three-dimensional curved space. Explicitly, the Hamiltonian function takes the form

$$\mathcal{H}_{\rm LB}^{\rm (classical)} \ = \ g^{\mu\nu}\,\Pi_\mu\,\Pi_\nu \ + \ V \ , \qquad (10)$$

where $\Pi_\mu\,,\ \mu = 12, 23, 13$ are the associated canonical conjugate momenta to the ρ-coordinates. The Hamilton-Jacobi equation, at vanishing potential $V = 0$ (free motion), is clearly integrable. However, a complete separation of variables is absent in the ρ-representation. The Poisson bracket between the kinetic energy $T = g^{\mu\nu}\,\Pi_\mu\,\Pi_\nu$ and the linear function in momentum variables

$$L_1^{(c)} \ = \ (\rho_{13}-\rho_{23})\Pi_{12}+(\rho_{23}-\rho_{12})\Pi_{13}+(\rho_{12}-\rho_{13})\Pi_{23}\ ,$$

is zero.

4. Three body harmonic oscillator system

In the spectral problem with Hamiltonian (3) we take the harmonic potential

$$V^{(HO)}(\rho) \ = \ 2\,\omega^2\left[\nu_{12}\,\rho_{12} + \nu_{13}\,\rho_{13} + \nu_{23}\,\rho_{23}\right] , \ (11)$$

$\omega > 0$ is frequency and $\nu_{12}, \nu_{13}, \nu_{23} > 0$ are constants with dimension of mass. This problem can be solved exactly [4]. In particular, in ρ-space the reduced operator (3) possesses multivariate polynomial eigenfunctions, see below. We call the above potential $V^{(HO)}(\rho)$ the 3-body oscillator system. We mention that in the case $d = 1$ (3 particles on a line), the corresponding spectral problem was studied in the paper [7]. In the current report, we analyze the $d-$dimensional case with $d > 1$.

In r-variables, $\rho = r^2$, the potential (11) can be interpreted as a three-dimensional (an)isotropic one-body oscillator. It is displayed in Figure 1. The configuration space is a subspace of the cube $\mathbb{R}^3_+(\rho)$ in E_3 ρ-space. The ρ-variables must obey the "triangle condition" $S^2_\triangle \geqslant 0$, namely the area of the triangle formed by the position vectors of the bodies is always positive.

4.1. Solution for the ground state

In the harmonic potential (11), the ground state eigenfunction reads

$$\Psi_0^{(HO)} \ = \ e^{-\omega\,(a_1\,\mu_{12}\,\rho_{12}\,+\,a_2\,\mu_{13}\,\rho_{13}\,+\,a_3\,\mu_{23}\,\rho_{23})}\ , \ (12)$$

where the parameters a_1, a_2, $a_3 \geq 0$ are introduced for convenience. They define the spring constants, see below. The associated ground state energy

$$E_0 \ = \ \omega\,d\,(a_1+a_2+a_3)\ , \qquad (13)$$

is mass-independent. There exists the following algebraic relations

$$\begin{aligned}
\nu_{12} &= a_1^2\,\mu_{12} \ + \ a_1\,a_2\,\frac{\mu_{12}\,\mu_{13}}{m_1} \ + \ a_1\,a_3\,\frac{\mu_{12}\,\mu_{23}}{m_2} \\
&\quad - \ a_2\,a_3\,\frac{\mu_{13}\,\mu_{23}}{m_3}\ , \\
\nu_{13} &= a_2^2\,\mu_{13} \ + \ a_1\,a_2\,\frac{\mu_{12}\,\mu_{13}}{m_1} \ + \ a_2\,a_3\,\frac{\mu_{13}\,\mu_{23}}{m_3} \\
&\quad - \ a_1\,a_3\,\frac{\mu_{12}\,\mu_{23}}{m_2}\ , \\
\nu_{23} &= a_3^2\,\mu_{23} \ + \ a_1\,a_3\,\frac{\mu_{12}\,\mu_{23}}{m_2} \ + \ a_2\,a_3\,\frac{\mu_{13}\,\mu_{23}}{m_3} \\
&\quad - \ a_1\,a_2\,\frac{\mu_{12}\,\mu_{13}}{m_1}\ .
\end{aligned}$$

5. Lie algebraic structure

Using the previous function $\Psi_0^{(HO)}$ (12) as a gauge factor, the transformed Hamiltonian $\mathcal{H}_{\rm rad}$ (3)

$$h^{\rm (algebraic)} \equiv \left(\Psi_0^{(HO)}\right)^{-1}\left[-\Delta_{\rm rad} + V - E_0\right]\Psi_0^{(HO)} \qquad (14)$$

is an algebraic operator, i.e. the coefficient are polynomials in the ρ-variables. The E_0 is taken from (13).

