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# RATIONAL EXTENSION OF MANY PARTICLE SYSTEMS

Bhabani Prasad Mandal

*Banaras Hindu University, Institute of Science, Physics Department, Lanka, 221005–Varanasi, Uttar Pradesh India*

correspondence: bhabani.mandal@gmail.com

Abstract.

In this talk, we briefly review the rational extension of many particle systems, and is based on a couple of our recent works. In the first model, the rational extension of the truncated Calogero-Sutherland (TCS) model is discussed analytically. The spectrum is isospectral to the original system and the eigenfunctions are completely expressed in terms of exceptional orthogonal polynomials (EOPs). In the second model, we discuss the rational extension of a quasi exactly solvable (QES) N-particle Calogero model with harmonic confining interaction. New long-range interaction to the rational Calogero model is included to construct this QES many particle system using the technique of supersymmetric quantum mechanics (SUSYQM). Under a specific condition, infinite number of bound states are obtained for this system, and corresponding bound state wave functions are written in terms of EOPs.

Keywords: Exceptional orthogonal polynomials, rational extensions, many particle systems, SUSYQM.

## 1. Introduction

Orthogonal polynomials play very useful and important roles in studying physics, particularly in electrostatics and in quantum mechanics. In quantum mechanics, only a few of the commonly occuring bound states problems, which have a wide range of applications and/or extensions, are solvable. Such systems generally bring into physics a class of orthogonal polynomials.These classical orthogonal polynomials have many properties common, such as (i) each constitutes orthogonal polynomials of successive increasing degree starting from $m = 0$, (ii) each satisfy a second order homogeneous differential equations, (iii) they satisfy orthogonality over a certain interval and with a certain non-negative weight function, etc. In 2009, new families of orthogonal polynomials (known as exceptional orthogonal polynomials (EOP)) related to some of the old classical orthogonal polynomials were discovered [1–3]. Unlike the usual classical orthogonal polynomials, these EOPs start with degree $m = 1$ or higher integer values and still form a complete orthonormal set with respect to a positive definite inner product defined over a compact interval. Two of the well known classical orthogonal polynomials, namely Laguerre orthogonal polynomials and Jacobi orthogonal polynomials, have been extended to EOPs category. $X_m$ Laguerre (Jocobi) EOP means the complete set of Laguerre (Jacobi) orthogonal polynomials with degree $\geq m$. $m$ is positive integer and can have values of $1, 2, 3, \ldots$ Attempts were made to also extend the classical Hermite polynomials [4]. Soon after this remarkable discovery, the connection of EOPs with the translationally shape invariant potential were established [5–9]. The list of exactly solvable quantum mechanical systems is enlarged and the wave functions for the newly obtained exactly solvable systems are written in terms of EOPs. Such systems are known as rational extension of the original systems. The study for the exactly solvable potentials has been boosted greatly due to this discovery of EOPs over the past decade [10–37].

There are several commonly used approaches to build the rationally extended models, such as SUSYQM approach [38, 39], Point canonical transformation approach [40, 41], Darboux-Crum transformation approach [42, 43], group theoretical approach [44], etc. These approaches have been used to study different problems in this field leading to a discovery of a large number of new exactly solvable systems, which are isospectral to the original system and the eigenfunctions are written in terms of EOPs. Further, quasi-exactly solvable (QES) systems [45–49] and conditionally exactly solvable (CES) systems [50, 51] attracted attention in literature due to the lack of many exactly solvable systems. Several works have been devoted to the rational extension of these QES/ CES systems [22, 24, 37]. Nowadays, the parity time reversal (PT) symmetric non-Hermitian systems [52–62] are among the exciting frontier research areas. Rational extensions have also been carried out for non-Hermitian systems [6, 19, 29–32]. Even though most of the rational extensions are for the one dimensional and/or one particle exactly solvable systems, the research in this field has also been extended to many particle systems [24, 25, 27]. We have done several works on rational extensions for many particle systems. In one of the works, the well known Calogero-Wolfes type 3-body problem on a line was extended rationally to show that exactly solvable wave functions are written in terms of $X_m$ Laguerre and $X_m$ Jacobi EOPs [26]. However, this article is based on two of our earlier works on rational extension of many particle systems [24, 25], which were central to the talk presented during the AAMP meeting. In the first work [25], we discuss the rational extension of the truncated Calogero-Sutherland model using a PCT approach. We indicate how to obtain rationally extended solutions, which are isospectral to the original system in terms of $X_m$ Laguerre EOPs. In the second model [24], we discuss the rational extension of a QES N-particle Calogero model with a harmonic confining interaction. New long-range interactions to the rational Calogero model are included to construct this QES many particle system using SUSYQM. The wavefunctions are expressed, again, in terms of exceptional orthogonal Laguerre polynomials.

