BIOMATH
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Biomath Forum
ORIGINAL ARTICLE

Well-posedness and qualitative analysis of a SEIR model with spatial diffusion for COVID-19 spreading

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Received: November 30, 2022, Accepted: July 20, 2023, Published: July 28, 2023

Abstract: **In this paper, we study the well-posedness and the qualitative behavior of equilibria of a SEIR epidemic models with spatial diffusion for the spreading of COVID-19. The well-posedness of the model is proved using both the Semigroup Theory of sectorial operators and existence results for abstract parabolic differential equations. The asymptotical local stability of both disease-free and endemic equilibria are established using standard linearization theory, and confirmed by illustrative numerical simulations. The asymptotical global stability of both disease-free and endemic equilibria are established using a Lyapunov function.**

Keywords: **Abstract differential equations, COVID-19, SEIR compartmental model, Semigroup theory**

I. Introduction

At the end of December 2019, a new disease caused a serious respiratory problem was identified in the city of Wuhan, Hubei province, China [1, 2]. This disease was named COVID-19 and in a short time an increasing number of patients were admitted to hospitals with it, prompting the authorities of the whole world to introduce measures to contain the spreading of the epidemic [3]. Despite these efforts, the virus has managed to spread fast quickly, infecting large numbers of people in various parts of the world. Since then, several countries have been severely affected by the disease, generating losses of human lives and a great impact on the economy and health systems.

With the purpose to help governments to manage disease control as well as understanding how they might affect the populations in the short and long terms, several mathematical models of this outbreak have been proposed. In this field, epidemic models of type SEIR with ordinary differential equations were used to describe the spread of COVID-19 and also to study the efficiency of non-pharmacological control measures such as the use of masks. Epidemic models with ordinary differential equations implicitly assume uniform encounters between the infectious and susceptible subpopulations, resulting in homogeneous spatial

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Citation: José Paulo Carvalho dos Santos, Evandro Monteiro, José Claudinei Ferreira, Nelson Henrique Teixeira Lemes, Diego Samuel Rodrigues, Well-posedness and qualitative analysis of a SEIR model with spatial diffusion for COVID-19 spreading, Biomath 12 (2023), 2307207, https://doi.org/10.55630/j.biomath.2023.07.207

distributions. However, in the human population, especially under different levels of mobility restrictions, this assumption is likely to fail. One way to get around this problem is to consider models with reaction-diffusion equations [4–12].

Reaction-diffusion equations are used to model a variety of physical and biological phenomena [13–17]. These equations describe how the concentration or density distributed in space varies under the influence of two processes: local interactions of the species, and diffusion, which causes the spread of species in space. In population dynamics, diffusion terms correspond to a random motion of individuals and reaction terms describe local interactions of the populations. On epidemic models for COVID-19 with spatial diffusion, in [18] Viguerie, et al. present an early version of a SEIRD mathematical model based on partial differential equations coupled with a heterogeneous diffusion model. The model describes the spatiotemporal spread of the COVID-19 pandemic and aims to capture dynamics also based on human habits and geographical features. Tsori and Granek [19] suggest an epidemic model with diffusion for COVID-19 and present numerical experiments to show remarkable similarity heat maps publicly available. In this paper, it is also demonstrated how localized lockdown/quarantine conditions can slow down the spreading of disease from epicenters. Zhu and Zhu [20] proved the existence of the global exponential attractor of general dissipative evolution systems of a SEIQR model for COVID-19. However, the well-posedness of SEIR diffusion models [5] is not addressed in the literature. Accordingly, in this paper, we study this subject by addressing the stability of disease-free and endemic equilibrium points for the COVID-19 model with diffusion being the main contribution of the paper.