In addition, this algebraic operator (14) is of Lie-algebraic nature. It admits a representation in terms of the generators

$$\begin{aligned}
\mathcal{J}_i^- \ &= \ \frac{\partial}{\partial y_i}\ , \\
\mathcal{J}_{ij}^0 \ &= \ y_i\,\frac{\partial}{\partial y_j}\ , \\
\mathcal{J}^0(N) \ &= \ \sum_{i=1}^3 y_i\,\frac{\partial}{\partial y_i} - N\,, \\
\mathcal{J}_i^+(N) \ &= \ y_i\,\mathcal{J}^0(N) \ = \ y_i\left(\sum_{j=1}^3 y_j\,\frac{\partial}{\partial y_j} - N\right)\ ,
\end{aligned}$$

$(i, j = 1, 2, 3)$ of the algebra $s\ell(4,\mathbb{R})$, see [8, 9] here N is a constant. The notation

$$y_1 \ = \ \rho_{12}\ , \qquad y_2 \ = \ \rho_{13}\ , \qquad y_3 \ = \ \rho_{23}\ ,$$

was employed for simplicity. If N is a non-negative integer, a finite-dimensional representation space takes place,

$$\mathcal{V}_N \ = \ \langle y_1^{n_1}\,y_2^{n_2}\,y_3^{n_3}\,|\ 0 \leq n_1+n_2+n_3 \leq N\rangle\ . \ (15)$$

6. Relation with the Jacobi oscillator

Now, we can indicate an emergent relation between the harmonic potential (11) and the Jacobi oscillator system

$$\mathcal{H}^{(\text{Jacobi})} \equiv \sum_{i=1}^{2}\left[-\frac{\partial^2}{\partial\mathbf{z}_i\partial\mathbf{z}_i} + 4\,\Lambda_i\,\omega^2\,\mathbf{z}_i\cdot\mathbf{z}_i\right]\ , \quad (16)$$

where $\omega > 0$, Λ_1, $\Lambda_2 \geq 0$, and

$$\mathbf{z}_1 = \sqrt{\frac{m_1\,m_2}{m_1+m_2}}\,(\mathbf{r}_1 - \mathbf{r}_2)$$
$$\mathbf{z}_2 = \sqrt{\frac{(m_1+m_2)\,m_3}{m_1+m_2+m_3}}\left(\mathbf{r}_3 - \frac{m_1\,\mathbf{r}_1 + m_2\,\mathbf{r}_2}{m_1+m_2}\right)$$

are standard Jacobi variables, see e.g. [10]. This Hamiltonian describes two decoupled harmonic oscillators in flat space, see [6]. Consequently, it is an exactly-solvable problem. The complete spectra and eigenfunctions can be calculated by pure algebraic means.

The solutions of the Jacobi oscillator that solely depend on the Jacoby distances $z_i = |\mathbf{z}_i|$ are governed by the operator,

$$\mathcal{H}^{(\text{Jacobi})}_{\text{rad}} = \sum_{i=1}^{2}\left[-\frac{\partial^2}{\partial z_i\partial z_i} - \frac{(d-1)}{z_i}\frac{\partial}{\partial z_i}\right] + 4\,\Lambda_1\,\omega^2\,z_1^2 + 4\,\Lambda_2\,\omega^2\,z_2^2\ . \quad (17)$$

In this case, the associated hidden algebra is given by $sl_2^{\otimes\,(2)}$ which acts on the two-dimensional space (z_1, z_2).

In particular, the eigenfunctions of $\mathcal{H}^{(\text{Jacobi})}$ (16) can be employed to construct approximate solutions for the n-body problem, for this discussion see [10].

