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ORIGINAL ARTICLE

Some properties of the Blumberg's hyper-log-logistic curve

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***Abstract*—The paper considers the sigmoid function defined through the hyper–log–logistic model introduced by Blumberg. We study the Hausdorff distance of this sigmoid to the Heaviside function, which characterises the shape of switching from 0 to 1. Estimates of the Hausdorff distance in terms of the intrinsic growth rate are derived. We construct a family of recurrence generated sigmoidal functions based on the hyper–log–logistic function. Numerical illustrations are provided.**

***Keywords*-Hyper–log–logistic model, Heaviside function, Hausdorff distance, upper and lower bounds**

Mathematics Subject Classifications (2010)
41A46; 68N30

I. Introduction

The logistic function belongs to the important class of smooth sigmoidal functions arising from population and cell growth models. *The logistic function* was introduced by Pierre François Verhulst [1]–[3], who applied it to human population dynamics. Verhulst proposed his logistic equation to describe the mechanism of the self-limiting growth of a biological population.

A number of models have been proposed to provide growth curve from 0 to 1 (or to some carrying capacity) of different shape, e.g. Gompertz [4], Pearl [5], Von Bertalanffy [6], Richards [7], Nelder [8], Blumberg [9], Turner and al. [10], Schnute [11], Tsoularis [12], Tsoularis and Wallace [13].

Analysis of continuous growth models in terms

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of generalized logarithm and exponential functions can be found in [14]. A very good kinetic interpretation of Log–logistic dose–time response curves is given in [15] (see also [16]).

In artificial neural networks, [17], the sigmoid functions are used as activation or transfer function between two states, usually 0 and 1.

In all of their application, the shape of the sigmoid functions is essential factor determining the properties of the underlying biological, chemical or artificial system. An important characteristic related to the shape of a sigmoid is how far it deviates from the Heaviside function, also referred to as step-function, binary switch, or binary activation depending on the context. As shown in [18]-[19], an appropriate measure of this deviation is the Hausdorff distance of the sigmoid to the interval Heaviside function. Some approximation and modelling aspects are discussed in [20]–[23]. In this paper we discuss the Hausdorff distance of the hyper–log–logistic sigmoid curve to the interval Heaviside function.

II. The Blumberg hyper–log–logistic model

In 1968 Blumberg [9] introduced a modified Verhulst logistic equation, the so called *hyper–log–logistic equation*:

$$\frac{dN(t)}{dt} = kN^{\alpha}(1-N)^{\gamma}, \qquad (1)$$

where k is the rate constant and α and γ are shape parameters. The equation (1) is consistent with the Verhulst logistic model when $\alpha = \gamma = 1$.

We will consider the following modification of the hyper–log–logistic equation (1) (see for instance [12]:

$$\frac{dN(t)}{dt} = kN^{1-\frac{1}{\beta}}(1-N)^{1+\frac{1}{\beta}} \qquad (2)$$

where β is a shape parameter. For $\beta \to \infty$ the equation (2) reduces to the Verhulst equation.

The equation (2), in essence, provides parametric interpolation between the logistic equation ($\beta \to \infty$) and second order kinetics ($\beta = 1$).

An explicit form of the solution is derived as follows.

Let the function $N(t)$ be defined by the following nonlinear equation:

$$\left(\frac{N}{1-N}\right)^{\frac{1}{\beta}} = 1 + \frac{kt}{\beta}. \qquad (3)$$

After differentiation of both sides of Eq. (3), we have

$$\frac{1}{\beta}\left(\frac{N}{1-N}\right)^{\frac{1}{\beta}-1}\frac{N'(1-N)+NN'}{(1-N)^2} = \frac{k}{\beta}.$$

From here it follows that

$$N' = kN^{1-\frac{1}{\beta}}(1-N)^{1+\frac{1}{\beta}}$$

and, therefore, the function $N(t)$ satisfies the hyper–log–logistic differential equation (2).