Now, we present the organisation of the article. In the next section, we present the TCS model and its solutions in brief to set the things for the section 3, where we consider the rational extension of the TCS model. In section 4, the QES solutions for the rationally extended Calogero type many particle system are presented. Section 5 is reserved for conclusions.

## 2. TCS model

In his work, Jain-Khare (JK) [63] exactly solved some variant of Calogero-Sutherland model (CSM) on the full line by taking only the nearest and next-to-nearest neighbor interactions through 2-body and 3-body interactions. Later, Pittman et al. [64] generalized this model by considering an N-body problem on a line with harmonic confinement with tunable inverse square as well as the three-body interaction extends over a finite number of neighbors and were able to solve it exactly. This model is known as truncated Calogero-Sutherland model (TCS). $N$-body TCS model [64], where particles are interacting through 2-body and 3-body potentials, is given by

$$H = \sum_{i=1}^{N}\Big[-\frac{1}{2}\frac{\partial^2}{\partial x_i^2}+\frac{1}{2}\omega^2 x_i^2\Big] + \sum_{\substack{i<j\\ |i-j|\le r}} \frac{\lambda(\lambda-1)}{\mid x_i - x_j\mid^2} + \sum_{\substack{i<j<k\\ |i-j|\le r\\ |j-k|\le r}} \frac{\lambda^2(x_i-x_j)\mathbf{x}\cdot(x_j-x_k)\mathbf{x}}{|x_j-x_j|^2|x_j-x_k|^2} \tag{1}$$

with $\lambda \neq 0$ and $\mathbf{x} = (x_1, x_2, \dots, x_N) \in R^N$. The 2- body interaction is attractive for $0 < \lambda < 1$ and is repulsive for $\lambda \ge 1$. Here, $r$ is the integer parameter and for $r = 1$, this system reduces to that of the Jain-Khare [63] model. However, for $r = N-1$, it corresponds to the CSM [65–67] model.

Using standard techniques in the case of many particle systems, the time independent Schrodinger equation (TISE) corresponding to the above system can be written in radial and angular parts as

$$\Phi''(\rho) + \Big(N + 2s - 1 + \lambda r(2N - r - 1)\Big)\frac{1}{\rho}\Phi'(\rho) + 2(E - \frac{1}{2}\omega^2\rho^2)\Phi(\rho) = 0\,, \tag{2}$$

(where $\rho = \sum_i^N x_i^2$, is the radial coordinate and the prime denotes the differentiation with respect to its arguments and this convention is adopted throughout this manuscript) and

$$\Bigg[\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + 2\lambda\sum_{i<j}^{N-1}\frac{1}{x_i - x_j}\Big(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\Big)\Bigg]P_s(\mathbf{x}) = 0. \tag{3}$$

(where the function $P_s$ denotes the homogeneous polynomial of angular variables of degree $s = 0, 1, 2, \dots$). To obtaine these Eqs. we have substituted the wave function

$$\Psi(x) = \prod_{i<j}(x_i - x_j)^\lambda\,\Phi(\rho)\;P_s(x) \tag{4}$$

in the TISE, $H\Psi = E\Psi$ .

This model is solved exactly and the solution is given by,

$$\text{the spectrum: } E_n = \omega\Big(2n + s + \frac{N}{2} + \frac{\lambda r}{2}(2N - r - 1)\Big), \tag{5}$$

and the corresponding radial wave function in terms of classical Laguerre polynomials is given as

$$\Phi(\rho) \simeq \exp(-\frac{\omega\rho^2}{2})L_n^{(\alpha)}(\omega\rho^2); \quad n = 0, 1, 2, \dots \tag{6}$$

where $\alpha = \big(s - 1 + \frac{N}{2} + \frac{\lambda r}{2}(2N - r - 1)\big)$. This result is consistent with JK and CSM models in the appropriate limit [64]. In the next section, we will extend this model by adding some interaction terms. Then, the extended model will be cast as a rational extension of the TCS model.

