The History of Mathematics as Scaffolding for Introducing Prospective Teachers into the Philosophy of Mathematics

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An Introductory Note

This paper claims that the awareness of crucial philosophical questions and controversies, which have arisen during the historical evolution of fundamental concepts, ideas and processes in mathematics, should be an essential component of the professional knowledge of student teachers who intend to teach children mathematics.

The starting point of this claim is the premise that the philosophy of mathematics may provide a unifying framework, which potentially supports an epistemological clarification of mathematical knowledge, as well as critical reflection on the beliefs and values about mathematics, that a teacher holds, which guide the practice of mathematics teaching, as relevant research has shown (Thompson, 1984, 1992).

Based upon the above perspective, a course has been designed and implemented in a Greek university. It uses themes from the history of mathematics as scaffolding for the introduction of prospective teachers to the questions and controversies raised by various philosophical approaches to mathematics throughout its historical development.

Reasons Why the Philosphy of Mathematics for Teaching Mathematics

In the project reported here, mathematics teaching is understood as a practice that intentionally works on students' learning of mathematics by paying attention to the following: firstly, the representation of mathematical knowledge; secondly, the students' mental processes of learning; and finally, the instructional media within which teachers and students interact (Cohen, 2011). This is exemplified by Herbst et.al. (2010), where in practice, the representation of mathematical knowledge includes teaching tasks where examples of mathematical ideas are selected and mathematical statements formulated; thus, providing mathematically persuasive explanations and choosing problems for students which promote the understanding of target mathematical concepts and more. Being aware of the students' mental processes of learning includes teaching techniques, for example, eliciting students' thinking, interpreting students' conceptions and identifying errors, etc., while simultaneously dealing with the instructional media and handling a number of diverse teaching skills and techniques associated with interpersonal dynamics and communication. This also includes handling the limitations and constraints of the institution where the teaching and learning activities are taking place.

The important question is, to discover the purpose of teaching mathematics in our schools, and the answer to this question, implicitly or explicitly, guides the work of the mathematics teacher. The *why* stems from the viewpoint of mathematics as a discipline and a subject to be learned, in addition to the perception of its role and purpose in society. Such points of view may be spontaneous and somewhat incoherent; however, they do, in fact, exist and affect the activities of mathematics teachers.

With reference to the above, it may be asserted that there is a direct association of a philosophy of mathematics adopted by mathematics teachers which will be combined with fundamental features of their teaching practices. Over forty years ago, Thom claimed that a philosophy of mathematics has powerful implications for educational practice, pointing out that: “In fact, whether one wishes it or not, all mathematical pedagogy, even if scarcely coherent, rests on a philosophy of mathematics” (Thom, 1973, p.204), and following this claim, Hersh emphasised that:

> One’s conception of what mathematics is affects one’s conception of how it should be presented. One’s manner of presenting it is an indication of what one believes to be most essential in it. ... The issue, then, is not, what is the best way to teach? But, what is mathematics really about? ...” (Hersh, 1979, p.33).

Nowadays, it is widely accepted, both on a philosophical level (Lerman, 1983, 1990; Steiner, 1987; Ernest, 1989) and on the basis of empirical evidence, (Thompson, 1984, 1992) that teachers’ assumptions about mathematics and its learning and teaching, implicitly reflect, or are related to, a philosophy of mathematics. In their turn, these assumptions play a significant role in shaping characteristic patterns of their teaching practices. Firstly, they include the origins of mathematical knowledge and the nature of this knowledge as a discipline. This is followed by the nature of mathematical problems and tasks and the relationships between mathematical knowledge and empirical reality, particularly in terms of the applicability and utility of mathematical knowledge and its nature as a subject taught in schools (Törner, 1996). Added to this is the notion of the teacher as a learner and user of mathematics and more generally, opinions about the process of learning mathematics (Ernest, 1989; Pehkonen, 1994).

Therefore, what teachers believe about mathematics reflects, or is related to, a philosophy of mathematics and, in fact, constitutes a kind of a *practical philosophy of mathematics*. This, as a complex, practically-oriented set of understandings, regulates and shapes to a great extent the teachers’ thoughts and practices within the classroom; it is however, subject to the constraints and contingencies of the school context. Moreover, the teachers’ *practical philosophy of mathematics* often takes precedence over knowledge, shaping the interpretation of their received knowledge and selectively admitting or rejecting new knowledge.