For the well-posedness of the SEIR model for COVID-19 with spatial diffusion, we used similar results to those found in [15, 21–24]. The stability of the equilibrium points for the SEIR model was studied using linearization and Lyapunov theories for partial differential equations [13, 14, 25–31]. This paper is organized as follows: in Section II, we introduce some notation and present relevant ideas to establish our results. Section III is devoted to abstract and parabolic differential equations to show the existence and uniqueness of a non-negative global solution, implying that the problem is well posed in a biological sense. Section IV is about the local stability of equilibria using linearization theory and the global stability of the equilibria using Lyapunov theory. Section V is devoted to illustrate these analytical results by presenting examples using numerical simulations. Finally, Section VI closes the paper with our conclusions.

II. Theoretical Background

In this section, we introduce notations and present key ideas for the work involving function spaces, semigroup of linear operators theory, abstract differential equations, and qualitative analysis of parabolic differential equations. For more details, the reader is referred to [16, 17, 32–35].

In this work, we denote by Ω a bounded domain in $\mathbb{R}^3$. For $1 \le p \le \infty$, the space of complex-valued L^p functions in Ω denoted by $L^p(\Omega)$ with the usual norm $\|\cdot\|_{L^p}$. The complex Sobolev space in Ω of order $k, k = 0, 1, 2, \ldots,$ is denoted by $H^k(\Omega)$ with norm $\|\cdot\|_{H^k}$. The space of complex-valued continuous functions on $\overline{\Omega}$ is denoted by $\mathcal{C}(\overline{\Omega})$ with norm $\|\cdot\|_{\mathcal{C}}$.

Let X be a Banach space with norm $\|\cdot\|$, we denote by $\mathcal{C}(\Omega; X)$ and $\mathcal{C}^1(\Omega; X)$ the space of X-valued continuous functions and of X-valued continuously differentiable functions, respectively. Additionally, let $\mathcal{B}(\Omega; X)$ be the space of X-valued bounded functions. The Sobolev space of fractional order $s > 0$ is denoted by $H^s(\Omega)$ with norm $\|\cdot\|_{H^s}$. We assume Ω has a $\mathcal{C}^2$ class boundary $\partial\Omega$, and for $\frac{3}{2} < s \le 2$ by $H^s_N(\Omega)$ we denote a closed subspace of $H^s(\Omega)$ such that

$$H^s_N(\Omega) = \{u \in H^s(\Omega) : \partial_n u = 0 \text{ on } \partial\Omega\}.$$

In what follows, for the sake of simplicity, we use the universal notation C to denote any constant that is determined for each specific occurrence of Ω. In cases in which C also depends on some parameter, say ξ, we use the notation C_ξ.

Let us comment on the existence theorem for local solutions to an abstract equation in a Banach space. We consider the following Cauchy problem for an abstract evolution equation in X

$$\begin{cases} \dfrac{dU}{dt} + AU &= F(U), \quad 0 < t \le T, \\ U(0) &= U_0. \end{cases} \tag{1}$$

Here, A is a sectorial operator of X with angle $0 \le \omega_A < \frac{\pi}{2}$. By definition,

$$\sigma(A) \subset \Sigma_\omega = \{\lambda \in \mathbb{C} : |\arg(\lambda)| < \omega\}, \ \omega_A < \omega < \frac{\pi}{2},$$

and

$$\|(\lambda - A)^{-1}\| \le \frac{M_\omega}{|\lambda|}, \quad \lambda \notin \Sigma_\omega, \quad \omega_A < \omega < \frac{\pi}{2}.$$

Let F be a nonlinear mapping from $\mathcal{D}(A^\eta)$ into X, where $0 \leq \eta < 1$, where A^η represents the fractional power of the operator A, for more details see [35]. F is assumed to satisfy the Lipschitz condition of the form

$$\begin{aligned}\|F(U) - F(V)\| \leq \varphi(\|U\| + \|V\|) \cdot \big(\|A^\eta(U - V)\| \\ + (\|A^\eta U\| + \|A^\eta V\|)\|U - V\|\big), \\ U, V \in \mathcal{D}(A^\eta), \end{aligned} \tag{2}$$

and $\varphi(\cdot)$ is some increasing continuous function. The initial value U_0 is taken in $\mathcal{D}(A^\eta)$.