## 3. Extended TCS model (ETCS)

We would like to find another system related to the TCS model, which is isospectral to the TCS and possibly, its wave functions are written in terms of EOPs. To find the rational extension of the TCS model, we start by adding a new interaction term, [25]

$$H_e = H + \frac{(\alpha_1 + \alpha_2\omega^2\rho^2)}{(\beta_1 + \beta_2\omega^2\rho^2)^2} = H + V_{new}, \tag{7}$$

where $\alpha_{1,2}$ and $\beta_{1,2}$ are unknown constants and will be fixed later. We would like to show that this $H_e$ will correspond to the rational extension of this TCS model for some specific values of the parameters $\alpha_{1,2}$ and $\beta_{1,2}$. Since $V_{new}$ depends only on radial coordinate, $\rho$, only the radial equation will be modified and angular equation will be the same as in the case of the TCS model. The radial part is obtained having the same substitution as in Eq. (4) in the earlier section for the Hamiltonian $H_e$

$$\Phi''_{ext}(\rho) + \big(N + 2s - 1 + \lambda r(2N - r - 1)\big)\frac{1}{\rho}\Phi'_{ext}(\rho) + 2\big(E - (\frac{1}{2}\omega^2\rho^2 + V_{new})\big)\Phi_{ext}(\rho) = 0, \tag{8}$$

with $P_s(\mathbf{x})$ satisfying the same generalised Laplace equation as in Eq. (3). Note that here, a prime on $\Phi_{ext}(\rho)$ indicates a derivative with respect to $\rho$.
We further substitute,

$$\Phi_{ext}(\rho) = f(\rho)\zeta(g(\rho)), \tag{9}$$

in Eq. 8 where $f(\rho)$ and $g(\rho)$ are two undermined functions and $\zeta(g)$ is a special function to obtain

$$\zeta''(g) + \left(\frac{2f'(\rho)}{f(\rho)g'(\rho)} + \frac{g''(\rho)}{g'(\rho)^2} + \frac{\tau}{\rho g'(\rho)}\right)\zeta'(g) + \frac{1}{g'(\rho)^2}\left(\frac{f''(\rho)}{f(\rho)} + \frac{\tau f'(\rho)}{\rho f(\rho)} + 2(E_{ext} - V_{ext})\right)\zeta(g) = 0, \tag{10}$$

where, $\tau = \big(N + 2s - 1 + \lambda r(2N - r - 1)\big)$ and $E_{ext}$ is exactly same as $E_n$ given in Eq. (5)

We now compare this differential equation satisfied by $\zeta(g(\rho))$ with the differential equation satisfied by the $X_1$ Laguerre polynomial $\hat{L}_n^{(\alpha)}(g)$

$$\hat{L}_n''^{(\alpha)}(g(\rho)) - \frac{(g-\alpha)(g+\alpha+1)}{g(g+\alpha)}\hat{L}_n'^{(\alpha)}(g(\rho)) + \frac{1}{g}\left(\frac{(g-\alpha)}{(g+\alpha)} + n - 1\right)\hat{L}_n^{(\alpha)}(g(\rho)) = 0; \quad n \geq 1, \tag{11}$$

to obtain (with $n \to n+1$,)

$$V_{ext} = \frac{1}{2}\omega^2\rho^2 + \frac{4\omega}{(2\omega\rho^2 + \tau - 1)} - \frac{8\omega(\tau - 1)}{(2\omega\rho^2 + \tau - 1)^2}, \tag{12}$$

and

$$f(\rho) \simeq (g'(\rho))^{-\frac{1}{2}}\rho^{-\frac{\alpha}{2}}\exp\left(\frac{1}{2}\int^g [-\frac{(g-\alpha)(g+\alpha+1)}{g(g+\alpha)}]dg\right). \tag{13}$$

for a given $g(\rho)$ as defined in the case of a conventional model

$$g(\rho) = \omega\rho^2; \quad \alpha = \frac{\tau}{2} - \frac{1}{2}. \tag{14}$$