The equation (3) can be rewritten as:

$$N(t) = 1 - \frac{1}{1+\left(1+\frac{kt}{\beta}\right)^{\beta}}. \qquad (4)$$

Further, we see that

$$N(0) = \frac{1}{2}. \qquad (5)$$

Since equation (2) satisfies the conditions for local existence and uniqueness while $N > 0$, the function $N(t)$ given in (4) is a unique solution of equation (2) satisfying the condition (5). The function is defined on $\left[\frac{\beta}{k}, +\infty\right)$. The definition can be extended in a unique way on the rest of the t-axis as zero.

III. Preliminaries

As stated in the Introduction, our main interest is the Hausdorff distance from the hyper–log–logistic function in (4) to the interval Heaviside function. We recall here the relevant definitions.

Definition 1. *The interval Heaviside function is defined as [24]:*

$$h(t) = \begin{cases} 0, & \textit{if} \quad t < 0, \\ [0,1], & \textit{if} \quad t = 0, \\ 1, & \textit{if} \quad t > 0. \end{cases} \qquad (6)$$

Definition 2. *[28] The one–sided Hausdorff distance $\overrightarrow{\rho}(f,g)$ between two interval functions f,g on $\Omega \subseteq \mathbb{R}$, is the one–sided Hausdorff distance between their completed graphs $\mathcal{F}(f)$ and $\mathcal{F}(g)$ considered as closed subsets of $\Omega \times \mathbb{R}$. More precisely,*

$$\overrightarrow{\rho}(f,g) = \sup_{B\in\mathcal{F}(g)} \inf_{A\in\mathcal{F}(f)} ||A-B||,$$

where $||\cdot||$ is a norm in $\mathbb{R}^2$.

We recall that completed graph of an interval function f is the closure of the graph of f as a subset of $\Omega\times\mathbb{R}$. If the graph of an interval function f equals $\mathcal{F}(f)$, then the f is called S-continuous. The Hausdorff distance

$$\rho(f,g) = \max\{\overrightarrow{\rho}(f,g), \overrightarrow{\rho}(g,f)\}$$

defines a metric in the set of all S-continuous interval functions. The topological and algebric structure of the space of S-continuous functions and its subspaces is studied in [24]–[27].

In this paper we apply only the concept of the one–sided Hausdorff distance.

IV. Main Results

Our main interest is characterizing the shape of N as a switching curve from 0 to 1. To this end, we use as a characteristic the one–sided Hausdorf distance from N to h as in [19].

The following theorem gives upper and lower bounds for $\overrightarrow{\rho}(N,h)$.

Theorem 3. *The one–sided Hausdorff distance $\overrightarrow{\rho}(N,h)$ from the function N given in (4) to the Heaviside function h given in (6) satisfies the following inequalities for $k>0$:*

$$d_l := \frac{1}{2+k} < \overrightarrow{\rho}(N,h) < \frac{1}{1+\sqrt{1+k}} =: d_r. \quad (7)$$

Proof: First we consider the interval $[0,+\infty)$. Taking into account the sigmoid shape of the function $N(t)$ in (4), the one–sided Hausdorff distance from N to the Heaviside function h on the interval $[0,+\infty)$ is a root of the equation

$$N(t) = 1-t,$$

or, equivalently,

$$F(t) := \left(1+\frac{kt}{\beta}\right)^{\beta} + 1 - \frac{1}{t} = 0. \quad (8)$$

Clearly, F is an increasing function of $t \in [0,+\infty)$. Hence, if (8) has a root, then it is unique. We use the well-known inequalities

$$1+\alpha < \left(1+\frac{\alpha}{x}\right)^{x} < e^{\alpha}, \quad (9)$$

where $\alpha \in \mathbb{R}$, $x>1$ and $\alpha + x > 0$. Using the first inequality in (9) we have

$$F(t) > 1+kt+1-\frac{1}{t} = \frac{kt^2+2t-1}{t}$$

The positive root of the quadratic in the numerator is

$$\frac{-1+\sqrt{1+k}}{k} = \frac{1}{1+\sqrt{k+1}} = d_r.$$

Then

$$F(d_r) > 0. \quad (10)$$

Using the second inequality in (9) we have

$$F(t) < e^{kt} + 1 - \frac{1}{t}.$$

Hence,

$$\begin{aligned} F(d_l) &= F\left(\frac{1}{k+2}\right) < e^{\frac{k}{k+2}} + 1 - k - 2 \\ &< (k+1)\left(\frac{e^{1-\frac{2}{k+2}}}{k+1} - 1\right). \end{aligned}$$