Then, from [35], we state the following local solution existence theorem to (1):

Proposition 1 ([35], Theorem 4.4)**.** *Under the above assumptions, for any $U_0 \in X$, the equation* (1) *has a unique local solution in the function space $U \in \mathcal{C}([0, T_{U_0}]; X) \cap \mathcal{C}^1((0, T_{U_0}]; X) \cap \mathcal{C}((0, T_{U_0}]; \mathcal{D}(A))$, with the bound*

$$\|U(t)\| + t\left\|\frac{dU(t)}{dt}\right\| + t\|AU(t)\| \leq C_{U_0},\ 0 < t \leq T_{U_0},$$

Here, T_{U_0} and C_{U_0} are positive constants depending only on the norm $\|U_0\|$.

Proposition 2 ([35], Corollary 4.3)**.** *Let the assumptions of Proposition 1 holds and $U_0 \in X$. Assume that any local solution U of* (1) *in the function space $\mathcal{C}([0, T_U]; X) \cap \mathcal{C}((0, T_U]; \mathcal{D}(A)) \cap \mathcal{C}^1((0, T_U]; X)$, satisfies the estimative*

$$\|U(t)\| \leq C_{U_0}, \quad 0 \leq t \leq T_U, \tag{3}$$

with some constant $C_{U_0} > 0$ independent of T_U. Then, (1) *possesses a unique global solution in the function space $\mathcal{C}([0, T]; X) \cap \mathcal{C}((0, T]; \mathcal{D}(A)) \cap \mathcal{C}^1((0, T]; X)$.*

Using the terminology of [32], define $z = (z_1, z_2, \ldots, z_k)$, $f = (f_1, f_2, \ldots, f_k)$, D representing the diagonal matrix $D = (D_1, D_2, \ldots, D_k)$, $\Delta z = (\Delta z_1, \Delta z_2, \ldots, \Delta z_k)$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$. Let

$$\frac{\partial z}{\partial t} = D\Delta z + f(z). \tag{4}$$

in $[0, \infty) \times \Omega$ where Ω is some bounded domain with smooth boundary with initial data $z(0, x) = \alpha(x)$ and Neumann boundary data $\partial_n z(t, x) = 0$ for x on $\partial\Omega$ and $0 \leq t < \infty$.

Remark 1. *The system* (4) *can be written in the abstract form* (1)*.*

Under certain assumptions, one might expect that solution to (4) would approach as $t \to \infty$ as a solution of steady state equations

$$D\Delta z + f(z) = 0, \tag{5}$$

in D with Neumann boundary data. Solutions of (5) with the Neumann boundary condition are called equilibrium solutions.

Definition 1. *An equilibrium solution $\beta(x)$ is:*

(i) *stable if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|z(0, x) - \beta(x)\| < \delta$, then $\|z(t, x) - \beta(x)\| < \varepsilon$ for all $t \geq 0$.*
(ii) *called asymptotically stable if* (i) *holds and there exists $\delta > 0$ such that if $\|z(0, x) - \beta(x)\| < \delta$, then $\|z(t, x) - \beta(x)\| \to 0$ as $t \to \infty$. If δ can be chosen arbitrarily large, then the solution is globally asymptotically stable.*
(iii) *unstable if* (i) *does not hold.*

In this paper, we are interested in constant equilibrium solutions of (5). Let $f(z) = Az + g(z)$ where $g(0) = 0, \nabla g(0) = 0$, and define linearized system to be

$$\frac{\partial z}{\partial t} = D\Delta z + Az, \tag{6}$$

with the same initial and boundary conditions as given with (4). Then the zero solution of (4) is (locally) asymptotically stable if the zero solution of (6) is asymptotically stable.

Let $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$, be the sequence of eigenvalues of the operator $-\Delta$ subject to the homogeneous Neumann conditions on Ω, where each λ_i has multiplicity $m_i \geq 1$. Also let $\Phi_{ij}, 1 \leq j \leq m_i$ (recall that Φ_0=constant $\lambda_i \to \infty$ as $i \to \infty$) be a normalized eigenfunction corresponding to λ_i. That is Φ_{ij} and λ_i satisfy $-\Delta\Phi_{ij} = \lambda_i\Phi_{ij}$ in Ω, with $\partial_n\Phi_{ij} = 0$ and $\int_\Omega \Phi_{ij}^2(x)dx = 1$. Under these conditions we can have the following result.