The energy eigenvalues $E_{ext}$ for the new system with the potential in Eq. 12 turn out to be the same as that of the conventional TCS model as discussed in Section 2 and are given by Eq. (5). However, the corresponding eigenfunction $\Phi_{ext}(\rho)$ is completely different. Using $f(\rho)$ and replacing $\zeta(g) \to \hat{L}_{n+1}^{(\alpha)}(g)$ in Eq. (9), the expressions for the energy eigenfunctions are obtained in terms of $X_1$ exceptional orthogonal Laguerre polynomials ($\hat{L}_{n+1}^{(\alpha)}(g)$) as

$$\Phi_{ext}(\rho) \simeq \frac{\exp(-\frac{\omega\rho^2}{2})}{(2\omega\rho^2 + \alpha)}\hat{L}_{n+1}^{(\alpha)}(\omega\rho^2); \; n = 0, 1, 2, \ldots \tag{15}$$

Note that the $X_1$ Laguerre polynomial ($\hat{L}_{n+1}^{(\alpha)}(g)$) is related to the classical Laguerre polynomials by

$$\hat{L}_{n+1}^{(\alpha)}(g) = -(g + \alpha + 1)L_n^{(\alpha)}(g) + L_{n-1}^{(\alpha)}(g). \tag{16}$$

The constant parameters $\alpha_{1,2}$ and $\beta_{1,2}$ for which the Hamiltonian (7) is exactly solvable can easily be determined by comparing Eqs. (7) and (12), and one finds that

$$\begin{aligned} \alpha_1 &= -4\omega(\tau-1); \qquad \alpha_2 = 8, \\ \beta_1 &= \tau - 1; \quad \text{and} \quad \beta_2 = 2/\omega. \end{aligned} \tag{17}$$

In the special cases of $r = 1$ and $r = N - 1$, we then obtain the rational extension of the JK model and the CSM, respectively.

$X_m$ **case:**

Similar to the $X_1$ case, we redefine Eqs. (8) and (9) by replacing $\Phi_{ext}(\rho) \to \Phi_{m,ext}(\rho)$ and $f(\rho) \to f_m(\rho), \zeta(g) \to \zeta_m(g)$, respectively. Now the differential Eq. (8) will also be $m$ dependent and can be written as

$$\Phi''_{m,ext}(\rho) + \big(N + 2s - 1 + \lambda r(2N - r - 1)\big)\frac{1}{\rho}\Phi'_{m,ext}(\rho) + 2\big(E - (\frac{1}{2}\omega^2\rho^2 + V_{m,new})\big)\Phi_{m,ext}(\rho) = 0. \tag{18}$$

Now, we proceed with the steps as in the case of $X_1$, by substituting $\Phi_{m,ext}(\rho) = f_m(\rho)\zeta_m(g(\rho))$ in the above equation to obtain the differential equation for $\zeta_m(g)$, which is exactly same as in Eq. (10).Then, we compare that equation with the $X_m$ exceptional Laguerre differential equation

$$\hat{L}''^{(\alpha)}_{n,m}(g(\rho)) + Q_m(g)\hat{L}'^{(\alpha)}_{n,m}(g(\rho)) + R_m(g)\hat{L}^{(\alpha)}_{n,m}(g(\rho)) = 0, \tag{19}$$

with

$$\begin{aligned} Q_m(g) &= \frac{1}{g}\left[(\alpha + 1 - g) - 2g\frac{L^{(\alpha)}_{m-1}(-g)}{L^{(\alpha-1)}_m(-g)}\right] \\ \text{and} \quad R_m(g) &= \frac{1}{g}\left[n - 2\alpha\frac{L^{(\alpha)}_{m-1}(-g)}{L^{(\alpha)}_m(-g)}\right] \end{aligned} \tag{20}$$

to get (replacing $n$ by $n + m$)

$$V_{m,new} = -2\omega^2\rho^2\frac{L^{(\alpha+1)}_{m-2}(-g)}{L^{(\alpha-1)}_m(-g)} + 2\omega(\alpha + \omega\rho^2 - 1)\frac{L^{(\alpha)}_{m-1}(-g)}{L^{(\alpha-1)}_m(-g)} + 4\omega^2\rho^2\left(\frac{L^{(\alpha)}_{m-1}(-g)}{L^{(\alpha-1)}_m(-g)}\right)^2 - 2m\omega, \tag{21}$$

and

$$f(\rho) \simeq (g'(\rho))^{-\frac{1}{2}}\rho^{-\frac{\alpha}{2}}\exp\left(\frac{1}{2}\int^g Q_m(g)dg\right). \tag{22}$$