Consequently, in assuming a connection between both a philosophy of mathematics and mathematics teaching practices, as well as a philosophy of mathematics and teachers’ philosophical and epistemological view of mathematics, the philosophy of mathematics per se would have to be considered as an essential component of teachers’ preparation in order to introduce children to the culture of mathematics even in early childhood education.

The aim is to enable teachers to develop a questioning stance towards dominant canons of mathematics education and lead them to be able to reflect critically on their personal didactical practices. It is believed that this will increase their professional autonomy in teaching mathematics and consequently is a vitally important part of a teachers’ preparation. Also of great importance is the necessity to create a teachers’ preparation which aims to support them, so that they can become reflective practitioners. It is equally essential that these teachers play an important role in the definition of the purposes and goals of their work, as well as the means to attain them, that will result in their participation in the production of knowledge about teaching mathematics, the possession of which, would place teachers in a position where they could critically assess the established standards.

The Subject of the Philosophy of Mathematics

Ernest (1991, 1998) described two contrasting philosophical views of mathematics, Absolutism and Fallibilism, which, although somewhat simplistic, are useful in our discussion of introducing the philosophy of mathematics into teachers' preparation courses.

The first is an absolutist view of mathematics, which stems from both Platonist and Formalist philosophies; these consider mathematics to be a consistent body of knowledge without errors or contradictions and are expressed in a formalized language. These philosophies regard mathematics as infallible, due either to its existence beyond humanity just waiting to be discovered (Platonist school), or due to its creation as a logical, closed set of rules and procedures (Formalist school).

The second view of mathematics originates from a fallibilistic philosophy or a philosophy of humanism, as Hersh, (1997) termed it, which regards mathematics as a human construction, consequently fallible and corrigible. It is this philosophical view of mathematics which has moved away from the dogmas of the traditional, foundationalist schools of formalism, logicism and intuitionism, each of which sought to establish the universal and absolute validity of mathematical knowledge, setting its epistemological status above all other forms of knowledge. This perspective has included the history and practice of mathematics in the topics studied by philosophy of mathematics. Lakatos, one of the founders of this philosophy of mathematics, stressed the necessity of a close relationship between the history and philosophy of mathematics, claiming that "The history of mathematics, lacking the guidance of philosophy has become blind, while the philosophy of mathematics turning its back on the...history of mathematics, has become empty" (1976, p. 2).

Following the same direction, Ernest (1998) builds on the need to consider the relationships between mathematics and its corporeal agents, i.e. humans, listing the following minimum number of aspects of mathematical knowledge and practices that a modern philosophy of mathematics should account for:

1. Mathematical knowledge; its character, genesis and justification, with special attention to the role of proof.

2. Mathematical theories, both constructive and structural; their character and development, and issues of appraisal and evaluation.

3. The objects of mathematics; their character, origins and relationship with the language of mathematics.

4. Mathematical practice; its character, and the mathematical activities of mathematicians, in the present and past.

5. Applications of mathematics; its relationship with science, technology, other areas of knowledge and values.

6. The learning of mathematics: its character and role in the onward transmission of mathematical knowledge, and in the creativity of individual mathematicians. (pp. 56-57)

Ernest stated that:

criteria 1 and 3 include the traditional epistemological and ontological focuses of the philosophy of mathematics but add a concern with the genesis of both mathematical knowledge and the objects of mathematics, as well as with the language in which mathematical knowledge is expressed and mathematical objects named. Criterion 2 adds a concern with the form that mathematical knowledge usually takes: that is in mathematical theories. It allows as admissible the notion that theories evolve over time and can be appraised. The discussion of standards and theory

> appraisal is evidently a matter for the philosophy of mathematics, even if theory choice is not the same kind of issue as it is in philosophy of science. This criterion also indicates the dual nature of mathematical theories and concepts, which can be either constructive or structural, and this is a philosophically significant distinction. Criteria 4 and 5 go beyond the traditional boundaries of the philosophy of mathematics by admitting the application of mathematics and human mathematical practice as legitimate philosophical concerns, as well as its relations with other areas of human knowledge. They unambiguously admit social aspects of mathematics as legitimate areas of philosophical enquiry. Criterion 6 adds a concern with how mathematics is transmitted onward from one generation to the next and, in particular, how it is learnt by individuals and the dialectical relation between individuals and existing knowledge in creativity. (p. 57)

Aligned with the two contrasting philosophical views of mathematics mentioned above, are ideas about mathematics teaching. In the analysis of Threlfall (1996), the absolutist view of mathematics is usually associated with a behaviorist approach, utilizing drills and the practice of discrete skills, individual activities, and an emphasis on procedures, while the fallibilist view of mathematics aligns itself with pedagogy consistent with constructivist theories, utilizing problem-based learning, real world application, and collaborative learning, with the emphasis on the process and not on product.