Theorem 1 ([32], Theorem 1)**.** *For the linearized system* (6)*:*

(i) *The zero solution is globally asymptotically stable if for each non-negative integer n the eigenvalues of $A - \lambda_n D$ have negative real parts. Further, there exist positive constants K, ω such that for any $t > 0$,*
$$\|z(t, x)\| \leq Ke^{-\omega t}\|\alpha(x)\|.$$
(ii) *The zero solution is stable if for each non-negative integer n the eigenvalues of $A - \lambda_n D$ have non-positive real parts and those with zero real parts have simple elementary divisors.*
(iii) *The zero solution is unstable if for some n there exists an eigenvalue of $A - \lambda_n D$ with either a positive real part or zero real part with a non-simple elementary divisor.*

Theorem 2 ([32], Theorem 2). *The zero solution of* (4) *is asymptotically stable if the zero solution of the linearized problem* (6) *is asymptotically stable.*

The next result is used a lot throughout the text.

Lemma 1 ([35], Proposition 1.4). *Let* $a \in \mathcal{C}([0,T];\mathbb{R})$ *and* $f \in \mathcal{C}([0,T];\mathbb{R})$, *and let a function* $u \in \mathcal{C}([0,T];\mathbb{R}) \cap \mathcal{C}^1((0,T];\mathbb{R})$ *satisfying the differential inequality*

$$\frac{du}{dt} + a(t)u \leq f(t), \quad 0 < t \leq T.$$

Then, for any fixed $0 < t \leq T$, *we get*

$$u(t) \leq e^{-\int_0^t a(\tau)d\tau} u(0) + \int_0^t e^{-\int_s^t a(\tau)d\tau} f(s)ds.$$

In particular, if $a(t) \equiv \delta > 0$ *and* $f(t) \equiv f > 0$, *then it follows that*

$$u(t) \leq e^{-\delta t} u(0) + f\delta^{-1}, \quad 0 < t \leq T.$$

III. Mathematical Modeling and Results

Using the methodology of compartmental models and spatial diffusion, we can study the spreading of COVID-19 in regions of interest. For that, the population can be divided into four subpopulations, which can be described with the following parameters and compartments:

- $S(t,x)$ denotes the susceptible population,
- $E(t,x)$ is the exposed population to COVID-19 by the contact with an infected,
- $I(t,x)$ denotes the infected population,
- $R(t,x)$ is the population of removed/recovered,
- $\beta > 0$ is the transmission rate from the susceptible population to the infected population,
- $\Lambda > 0$ comprises new births and new residents per unit value of time,
- $\mu > 0$ is the rate of death,
- $\gamma > 0$ is the transmission rate of confirmed infected people from the exposed population ($1/\gamma$ is the duration of the latent period).
- $\alpha > 0$ is the removed/recovery rate from the infected population (the time spent in the "infectious" compartment is $1/\alpha$).
- d_S, d_E, d_I and d_R denotes the diffusion coefficient of the populations $S(t,x), E(t,x), I(t,x)$ and $R(t,x)$, respectively.

Using the arguments of [12, 36], we can describe the disease epidemic models with spatial diffusion given by the following partial differential equations of parabolic type with Neumann condition:

$$\begin{cases} \dfrac{\partial S(t,x)}{\partial t} &= d_S \Delta S(t,x) - \beta S(t,x) I(t,x) \\ & \quad -\mu S(t,x) + \Lambda, \\ \dfrac{\partial E(t,x)}{\partial t} &= d_E \Delta E(t,x) + \beta S(t,x) I(t,x) \\ & \quad -(\mu+\gamma) E(t,x), \\ \dfrac{\partial I(t,x)}{\partial t} &= d_I \Delta I(t,x) + \gamma E(t,x) \\ & \quad -(\mu+\alpha) I(t,x), \\ \dfrac{\partial R(t,x)}{\partial t} &= d_R \Delta R(t,x) + \alpha I(t,x) \\ & \quad -\mu R(t,x), \\ & (t,x) \in (0,\infty)\times\Omega, \end{cases} \tag{7}$$