For the derivative of

$$\varphi(k) = \frac{e^{1-\frac{2}{k+2}}}{k+1} - 1$$

we have

$$\begin{aligned} \varphi'(k) &= e^{1-\frac{2}{k+2}} \frac{2}{(k+2)^2}\frac{1}{k+1} - \frac{1}{(k+1)^2}e^{1-\frac{2}{k+2}} \\ &= -\frac{k^2+2}{(k+1)^2(k+2)^2}e^{1-\frac{2}{k+2}} < 0. \end{aligned}$$

Therefore φ is a decreasing function of k. Using that $k>0$ we have

$$F(d_l) < (k+1)\varphi(k) < (k+1)\varphi(0) = 0. \quad (11)$$

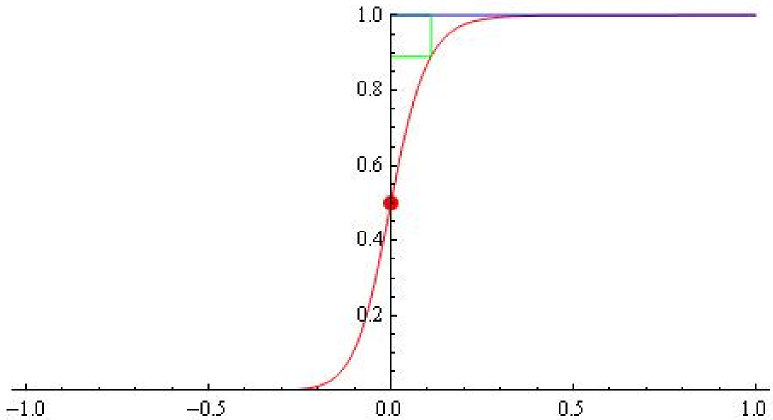

Fig. 1. The model (5) for $\beta = 21$, $k = 20$; H–distance = 0.109948, $d_l = 0.0454545$, $d_r = 0.179129$.

Since F is an increasing function, the inequalities (10) and (11) imply that (8) has a unique root in the interval (d_l, d_r).

Secondly, we consider the interval $(-\infty, 0]$. Similarly to the interval $[0, +\infty)$, using the shape of the sigmoid, the Hausdorff distance from N to h is a root of the equation

$$N(-\theta) = \theta,$$

or, equivalently,

$$G(\theta) := \left(1 - \frac{k\theta}{\beta}\right)^{\beta} + 1 - \frac{1}{1-\theta} = 0. \qquad (12)$$

Clearly, G is a decreasing function of $\theta \in [0, \min\{\frac{\beta}{k}, 1\}]$. Hence, if (12) has a root, then it is unique. Using the first inequality in (9) we have

$$G(\theta) > 2 - k\theta - \frac{1}{1-\theta}.$$

Then

$$G(d_l) = G\left(\frac{1}{k+2}\right) > 2 - \frac{k}{k+2} - \frac{k+2}{k+1} = \frac{k}{(k+1)(k+2)} > 0. \qquad (13)$$

Using the second inequality in (9) we have

$$G(\theta) < e^{-k\theta} + 1 - \frac{1}{1-\theta}.$$

Then

$$G(d_r) = G\left(\frac{1}{1+\sqrt{1+k}}\right) < e^{-\frac{k}{1+\sqrt{1+k}}} + 1 - \frac{1+\sqrt{1+k}}{\sqrt{1+k}} = \frac{1}{\sqrt{1+k}}\left(\sqrt{1+k}e^{1-\sqrt{1+k}} - 1\right) \qquad (14)$$