However, although there have been numerous calls to change and adapt the teaching of mathematics through the adoption of a constructivist epistemology, little has been done to challenge teachers' conceptions of mathematics but as Ernest (1989) has underlined, "Teaching reforms cannot take place unless teachers' deeply held beliefs about mathematics and its teaching and learning change" (p. 249).

The History vs. the Philosphy of Mathematics

The history and philosophy of mathematics are academic disciplines which have different objectives, conceptual structures, interests and methods and they perceive the world of mathematics from a different point of view. Some of their differences are not important when attempting to interweave themes from the one, with questions raised by the other, in a course designed for student teachers who intend to teach mathematics in schools; however, others are crucial (Aspray & Kitcher, 1988). For example, history deals with the particular and the temporal, while philosophy deals with the universal and a-temporal; the first focuses on selected episodes from the past, while the latter is concerned with universality building upon abstractness.

In actual fact, the connection between history and philosophy is a complex, dynamic and purpose-sensitive relationship. History employs a variety of practices, and philosophy is so heterogeneous that it is extremely difficult to draw simple and uniform links between them. At the same time, any connections drawn are dependent on the approach adopted in drawing them. For this reason, the history and philosophy of mathematics may be related in different ways for different purposes.

In our project and for our purposes, we have found such a relationship based on the following premises:

- The philosophy of mathematics involves a temporal dimension. As exemplified by Lakatos, in his *Proofs and Refutations*, he stated that mathematical definitions, proofs and theories are not created at the outset in their full format, ready to meet the requirements of mathematical research and development. As a consequence, the concept of truth in mathematics is actually conditional and domain-specific and as such, this approach has significant implications for questions posed and answered by the philosophy of mathematical practice.

- The philosophy of mathematics studies, mathematical procedures, representation, and ontology in comparison with the history of mathematics which studies changes in mathematical procedures, representation, and ontology. Explanations of changes presented by history provide philosophy the grounds for testing its general hypotheses and theories and for interpreting their transformations over time.

– Mathematical ideas are not developed in isolation from their wider intellectual context or the institutional factors regulating the mathematical practice. Mathematical definitions, proofs and theories often develop in response to practical problems and these practicalities condition them for as long as mathematics continues to develop. As a result, many questions related to the epistemology of mathematics may be answered by an inquiry which transcends the boundaries of the strictly mathematical or even intellectual fields. Adopting the above mentioned view of the connection between the history and the philosophy of mathematics creates the need for a key question to be answered: which of the two disciplines will provide the template for the syllabus of a course aiming to introduce students to the philosophy of mathematics?

A series of in-depth interviews with students, during the planning of the course being reported on here, showed that the history of mathematics was more familiar and a more appealing subject matter to the students than the philosophy of mathematics. The latter was referred to as a field dealing with abstract, incomprehensible theories about mathematical knowledge, focused on the 'logic' of mathematics or with the formal logic of mathematical thinking. Thus, the history of mathematics was chosen to be used as scaffolding for the introduction of prospective teachers to the questions and controversies raised by various philosophical approaches to mathematics in the course of its development.

The Course

With this background, the introduction of any aspects of the philosophy of mathematics to a course offered to student teachers, has to meet additional prerequisites. Firstly, the selected topics from the history and philosophy of mathematics should have a clear relevance to the topics of school mathematics, in order to be a crucial motivating factor for teachers. Secondly, these issues of the philosophy of mathematics ought to introduce thought provoking questions related, directly or indirectly, to the mathematics taught in schools, thereby functioning as a catalyst in attracting teachers' interest and involvement in the philosophy of mathematics. In addition, a preliminary study has produced evidence of student teachers' poor mathematical background and rather narrow view of mathematics as a discipline, inherited from their own school experience, and this needs to be considered carefully.

Taking into account the above outlined issues, the course reported here has been developed and implemented on the following rationale.