$$\begin{aligned} &S(0,x) = S_0(x), \quad E(0,x) = E_0(x), \\ &I(0,x) = I_0(x), \quad R(0,x) = R_0(x), \quad x \in \Omega, \end{aligned} \tag{8}$$

$$\begin{aligned} &\partial_n S(t,x) = \partial_n E(t,x) = \partial_n I(t,x) \\ &\quad = \partial_n R(t,x) = 0 \text{ for } x \text{ on } \partial\Omega \text{ and } t \in (0,\infty), \end{aligned} \tag{9}$$

$$\int_\Omega S(0,x) + E(0,x) + I(0,x) + R(0,x) dx \equiv N. \tag{10}$$

Let $\mathcal{U} = \begin{pmatrix} S \\ E \\ I \\ R \end{pmatrix}$. The problem (7)-(10) can be formulated as an abstract Cauchy problem:

$$\begin{aligned} \mathcal{U}'(t) + \mathcal{A}\mathcal{U}(t) &= \mathcal{F}(\mathcal{U}), \quad t > 0, \\ \mathcal{U}(0) &= \mathcal{U}_0 \in X, \end{aligned} \tag{11}$$

where

$$\mathcal{U}_0 = \begin{pmatrix} S_0 \\ E_0 \\ I_0 \\ R_0 \end{pmatrix}$$

$$\mathcal{A} = \begin{pmatrix} 1 - d_S\Delta & 0 & 0 & 0 \\ 0 & 1 - d_E\Delta & 0 & 0 \\ 0 & 0 & 1 - d_I\Delta & 0 \\ 0 & 0 & 0 & 1 - d_R\Delta \end{pmatrix}$$

$$\mathcal{F}(\mathcal{U}) = \begin{pmatrix} \Lambda - \beta S(t,x)I(t,x) - \mu S(t,x) + S(t,x) \\ \beta S(t,x)I(t,x) - (\gamma+\mu)E(t,x) + E(t,x) \\ \gamma E(t,x) - (\alpha+\mu)I(t,x) + I(t,x) \\ \alpha I(t,x) - \mu R(t,x) + R(t,x) \end{pmatrix}$$

In (11), the space X is defined by $X = L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ under the norm

$$\left\| \begin{pmatrix} S \\ E \\ I \\ R \end{pmatrix} \right\| = \left(\int_\Omega |S|^2 + |E|^2 + |I|^2 + |R|^2 dx \right)^{\frac{1}{2}}.$$

It is easy to verify that X is a Hilbert space. We define the $D(\mathcal{A})$ by

$$D(\mathcal{A}) = \left\{ \begin{pmatrix} S \\ E \\ I \\ R \end{pmatrix} \in X : S, E, I, R \in H^2_N(\Omega) \right\}.$$

A. Global Existence, Positivity of Solutions

In the sequel, we show the existence of local solutions associated with the system (7)-9.

Theorem 3. *For each initial function*

$$(S_0, E_0, I_0, R_0) \in L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega),$$

the problem (11) *admits a unique local-in-time solution* $\mathcal{U} = (S, E, I, R)$ *in the space*

$$\begin{aligned} S, E, I, R \in & \mathcal{C}([0, T_{\mathcal{U}_0}]; L^2(\Omega)) \cap \\ & \mathcal{C}^1((0, T_{\mathcal{U}_0}]; L^2(\Omega)) \cap \mathcal{C}((0, T_{\mathcal{U}_0}]; H^2_N(\Omega)), \end{aligned}$$

where $T_{\mathcal{U}_0}$ *is a positive constant depending only on the norm* $\|S_0\|_{L^2} + \|E_0\|_{L^2} + \|I_0\|_{L^2} + \|R_0\|_{L^2}$.