Assuming any of the two conditions

$$\frac{m_2}{m_3} = \frac{\nu_{12}}{\nu_{13}} \quad ; \quad \frac{m_1}{m_2} = \frac{\nu_{13}}{\nu_{23}}\ ,$$

in the harmonic oscillator potential $V^{(HO)}$ (11), we obtain

$$\begin{aligned} U_J^{(HO)} &\equiv 4\,\Lambda_1\,\omega^2\,z_1^2 + 4\,\Lambda_2\,\omega^2\,z_2^2 \\ &= 2\,\omega^2\left[\nu_{12}\,\rho_{12} + \nu_{13}\,\rho_{13} + \nu_{23}\,\rho_{23}\right] \\ &= V^{(HO)} \end{aligned} \quad (18)$$

with

$$\Lambda_1 = \Lambda_2 = \frac{m_1+m_2+m_3}{2\,m_1\,m_3}\,\nu_{13}\ ,$$

hence, in this case the three-body oscillator potential coincides with the two-body Jacobi oscillator potential. In fact, imposing the singly condition $m_2\,\nu_{13} = m_3\,\nu_{12}$ the equality (18) is still valid but $\Lambda_1 \neq \Lambda_2$ and the system is not maximally superintegrable any more.

6.1. Identical particles: hyperradious

A remarkable simplification occurs in the case of three identical particles with the same common spring constant, namely

$$m_1 = m_2 = m_3 = 1 \quad , \quad a_1 = a_2 = a_3 \equiv a\ . \quad (19)$$

Thus, the potential (11) reduces to

$$\begin{aligned} V^{(HO)} &= \frac{3}{2}\,a^2\,\omega^2\,(\rho_{12} + \rho_{13} + \rho_{23}) \\ &= \frac{3}{2}\,a^2\,\omega^2\,\tau_1\ . \end{aligned}$$

Consequently, the ground state solutions (12) and (13) read

$$\begin{aligned} \Psi_0^{(3a)} &= e^{-\frac{\omega}{2}\,a\,(\rho_{12} + \rho_{13} + \rho_{23})} \\ &= e^{-\frac{\omega}{2}\,a\,\tau_1}\ , \end{aligned} \quad (20)$$

$$E_0 = 3\,\omega\,d\,a\ , \quad (21)$$

respectively. Moreover, from (6) it follows that in this case there exists an infinite family of eigenfunctions

$$\Psi_N(\tau_1) = e^{-\frac{1}{2}\,a\,\omega\,\tau_1}\,L_N^{(d-1)}(a\,\omega\,\tau_1)\ ,$$

with energy

$$E_N = 3\,a\,\omega\,(\,d + 2\,N\,)\ ,$$

$N = 0, 1, 2, 3, \ldots$, that solely depend on the variable τ_1, the so called *hyperradious*, here $L_N^{(d-1)}(x)$ denotes the generalized Laguerre polynomial. These solutions are associated with a hidden $s\ell(2,\mathbb{R})$ Lie-algebra.

6.2. Arbitrary masses: moment of inertia

A generalization of the results presented in Section 6.1 can be derived from the decomposition of Δ_{rad} (4)

$$\Delta_{\text{rad}} = \Delta_{\mathcal{I}} + \tilde{\Delta}\ , \quad (22)$$

where $\Delta_{\mathcal{I}} = \Delta_{\mathcal{I}}(\mathcal{I})$ is an algebraic operator for arbitrary $d \geq 1$. It depends on the moment of inertial $\mathcal{I}$ only. Explicitly, we have

$$\Delta_{\mathcal{I}} = 2\,\mathcal{I}\,\partial^2_{\mathcal{I},\mathcal{I}} + 2\,d\,\partial_{\mathcal{I}}\ . \quad (23)$$

The operator $\tilde{\Delta} = \tilde{\Delta}(\mathcal{I}, q_1, q_2)$ depends on $\mathcal{I}$ and two more (arbitrary) variables q_1, q_2 for which the coordinate transformation $\{\rho_{ij}\} \to \{\mathcal{I}, q_1, q_2\}$ is invertible (not singular). Since such an operator $\tilde{\Delta}$ annihilates any function $F = F(\mathcal{I})$, i.e. $\tilde{\Delta}\,F = 0$, the splitting (22) indicates that for any potential of the form

$$V = V(\mathcal{I})\ , \quad (24)$$

the eigenvalue problem for the operator $\mathcal{H}_{\text{rad}} = -\Delta_{\text{rad}} + V$ is further reduced to a one-dimensional spectral problem, namely

$$[\,-\Delta_{\mathcal{I}} + V(\mathcal{I})\,]\,\psi = E\,\psi\ , \quad (25)$$

which can be called the $\mathcal{I}-$representation.