1. As a first step, themes from the history of mathematics were selected for the syllabus of the course. The criterion for the choice was based, both on the crucial role played in the development of mathematics and its philosophy and on the need to incorporate into these themes the fundamental concepts, processes and application of mathematics, which are included in the curricula of pre-school and primary school mathematics.

2. The second step, selected issues from the philosophy of mathematics and integrated them into each theme which was introduced by challenging and provocative questions aiming to create dissonance within the group of teachers; thus, stimulating them to be actively involved in discussions about learning and reflective thinking about mathematics, e.g., how it is taught and learnt. So for every issue or theme an attempt was made to include questions that would provoke discus-

sion which would meet the previously mentioned criteria for a modern philosophy of mathematics.

The organising concepts of this rationale are “themes” and “thought provoking questions.” By “themes” here is meant, collections of learning experiences from the history of mathematics, that assist students to relate their learning to questions that are important and meaningful for them, as well as being clearly linked to practice (Freeman & Sokoloff, 1996). Themes are the means with which to organize the philosophy of mathematics content, which is presented and discussed by using questions that are meaningful to the teachers as starting-points, and they are intended to give meaning and direction to the reflection and learning process (Perfetti & Goldman, 1975). This thematic approach seems to provide an environment where knowledge can be individually and socially constructed, so it may be considered to be associated with constructivist ideas of knowing.

“Thought provoking questions” are those which create discordant situations for the teachers and thus motivate them to be actively involved in approaching the philosophy of mathematics; they are in fact, questions which create perplexity, challenge beliefs, point out questionable issues and potentially foster a conceptual reconstruction of mathematical knowledge and pedagogy of the teachers.

As a result, the following twelve teaching units were developed using historical themes as scaffolding for introducing key questions and crucial controversies from the philosophy of mathematics; challenging and provocative questions were also provided for each theme:

1. Numbers and numerals: from the Ishango bone to the Arithmetic of Stevin.

- *Are numbers: objects or properties?*

- *What differences and similarities can you point out in the following definitions of number (Definition of number given by Pythagoreans, Cantor, Peano and Frege are offered)?*

- *Do different number definitions impact on any differences in teaching number concept?*

- *Why were the various numeration systems invented and used over time and across cultures?*

2. Discrete and continuous quantities: from Zeno’s paradoxes to Dedekind’s cuts.

- *How is the difference between classes and sets conceived?*

- *What happens when ordering rational numbers?*

3. Zero and infinity

- *If zero is the representation of nothing, then nothing must mean something because it is being represented, is this correct?*

- *Is any collection of objects sharing a common characteristic a set?*

- *What is the relation between a whole and one of its parts and may it be divided?*

- *Could you give an example of an infinite set?*

- *What is the difference between an ordinal number and a cardinal number?*

- *Have you ever heard about the continuum hypothesis? If not, could you make a guess based on your understanding of the term "continuum."*

4. Equality and equation: Algebra from Diophantus, to Cardano and then to Dr. Noether.

- *May mathematical statements be understood in the same way as ordinary statements? For instance, statements of equality or equation?*

5. Euclidean geometry and non-Euclidean geometries: from the ancient Egypt surveyors to Lobachevski's creation.

- *Can you see any difference in the statements, "Between any two points one can draw a straight line" (Euclid) and "Between any two points there is a straight line" (Hilbert)?*

- *Why is the Golden Ratio so prevalent in nature?*

- *How many dimensions have a curved plane?*

- *Are mathematical concepts and truths discovered or invented?*

6. Existence and construction in mathematics.

- *Do mathematical objects (sets, numbers, lines, functions, circles, etc.) actually exist?*

- *Are there mathematical properties which a mathematical object might have only contingently?*

- *Does the existence of a Mobius strip prove that mathematical objects are true?*

- *How can we rationalize the fact that complex numbers exist?*

7. Geometry, arithmetic and their mutual relationships.

- *Newton declared that "geometry is founded on mechanical practice." What could he have meant?*

- *Descartes postulated that "geometrical" curves are "those which are amenable to precise and exact measurement." Compare this statement with the Euclidean concept of a curve.*

8. Randomness, probability and finally statistics.

- *Is there any difference between "randomness" and "anything can happen"?*

- *What does "probable" mean? Can every statement be assigned a numerical probability related to the given evidence?*