Proof: We denote by $\mathcal{L}^\infty(\Omega) = L^\infty(\Omega) \times L^\infty(\Omega) \times L^\infty(\Omega) \times L^\infty(\Omega)$ with the norm $\|\cdot\|_{\mathcal{L}^\infty} = \|\cdot\|_{L^\infty} + \|\cdot\|_{L^\infty} + \|\cdot\|_{L^\infty} + \|\cdot\|_{L^\infty}$ and

$$\mathcal{A} = \begin{pmatrix} A_S & 0 & 0 & 0 \\ 0 & A_E & 0 & 0 \\ 0 & 0 & A_I & 0 \\ 0 & 0 & 0 & A_R \end{pmatrix}.$$

Where A_ξ are realizations of operators $1 - d_\xi \Delta \xi$, with $\xi = S, E, I$ and R, respectively, in $L^2(\Omega)$ under the Neumann boundary conditions on $\partial\Omega$.

By [35, Theorem 2.4] we get that $\mathcal{A}$ is a sectorial operator of X with angle $0 \leq \omega_A < \frac{\pi}{2}$. Furthermore, according to [35, Theorem 16.7],

$$\begin{aligned} D(A^\eta_\xi) &= H^{2\eta}(\Omega), && \text{if } 0 \leq \eta < \frac{3}{4} \\ D(A^\eta_\xi) &= H^{2\eta}_N(\Omega), && \text{if } \frac{3}{4} < \eta \leq 1. \end{aligned}$$

Let $\mathcal{F}(\mathcal{U}_i)$, $i = 1, 2$, be

$$\mathcal{F}(\mathcal{U}_i) = \begin{pmatrix} \Lambda - \beta S_i(t,x) I_i(t,x) - \mu S_i(t,x) + S_i(t,x) \\ \beta S_i(t,x) I_i(t,x) - (\gamma + \mu) E_i(t,x) + E_i(t,x) \\ \gamma E_i(t,x) - (\alpha + \mu) I_i(t,x) + I_i(t,x) \\ \alpha I_i(t,x) - \mu R_i(t,x) + R_i(t,x) \end{pmatrix}.$$

By $H^{2\eta}_N(\Omega) \subset L^\infty(\Omega)$ with $\frac{3}{4} < \eta \leq 1$, we get