In the case of equal masses $m_1 = m_2 = m_3$ the coordinate $\mathcal{I}$ is proportional to the hyperspherical radius (hyperradious). Also, $H_{\mathcal{I}}$ (25) is gauge-equivalent to a one-dimensional the Schrödinger operator.

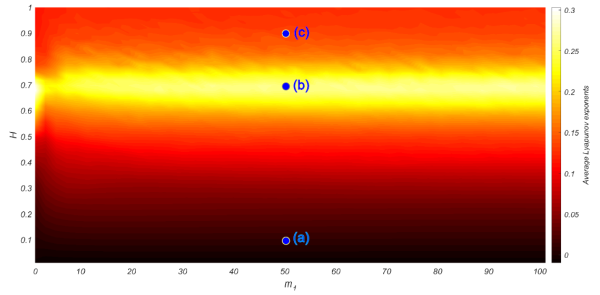

Figure 2. Classical generalized three-body harmonic oscillator system: average Lyapunov exponent in the space of parameters $(H,\, m_1)$. The values $m_2 = m_3 = 1$, $\omega = 1$, $\nu_{12} = \nu_{13} = \nu_{23} = 1$ and $R_{12} = R_{13} = R_{23} = 1$ were used.

7. Generalized three body harmonic oscillator system

Now, let us consider the following potential

$$V^{(R)} = 2\,\omega^2 \bigg[\nu_{12}\,(\sqrt{\rho_{12}} \,-\, R_{12})^2 \,+\, \nu_{13}\,(\sqrt{\rho_{13}} \,-\, R_{13})^2 \,+\, \nu_{23}\,(\sqrt{\rho_{23}} \,-\, R_{23})^2 \bigg] \ , \tag{26}$$

where $R_{12}, R_{13}, R_{23} \geqslant 0$ denote the rest lengths of the system. At $R_{12} = R_{13} = R_{23} = 0$ we recover the exactly solvable 3-body oscillator system, $V^{(R)} \to V^{(HO)}$. The relevance of $V^{(R)}$ comes from the fact that any arbitrary potential $V = V(r_{ij})$ can be approximated, near its equilibrium points, by this generalized 3-body harmonic potential.

However, the existence of non-trivial exact solutions is far from being evident. Even for the most symmetric case of equal masses and equal spring constants, we were not able to find a hidden Lie algebra in the corresponding spectral problem (3). Moreover, at the classical level such a system is chaotic. This can be easily seen by computing the average Lyapunov exponent in the space of parameters $(H,\, m_1)$, see Figure 2, where H is the value of the classical Hamiltonian (energy) with potential $V^{(R)}$ (26).

Also, for one-dimensional systems it is said (see [11]) that a classical orbit is $\mathcal{PT}$-symmetric if the orbit remains unchanged upon replacing $x(t)$ by $-x^*(-t)$. There are several classes of complex $\mathcal{PT}$-symmetric non-Hermitian quantum-mechanical Hamiltonians whose eigenvalues are real and with unitary time evolution [12, 13]. However, while the corresponding quantum three-body oscillator Hamiltonian is Hermitian, it can still have interesting complex classical trajectories.

7.1. Identical particles

In order to simplify the problem one can consider the simplest case of equal masses and equal spring constants (19) with $\omega = 1$. Also, we will assume equal rest lengths

$$R_{12} \ = \ R_{13} \ = \ R_{23} \ = \ R > 0 \ .$$

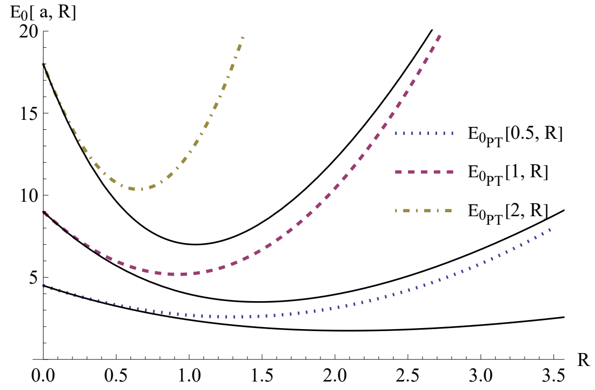

Figure 3. Ground state energy of the generalized 3-body harmonic oscillator *vs* R for different values of the parameter a which defines the spring constant, see text. The solid lines correspond to the variational result whilst the dashed ones refer to the value calculated by perturbation theory up to first order.