We note that the spectrum for the potential in Eq. (21) is exactly same as that for the potential in Eq. (12) and for a usual TCS system. However, the eigen functions in all three cases are different. For a usual TCS model, these are in terms of classical Laguerre polynomials, in case of the potential in Eq. (12), that is, in the $X_1$ case, these are in terms of $X_1$ Laguerre polynomials and finally the wave functions for the system with the potential in Eq. (21) are in terms of $X_m$ Laguerre polynomials. The wave functions for the system described by the potential in Eq. (21) are given by

$$\Phi_{m,ext}(\rho) \simeq \frac{\exp(-\frac{\omega\rho^2}{2})}{\hat{L}^{(\alpha-1)}_m(-\omega\rho^2)}\hat{L}^{(\alpha)}_{n+m}(\omega\rho^2); \quad n, m = 0, 1, 2, \ldots \tag{23}$$

where the $X_m$ Laguerre polynomial ($\hat{L}^{(\alpha)}_{n+m}(g)$) is related to the classical Laguerre polynomials by

$$\hat{L}^{(\alpha)}_{n+m}(g) = L^{(\alpha)}_m(-g)L^{(\alpha-1)}_n(g) + L^{(\alpha-1)}_m(-g)L^{(\alpha)}_{n-1}(g). \tag{24}$$

As expected, for $m = 1$, the above results reduce to the corresponding $X_1$-case, while for the $m = 0$ case one gets back the conventional TCS model. In the next section, we will consider a rational extension of Calogero like many particle systems.

## 4. QES many particle system

In this section, we would like to discuss a rational extension of Calogero like many particle systems [24]. We start with a many particle Calogero like Hamiltonian with an arbitrary potential $U(\sqrt{N}\rho)$, which depends only on the 'radial' coordinate $\rho$

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{i<j}^{N} \frac{\lambda}{(x_i - x_j)^2} + U(\sqrt{N}\rho);\ \lambda \geq -\frac{1}{2},\ \rho = \sqrt{\frac{1}{N}\sum_{i<j}^{N}(x_i - x_j)^2} \tag{25}$$

Our aim is to construct a possible structure of $U(\sqrt{N}\rho)$, for which the many particle model is exactly solvable. To this end, we follow the standard method [65–67] and take a trial wavefunction for $\psi(x)$ in a sector of the configuration space corresponding to a definite ordering of particles (e.g., $x_1 \geq x_2 \geq \cdots \geq x_N$) as

$$\psi(x) = \prod_{i<j}(x_i - x_j)^{a+\frac{1}{2}} P_{k,q}(x)\phi(\rho), \text{with } a = \frac{1}{2}\sqrt{1+2\lambda} \tag{26}$$

and obtain the radial part differential equation,

$$-[\phi''(\rho) + 2(k+b+1)\frac{1}{\rho}\phi'(\rho)] + [U(\sqrt{N}\rho) - E]\phi(\rho) = 0 \tag{27}$$

with $b = \frac{N(N-1)}{2}a + \frac{N(N+1)}{4} - 2$. The angular part is described by $P_{k,q}(x)$, which are translationally invariant, symmetric, $k$-th order homogeneous polynomials satisfying the differential equations

$$\sum_{j=1}^{N} \frac{\partial^2 P_{k,q}(x)}{\partial x_j^2} + \left(a + \frac{1}{2}\right) \sum_{j\neq k} \frac{1}{(x_j - x_k)} \left(\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_k}\right) P_{k,q}(x) = 0\,. \tag{28}$$

Note that the index $q$ in $P_{k,q}(x)$ can take any integral value ranging from 1 to $\lambda(N,k)$, where $\lambda(N,k)$ is the number of independent polynomials. The existence of such translationally invariant, symmetric and homogeneous polynomial solutions of (28) has been discussed in the original work by Calogero [65, 66].