$$\begin{aligned} &\|\mathcal{F}(\mathcal{U}_1) - \mathcal{F}(\mathcal{U}_2)\|^2 \\ &\quad \leq C(\|S_1\|^2_{L^\infty} \|I_1 - I_2\|^2_{L^2} + \|I_2\|^2_{L^\infty} \|S_1 - S_2\|^2_{L^2} \\ &\qquad + \|S_1 - S_2\|^2_{L^2} + \|E_1 - E_2\|^2_{L^2} + \|I_1 - I_2\|^2_{L^2} \\ &\qquad + \|R_1 - R_2\|^2_{L^2}) \\ &\quad \leq C\big((\|S_1\|_{L^\infty} + \|E_1\|_{L^\infty} + \|I_1\|_{L^\infty} + \|R_1\|_{L^\infty})^2 \\ &\qquad \cdot \|I_1 - I_2\|^2_{L^2} + \|S_1 - S_2\|^2_{L^2} \\ &\qquad \cdot (\|S_2\|_{L^\infty} + \|E_2\|_{L^\infty} + \|I_2\|_{L^\infty} + \|R_2\|_{L^\infty})^2 \\ &\qquad + \|S_1 - S_2\|^2_{L^2} + \|E_1 - E_2\|^2_{L^2} \\ &\qquad + \|I_1 - I_2\|^2_{L^2} + \|R_1 - R_2\|^2_{L^2}\big) \\ &\quad \leq C\big((\|S_1\|_{L^\infty} + \|E_1\|_{L^\infty} + \|I_1\|_{L^\infty} + \|R_1\|_{L^\infty} \\ &\qquad + \|S_2\|_{L^\infty} + \|E_2\|_{L^\infty} + \|I_2\|_{L^\infty} + \|R_2\|_{L^\infty})^2 \\ &\qquad \cdot (\|S_1 - S_2\|^2_{L^2} + \|E_1 - E_2\|^2_{L^2} + \|I_1 - I_2\|^2_{L^2} \\ &\qquad + \|R_1 - R_2\|^2_{L^2}) + \|S_1 - S_2\|^2_{L^2} + \|E_1 - E_2\|^2_{L^2} \\ &\qquad + \|I_1 - I_2\|^2_{L^2} + \|R_1 - R_2\|^2_{L^2}\big) \\ &\quad \leq C\big((\|A^\eta_S S_1\|_{L^2} + \|A^\eta_E E_1\|_{L^2} + \|A^\eta_I I_1\|_{L^2} \\ &\qquad + \|A^\eta_R R_1\|_{L^2} + \|A^\eta_S S_2\|_{L^2} + \|A^\eta_E E_2\|_{L^2} \\ &\qquad + \|A^\eta_I I_2\|_{L^2} + \|A^\eta_R R_2\|_{L^2})^2 \cdot (\|S_1 - S_2\|^2_{L^2} \\ &\qquad + \|E_1 - E_2\|^2_{L^2} + \|I_1 - I_2\|^2_{L^2} + \|R_1 - R_2\|^2_{L^2}) \\ &\qquad + \|S_1 - S_2\|^2_{L^2} + \|E_1 - E_2\|^2_{L^2} \\ &\qquad + \|I_1 - I_2\|^2_{L^2} + \|R_1 - R_2\|^2_{L^2}\big) \\ &\quad \leq C(\|\mathcal{A}^\eta \mathcal{U}_1\| + \|\mathcal{A}^\eta \mathcal{U}_2\| + 1)^2 \|\mathcal{U}_1 - \mathcal{U}_2\|^2, \end{aligned}$$

with $\mathcal{U}_1, \mathcal{U}_2 \in D(\mathcal{A}^\eta)$, where

$$D(\mathcal{A}^\eta) = \left\{ \begin{pmatrix} S \\ E \\ I \\ R \end{pmatrix} \in X : S, E, I, R \in H^{2\eta}_N(\Omega) \right\}.$$

By Proposition 1, the problem (7)-(9) has a unique local solution in the function space

$$\begin{aligned} S, E, I, R \in & \mathcal{C}([0, T_{\mathcal{U}_0}]; L^2(\Omega)) \cap \\ & \mathcal{C}^1((0, T_{\mathcal{U}_0}]; L^2(\Omega)) \cap \mathcal{C}((0, T_{\mathcal{U}_0}]; H^2_N(\Omega)). \end{aligned}$$

■

Theorem 4. *For any given initial data satisfying the condition* (7)-(10)*, there exists a unique solution to the problem defined on* $t \in [0, T_{\mathcal{U}_0}]$ *and this solution remains nonnegative for all* $t \in [0, T_{\mathcal{U}_0}]$.

Proof: We will show that $S(t) \geq 0$, $E(t) \geq 0$, $I(t) \geq 0$ and $R(t) \geq 0$ for all $0 < t \leq T_{\mathcal{U}_0}$. For this purpose, however, we have to introduce the modified nonlinear operator

$$\mathcal{F}(\widetilde{\mathcal{U}}) = \begin{pmatrix} \Lambda - \beta\widetilde{S}(t,x)\widetilde{I}(t,x) - \mu\widetilde{S}(t,x) + \widetilde{S}(t,x) \\ \beta\widetilde{S}(t,x)\widetilde{I}(t,x) - (\gamma+\mu)\chi(Re\widetilde{E})(t,x) \\ +\chi(Re\widetilde{E})(t,x) \\ \gamma\chi(Re\widetilde{E})(t,x) - (\alpha+\mu)\widetilde{I}(t,x) + \widetilde{I}(t,x) \\ \alpha\widetilde{I}(t,x) - \mu\widetilde{R}(t,x) + \widetilde{R}(t,x) \end{pmatrix}$$

where $\chi(u)$ denotes a function such that $\chi(u) \equiv 0$ for $-\infty < u < 0$ and $\chi(u) = u$ for $0 \le u < \infty$. We have to consider the auxiliary problem

$$\begin{aligned} \widetilde{\mathcal{U}}'(t) + \mathcal{A}\widetilde{\mathcal{U}}(t) &= \widetilde{\mathcal{F}}(\mathcal{U}), \quad t > 0, \\ \widetilde{\mathcal{U}}(0) &= \mathcal{U}_0 \in X, \end{aligned} \tag{12}$$