- *Can we assign probability to a person's rational act?*

9. Justification and proof in mathematics

- *What truth is expressed by a mathematical statement?*

- *Is a truth in mathematics about objects, concepts or neither?*

- *From where is the indispensability of a mathematical argument derived?*

10. Axiomatic systems and the foundations of mathematics: from Euclid to the foundational schools of the 19th century.

- *What do mathematical statements mean?*

- *What is the organization of mathematics concepts in axiomatic systems?*

- *What differences and similarities can you identify in Euclid's axioms for geometry and in Peano's axioms for natural numbers?*

- *What do the axiomatic organization of mathematics concepts mean for their relation to real world situations?*

- *Have you ever heard the terms "logicism," "formalism" or "intuitionism"? What do you think they mean in mathematics?*

- *What are the foundations of mathematics?*

11. The story continued: the foundations of mathematics and the strange Dr. Gödel.

- *Is the view that "mathematics is a game with signs" tenable or not?*

12 …What shall we do with mathematics now Dr. Lakatos?

- *Do you think that mathematics has any similarities to physics?*

In the above course of teaching units, there may be the possibility that a fragmented introduction to the philosophy of mathematics could lead students to confusion and misunderstanding of some issues concerning the philosophical views of mathematics. This is actually a risk underlying the implementation of this course. However, it is compensated by the power to raise and to keep alive the interest of the participants through the variety of presented and discussed topics. Briefly, this course suffers from the lack of a comprehensive introduction to the philosophy of mathematics but has the advantage of offering an anthology of the philosophy of mathematical issues.

Feedback and Evaluation of the Course

The course outlined above has been running for the last four years in the Faculty of Early Childhood Education at the University of Athens in Greece. After the completion of the course the students who have participated are asked to write a short anonymous report evaluating the main aspects of the course. They are requested to describe the most important benefits that they had from following the course, and identify the main obstacles which they encountered during the lectures and in participating in the discussions taking place as part of the course. All the reports were carefully examined by a team, so that the evaluative comments made by the students could be clearly elicited from their writing and the resulting data could be thoroughly analyzed.

From the analysis of the evidence collected over the four years of the implementation of this course the following conclusions were drawn, here briefly presented:

(a) Most of the students pointed out that the main benefit from attending the course was the actual conceptual reconstruction they engaged in, which was vaguely described as a "change of mind about mathematics," a "change of my view of mathematics" or a "change of my attitudes towards mathematics." In every case, these "changes" which can be interpreted to mean a re-conceptualization of the nature of mathematics leading them to look at the discipline from a position different to their own previously held perspective. It was shown in their reports that the perspective of mathematics that is "new" for them is that of a critical approach towards the dominant images of mathematics in schools and the teaching practices they have experienced. This critical approach has mostly emanated from the possibilities of questioning what has been taken for granted in school mathematics. Therefore this "change" appears to have been brought about by their introduction to and involvement in the philosophy of mathematics.

(b) The issues from the philosophy of mathematics that were obviously appreciated by the teachers and which seem to have the greatest effect on their thinking were those which offered explicit opportunities for challenging their conceptions about teaching particular concepts and processes of school mathematics (e.g. numbers or proofs) or elements of their teaching models (e.g. the use of manipulatives as embodiments of mathematical concepts or the employment of diagrams as depictions of mathematical relations).

(c) The most serious problems encountered during the implementation of the course were created as a result of the poor mathematical backgrounds of the students, which is mainly attributable to secondary school mathematics. The knowledge deficits about particular mathematical concepts, processes and theorems did not permit them to comprehend specific questions and ideas from the philosophy of mathematics and delimited the extent, depth and quality of relevant discussions. Infinity and continuity issues being the most characteristic examples. Therefore, the outcome of this course, and I believe any course concerning the philosophy of mathematics, will be highly dependent on students' mathematical backgrounds.

A final, highly relevant issue from the instructor who has been running this course for three years, is that of the demands, in terms of time and effort, placed on the tutor both for the preparation, and even more for the managing, of each session of this course. I suppose that this would also apply to similar courses that attempt to introduce student teachers to the philosophy of mathematics while having to meet other pedagogical goals.