It is easy to see that the function

$$\phi(k) = \sqrt{1+k}e^{1-\sqrt{1+k}} - 1$$

is decreasing. Indeed,

$$\phi'(k) = \frac{1}{2\sqrt{1+k}}e^{1-\sqrt{1+k}} - \sqrt{1+k}\frac{1}{2\sqrt{1+k}}e^{1-\sqrt{1+k}} = \frac{1}{2\sqrt{1+k}}e^{1-\sqrt{1+k}}(1-\sqrt{1+k}) < 0.$$

Hence, $G(d_r)$ in (14) is also e decreasing function of k. Using that $k > 0$ we have

$$G(d_r) < \frac{1}{\sqrt{1}}\left(\sqrt{1}e^{1-\sqrt{1}} - 1\right) = 0. \qquad (15)$$

Since G is a decreasing function of θ, the inequalities (13) and (15) imply that (12) has a unique root in the interval (d_l, d_r).

This completes the proof. ■

The model (4) for $\beta = 21$, $k = 20$ is visualized on Fig. 1. From the equations (8) and (12) as

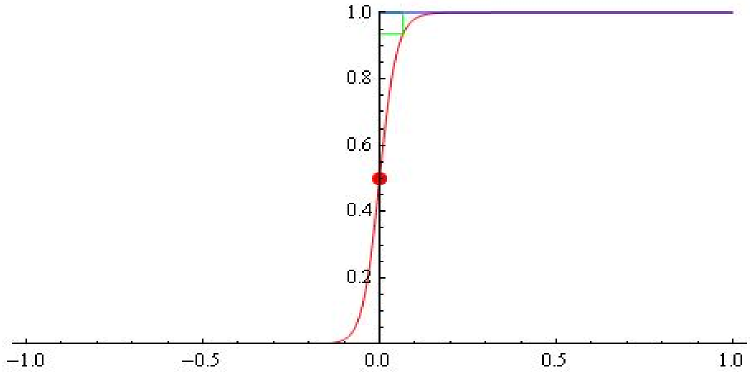

Fig. 2. The model (5) for $\beta = 61$, $k = 41$; H–distance $= 0.0660383$, $d_l = 0.0232558$, $d_r = 0.133677$.

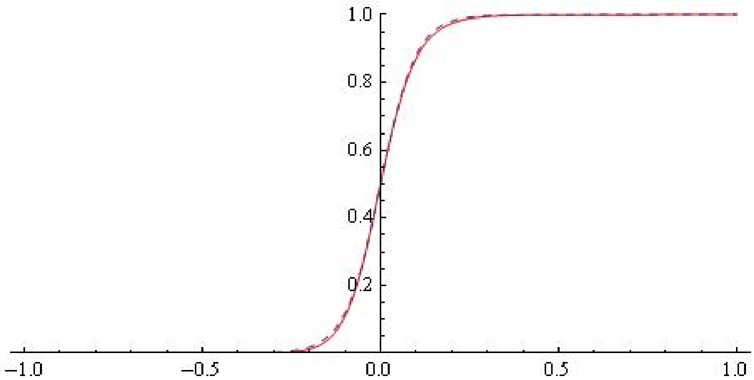

Fig. 3. Comparison between function $V(t)$ (dashed) and $N(t)$ (red) at fixed $k = 20$ and $\beta = 21$.

well as the inequalities (7) we have: $\overrightarrow{\rho}(N,h) = 0.109948$, $d_l = 0.0454545$, $d_r = 0.179129$.

The model (4) for $\beta = 61$, $k = 41$ is visualized on Fig. 2.

The estimates (7) of the one–sided Hausdorff distance of the Blumberg sigmoidal function to the Heaviside function, match those obtained for the Vehulst sigmoidal function. This should not surprise us. We already mentioned that the equation (2) is consistent with the Verhulst logistic model when $\beta \to +\infty$. As it is known, the Verhulst logistic function is of the form

$$V(t) = \frac{1}{1+e^{-kt}}.$$

A comparison between function $V(t)$ and $N(t)$ at fixed $k = 20$ and $\beta = 21$ is shown in Fig. 3.