For our purpose, we will look for a solution of the radial part and proceed with the substitution

$$\phi(\rho) = \rho^{-(l+1)}\chi(\rho), \quad \text{with } l = k + b \tag{29}$$

in Eq. (27) to obtain

$$-\frac{d^2}{d\rho^2}\chi(\rho) + U_k(\sqrt{N}\rho)\chi(\rho) = E\chi(\rho) \tag{30}$$

where $U_k(\sqrt{N}\rho)$ is k dependent (through l) and is written as

$$U_k(\sqrt{N}\rho) = \frac{l(l+1)}{\rho^2} + U(\sqrt{N}\rho)\,. \tag{31}$$

Our aim here is to find a solution of Eq. (30) with a possible general structure of $U(\sqrt{N}\rho)$. This will provide an exact solution of a many particle system given in Eq. (25). This can be done in various ways, but we would like to use the technique of SUSYQM, details of which can be found in [38, 39]

We consider a specific superpotential of the form [22]

$$W(\rho) = \rho + \frac{2g_1\rho}{1+g_1\rho^2} + \frac{\alpha+1}{\rho}\,, g_1(\alpha) = \frac{2}{2\alpha+3}\,.\quad \alpha \in \mathcal{R}^+ \tag{32}$$

for which one of the partner potentials is a radial oscillator. The partner potentials can be calculated as,

$$V_+(\rho) = \rho^2 + \frac{\alpha(\alpha+1)}{\rho^2} + 2\alpha + 7\,, \tag{33a}$$

$$V_-(\rho) = \rho^2 + \frac{(\alpha+1)(\alpha+2)}{\rho^2} - \frac{4g_1}{1+g_1\rho^2} + \frac{8g_1^2\rho^2}{(1+g_1\rho^2)^2} + 2\alpha + 5\,. \tag{33b}$$

The potential $V_+$ is the potential for a radial oscillator model with a constant term. The complete solution for this potential is given by

$$E_n^+ = 4\left(n + \alpha + \frac{5}{2}\right), \quad n = 0, 1, 2, \ldots \tag{34}$$

$$\chi_n^+(\rho) = \sqrt{\frac{n!}{\Gamma(n+\alpha+\frac{3}{2})}}\,\rho^{\alpha+1} e^{-\rho^2/2} L_n^{\alpha+1/2}(\rho^2)\,. \tag{35}$$

Where $L_n^{\alpha+1/2}(\rho^2)$ is a usual Laguerre polynomial. Using the results of SUSYQM, we can obtained the solution for the partner potential $V_-(\rho)$ given in Eq. (33b) as

$$E_n^- = 4\left(n + \alpha + \frac{5}{2}\right), \tag{36}$$

which is the same as in the case of $V_+$, the radial oscillator potential. The radial part of the wave function is written as

$$\chi_n^-(\rho) = \sqrt{\frac{4(n!)}{E_n^+\Gamma(n+\alpha+\frac{3}{2})}}\rho^{\alpha+2} e^{-\rho^2/2} \frac{1}{L_1^{\alpha+1/2}(-\rho^2)} \hat{L}_{n+1,1}^{\alpha+3/2}(\rho^2). \tag{37}$$

This solution for $V_-(\rho)$ is possible only when the parameter $g_1$ depends on $\alpha$ in a particular fashion and hence, the model with $V_-$ is conditionally exactly solvable. Note that $V_+(\rho)$ can be used to generate the exactly solvable Calogero model with a harmonic confining interaction as in this case

$$U(\sqrt{N}\rho) = U_k(\sqrt{N}\rho) - \frac{l(l+1)}{\rho^2} = \rho^2 + \frac{\alpha(\alpha+1) - l(l+1)}{\rho^2}, \tag{38}$$

apart from an overall constant term. Now, $l = k + b$ and $\alpha$ are free parameters and we can chose the parameter $\alpha = l$ such that $U(\sqrt{N}\rho) = \rho^2$ is independent of k. Calogero has shown that the many particle system (25) with $U(\sqrt{N}\rho) = \rho^2$ can be solved exactly for all possible values of $k$. However, unlike this case, $V_-(\rho)$ represents a QES many particle system as we explain below.

$$U(\sqrt{N}\rho) = \rho^2 + \frac{(\alpha+1)(\alpha+2) - l(l+1)}{\rho^2} - \frac{4g_1}{1+g_1\rho^2} + \frac{8g_1^2\rho^2}{(1+g_1\rho^2)^2} + 2\alpha + 5\,, \tag{39}$$