In this case, approximate solutions for the Schrödinger equation can be obtained using perturbation theory in powers of R.

7.1.1. Ground state

Taking the $R-$dependent terms in (26) as a small perturbation, the first correction $E_{1,0}$ to the ground state energy takes the form

$$E_{1,0} \ = \ \frac{3\,a}{2\,\pi}(\,3\,\pi\,a\,R^2 \ - \ 4\,R\,\sqrt{6\,\pi\,a}\,) \ .$$

The domain of validity of this perturbative approach is estimated by means of the variational method. The use of the simple trial function

$$\Psi_0^{\rm trial} \ = \ e^{-\frac{\omega}{2}\,a\,\alpha\,(\,\rho_{12} \ + \ \rho_{13} \ + \ \rho_{23}\,)}$$

c.f. (20), where α is a variational parameter to be fixed by the procedure of minimization, leads to the results shown in Figure 3.

7.1.2. First excited state

It is important to mention that for the 3-body harmonic oscillator $(R = 0)$ the exact first excited state possesses a degeneracy equal to 3. For $R > 0$, the perturbation theory partially breaks this degeneracy. The energy of the approximate first excited state calculated by perturbation theory, up to first order, is displayed in Figure 4.

8. Conclusions

In this report for a 3-body harmonic oscillator in Euclidean space $\mathbb{R}^d$ we consider the Schrödinger operator in ρ-variables $\rho_{ij} = r_{ij}^2$,

$$\mathcal{H}_{\rm LB} \ = \ -\Delta_{\rm LB}(\rho_{ij}) + V^{(HO)}(\rho_{ij}) + V^{(\rm eff)}(\rho_{ij}) \ , \tag{27}$$

where the kinetic energy corresponds to a 3-dimensional particle moving in a non-flat space. The

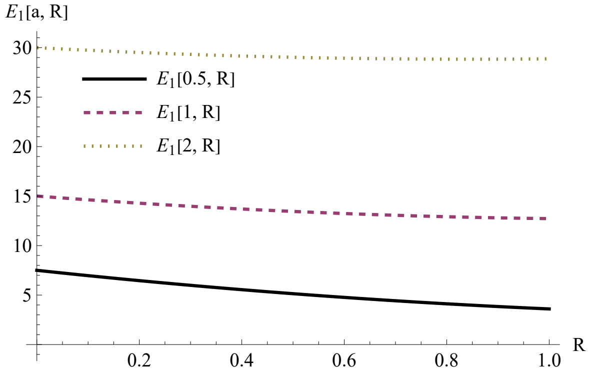

FIGURE 4. First excited state of the generalized 3-body harmonic oscillator *vs* R.

Schrödinger operator (27) governs the $\mathcal{S}$-states solutions of the original three-body system (1), in particular, it includes the ground state. It implies that the solutions of corresponding eigenvalue problem depend solely on three coordinates, contrary to the ($3d$)-dimensional Schrödinger equation. The reduced Hamiltonian $\mathcal{H}_{\rm LB}$ is an Hermitian operator, where the variational method can be more easily implemented (the energy functional is a 3-dimensional integral only). The classical analogue of (27) was presented as well. The operator (27) up to a gauge rotation is equivalent to an algebraic operator with hidden algebra $s\ell(4,\mathbb{R})$, thus, becoming a Lie-algebraic operator.

In the case of identical masses and equal frequencies the aforementioned model was generalized to a 3-body harmonic system with a non-zero rest length $R > 0$. In this case, no hidden algebra nor exact solutions seem to occur. An indication of the lost of integrability is the fact that the classical counterpart of this model exhibits chaotic motion. Using perturbation theory complemented by the variational method it was shown that the ground state energy *vs* R develops a global minimum, hence, defining a configuration of equilibrium.

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