The Hausdorff distance from the Verhulst function to the interval Heaviside function by is studied in detail in [19], [24]. Specifically, in the article [19], one may find more accurate estimates.

The hyper–log–logistic function can be used to recurrently generate a family of sigmoidal function:

$$N_{i+1}(t) = 1 - \frac{1}{1+\left(1+\frac{k}{\beta}\left(t-\frac{1}{2}+N_i(t)\right)\right)^{\beta}}, \quad (16)$$
$$i = 0, 1, 2, \ldots,$$

with

$$N_{i+1}(\alpha) = \frac{1}{2}, \quad i = 0, 1, 2, \ldots, \qquad (17)$$

where $N_0(t) = N(t)$ – the function given in

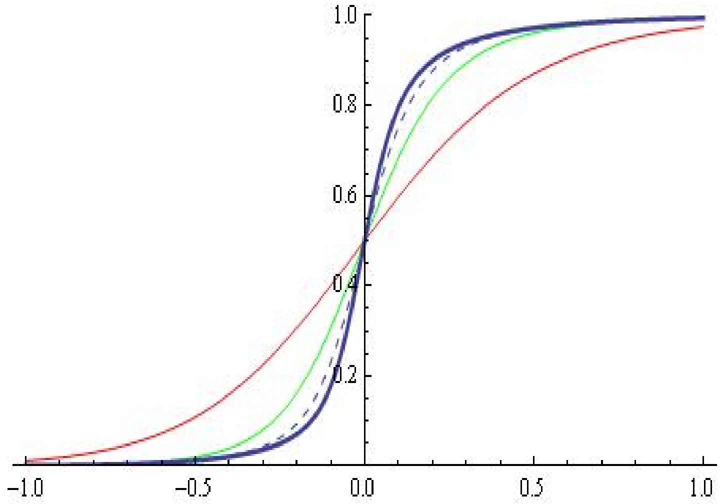

Fig. 4. The recurrence generated sigmoidal hyper–log–logistic functions: $N_0(t)$ (red); $d_0 = 0.21821$, $N_1(t)$ (green); $d_1 = 0.134208$, $N_2(t)$ (dashed); $d_2 = 0.095564$ and $N_3(t)$ (thick); $d_3 = 0.0749788$.

(4). We refer to this family shortly as recurrence generated sigmoidal hyper–log–logistic functions.

The recurrence generated sigmoidal hyper–log–logistic functions: $N_0(t), N_1(t), N_2(t)$ and $N_3(t)$ for $k = 4$ and $\beta = 21$ are visualized on Fig. 4. This type of family of functions can find application in the field of debugging and test theory [39]–[40]. Further, the results can be of interest to specialists working in the field of constructive approximation by superposition of sigmoidal functions [29]–[38].

V. Conclusions

In the areas of population dynamics, chemical kinetics or neural networks it is important to study the shape of the involved sigmoidal curve, since it relates to the fundamental properties of the respective system. In order to study the shape usually the curve is divided into lag phase, growth phase and saturation phase, [41]. These are defined in different ways in the literature, but in essence in the lag phase and in the saturation phase there is little or no growth, while most of the growth occurs in the growth phase. Hence the latter one is also called exponential phase. In [19] the Hausdorff distance to the interval Heaviside function is considered as a rigorously defined characteristic of the shape. One may consider that the points, where the value of the one–sided Hausdorff distance is attained, are precisely the points dividing the curve into the three mentioned segments. Then, the time-length of the growth phase is exactly twice the value of this distance.

In this paper we study the properties of the hyper-log-logistic curve produced by the Blumberg model through the one–sided Hausdorff distance of this curve to the interval Heaviside function. Lower and upper estimates of this distance are derived in terms of the intrinsic growth parameter and some possible applications are discussed.

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