In this case, $\alpha$ depends on $g_1$ and hence can't be chosen as such that $U(\sqrt{N}\rho)$ is independent of k. Hence, the many particle system with $U(\sqrt{N}\rho)$ is QES as it is solvable only for a particular value of k. The eigenvalues are

$$E_n = 4\left(n + \alpha + \frac{5}{2}\right), \quad n = 0, 1, 2, \ldots. \tag{40}$$

and the corresponding (unnormalized) QES eigenfunctions in terms of $X_1$ Laguerre polynomials are written as

$$\psi_n(x) = \rho^{\alpha-l+1} e^{-\rho^2/2} \frac{1}{L_1^{\alpha+1/2}(-\rho^2)} \hat{L}_{n+1,1}^{\alpha+3/2}(\rho^2) \prod_{i<j}(x_i - x_j)^{a+\frac{1}{2}} P_{\tilde{k},q}(x)\,. \tag{41}$$

Note that the solution of the angular part $P_{\tilde{k},q}(x)$ is only for a specific value of the degree of the polynomial, i.e. for $k = \tilde{k}$. We can't find the solution of the radial part for $k \neq \tilde{k}$. Thus, the solution is not complete, we have obtained a part of it and thus, in this sense, we called the solutions QES. Now, we have obtained another potential given in Eq. (33b), which is isospectral to a radial oscillator potential but the wave functions are completely different and are written in terms of $X_1$ Laguerre polynomials. Thus, we have achieved the rational extension of Calogero like many particle system with a $U(\sqrt{N}\rho)$ given in Eq. (39).

We would like to point out that exactly same result can also be obtained using the other method, like PCT method. Furthermore, using the PCT approach, one can obtain the most general rationally extended radial oscillator potential

$$V_m(\rho) = \rho^2 + \frac{l(l+1)}{\rho^2} - \frac{4\rho^2 L_{m-2}^{l+3/2}(-\rho^2)}{L_m^{l-1/2}(-\rho^2)} + 2(2\rho^2 + 2l - 1)\frac{4\rho^2 L_{m-2}^{l+1/2}(-\rho^2)}{L_m^{l-1/2}(-\rho^2)} + 8\rho^2\left[\frac{4\rho^2 L_{m-2}^{l+3/2}(-\rho^2)}{L_m^{l-1/2}(-\rho^2)}\right]^2 - 4 \tag{42}$$

whose bound state solutions for the energy eigenvalues $E_n = (4n + 2l + 3)\quad n = 0, 1, 2, \dots$ and eigenfunctions are written in terms of $X_m$ exceptional Laguerre Polynomials as

$$\chi_{n,m}(\rho^2) = [\frac{(n-m)!}{(l+1/2+n)\Gamma(l+1/2+n-m)}]^{1/2} \frac{x^{l+1}\exp(-\rho^2/2)}{L_m^{l-1/2}(-\rho^2)} \hat{L}^{l+1/2}_{n+m,m}(\rho^2) \tag{43}$$

Where $\hat{L}^{l+1/2}_{n+m,m}(\rho^2)$ is $X_m$ exceptional orthogonal Laguerre polynomial, $m = 0$ corresponds to usual Laguerre polynomials. Now, we note that $m = 1$ corresponds to the case we discus, in the context of Calogero Model as the potential

$$V_1 = \rho^2 + \frac{l(l+1)}{\rho^2} - \frac{8}{2\rho^2 + 2l + 1} + \frac{32\rho^2}{(2\rho^2 + 2l + 1)^2} \tag{44}$$

calculated from Eq. (42) is the same as our $V_-(\rho)$ in Eq. (33b) when $l = (\alpha + 1)$ and $g_1 = \frac{2}{(2\alpha+3)}$ apart from an overall constant term. The solution obtained through the PCT approach in Eq. (43) for $m = 1$ is exactly the same as we obtained through the supersymmetric approach.

## 5. Conclusions

In this article, we have reviewed two of our old works [24, 25] on a rational extension of many particle systems. In the first model [25], we have considered the TCS model with pairwise 2-body and 3-body interactions, and using the well known PCT approach, we have extended the model rationally. The spectrum is isospectral to the original TCS system and the wave functions are written in terms of $X_m$ Laguerre polynomials. This means that we have a family of isospectral systems for $m = 1, 2, \dots$ related to the TCS model with different potentials. The eigen functions are different and are written in terms of EOPs. In the other model, we have considered the QES Calogero like many particle system, and using SUSYQM techniques, obtained the general structure of the QES potential for which we can find the QES solutions. The wave functions are written in terms of EOPs. We have considered broken SUSYQM in our approach. We feel it would be of interest to investigate similar many particle systems for which supersymmetry is unbroken.

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