Concluding Comments

The growing philosophical investigations of mathematics in the past thirty years (see, e.g., Davis & Hersh, 1981; Hersh, 1997; Restivo, Van Bendegen, & Fischer, 1993; Tymoczko, 1986) have not been properly addressed in mathematics education. Even at present, rationales and practical proposals in mathematics education are mainly informed by educational and psychological research, disregarding ideas coming from disciplines that study the nature of mathematics, as do philosophy and history of mathematics.

In this paper, I have presented an attempt to use themes from the history of mathematics as scaffolding for the introduction of prospective teachers to the questions and controversies arisen by the philosophy of mathematics, claiming that an introduction to the philosophy of mathematics will support student teachers to become more reflective practitioners. On the other hand, it must be recognized that no single course can resolve all the issues associated with reflective thinking and the effectiveness of prospective teachers. However, if nothing else, it seems likely that if student teachers are never asked to reflect on the philosophical basis of their perceptions of mathematics, then they will continue to produce traditional, and mostly ineffective, teaching practices and remain resist to change.

References

Aspray, W. and P. Kitcher (1988). *History and philosophy of modern mathematics*. Minneapolis: University of Min nesota Press.

Cohen, D. (2011). *Teaching: Practice and its predicaments*. Cambridge, MA: Harvard University Press.

Davis, P. J., & Hersh, R. (1981). *The mathematical experience*. Boston: Birkhauser.

Ernest P. (1998). *Social Constructivism as a Philosophy of Mathematics*. Albany: State University of New York Press.

Ernest, P. (1991). *The philosophy of mathematics education*. London: Falmer.

Ernest, P. (1989). The Impact of Beliefs on the Teaching of Mathematics. In P. Ernest (ed.), *Mathematics Teach ing: The State of the Art* (pp. 249-254). London: Falmer Press.

Freeman, C., & Sokoloff, H.J. (1996). Toward a theory of theory of thematic curricula: Constructing new learning environments for teachers & learners. *Education Policy Analysis Archives*, 3(14), 1-19.

Herbst, P., Bieda, K., Chazal D. & González, G. (2010). *Representations of mathematics teaching and their use in teacher education: What do we need in a pedagogy for the 21st century?* Retrieved from http://hdl.handle.net/2027.42/78158 (25-04-2013).

Hersh, R. (1997). *What is mathematics, really?* New York: Oxford University Press.

Hersh, R. (1979). Some proposals for revising the philosophy of mathematics. *Advances in Mathematics*, 31(1), 31-50.

Lakatos, Imre (1976). *Proofs and Refutations*. Cambridge: Cambridge University Press

Lerman, S. (1983). Problem solving or knowledge centred: The influence of philosophy on mathematics teach ing. *International Journal of Mathematics Education in Science and Technology*, 14(1), 59–66.

Lerman, S. (1990). Alternative perspectives of the nature of mathematics and their influence on the teaching of mathematics. *British Educational Research Journal* 16(1), 53–61.

Pehkonen, E. (1994). On teachers' beliefs and changing mathematics teaching. *Journal für Mathematik-Didaktik*, 15(3/4), 177-209.

Perfetti, C.A., & Goldman, S.R. (1975), Discourse functions of thematization and topicalization, *Journal of Psycholinguistic Research*, 4(3), 257-271.

Restivo, S., Van Bendegen, J. P., & Fischer, R. (Eds.) (1993). *Math worlds: Philosophical and social studies of mathematics and mathematics education.* Albany, NY: State University of New York Press.

Steiner, H.-G. (1987). Philosophical and epistemological aspects of mathematics and their interaction with theory and practice in mathematics education. *For the Learning of Mathematics* 7(1), 7–13.

Thom, R. (1973). Modern mathematics: does it exist? In A.G. Howson (Ed.), *Developments in Mathematics Education: Proceedings of the Second International Congress on Mathematics Education* (pp. 194-209). Cambridge: Cambridge University Press.

Thompson, A. (1984). The relationship of teachers' conceptions of mathematics teaching to instructional practice. *Educational Studies in Mathematics*, 15, 105-127.

Thompson, A. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp.127-146). New York: Macmillan.

Törner, G. (1996). Views of German mathematics teachers on mathematics. In H.-G. Weigand, et. al. (Eds.), *Developments in mathematics education in Germany. Selected Papers from the Annual Conference on Didactics of Mathematics, Regensburg 1996* (pp. 116-129). Hildesheim: Franzbecker.

Tymoczko T. (1986). *New directions in the Philosophy of Mathematics.* Boston: Brikhauser.

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