It is clear that the new nonlinear operator $\widetilde{\mathcal{F}}$ also satisfies (2) with the same exponent η because

$$\|\chi(Re\ u) - \chi(Re\ v)\|_{L^2} \le \|u - v\| \text{ for } u, v \in L^2(\Omega).$$

Therefore, (12) possesses a unique local solution $\mathcal{U} = (\widetilde{S}, \widetilde{E}, \widetilde{I}, \widetilde{R})$ on an interval $[0, \widetilde{T}_{\mathcal{U}_0}]$ in the same functions spaces $\widetilde{S}, \widetilde{E}, \widetilde{I}, \widetilde{R} \in \mathcal{C}([0, \widetilde{T}_{\mathcal{U}_0}]; L^2(\Omega)) \cap \mathcal{C}^1((0, \widetilde{T}_{\mathcal{U}_0}]; L^2(\Omega)) \cap \mathcal{C}((0, \widetilde{T}_{\mathcal{U}_0}]; H^2_N(\Omega))$. First, we will show that $\widetilde{S}(t) \ge 0$, $\widetilde{E}(t) \ge 0$, $\widetilde{I}(t) \ge 0$ and $\widetilde{R}(t) \ge 0$ for all $0 < t \le \widetilde{T}_{\mathcal{U}_0}$.

We note that $\widetilde{\mathcal{U}}(t)$ is real-valued. Indeed, the complex conjugate $\overline{\widetilde{\mathcal{U}}(t)}$ of $\widetilde{\mathcal{U}}(t)$ is also a local solution of (12) with the same initial value $\mathcal{U}_0$. From the uniqueness of solutions, $\overline{\widetilde{\mathcal{U}}(t)} = \widetilde{\mathcal{U}}(t)$, hence $\widetilde{\mathcal{U}}(t)$ is real-valued.

Let $H(\cdot)$ be a $\mathcal{C}^{1,1}$ custoff function such that $H(u) = \frac{u^2}{2}$ for $-\infty < u < 0$ and $H(u) \equiv 0$ for $0 \le u < \infty$. By Yagi [35, page 52], the function $\psi(t) = \int_\Omega H(u(t))dx$ is continuously differentiable.

Computing the derivative of $\psi(t)$ with $u(t) = \widetilde{S}(t)$, we get

$$\begin{aligned} \psi'(t) &= \int_\Omega H'(\widetilde{S})\widetilde{S}'(t)dx \\ &= \int_\Omega H'(\widetilde{S})\Big(\Lambda + d_S\Delta\widetilde{S}(t) - \beta\widetilde{S}(t)\widetilde{I}(t) - \mu\widetilde{S}(t)\Big)dx \\ &= d_S\int_\Omega H'(\widetilde{S})\Delta\widetilde{S}(t)dx + \Lambda\int_\Omega H'(\widetilde{S})dx \\ &\quad - \int_\Omega H'(\widetilde{S})\widetilde{S}(t)(\beta\widetilde{I}(t) + \mu)dx, \end{aligned}$$

By the property (1.96) from [35], we can get

$$\begin{aligned} \int_\Omega H'(\widetilde{S})\Delta\widetilde{S}dx &= -\int_\Omega \nabla H'(\widetilde{S}) \cdot \nabla\widetilde{S}(t)dx \\ &= -\int_\Omega \nabla H'(\widetilde{S}) \cdot \nabla H'(\widetilde{S})dx \\ &= -\int_\Omega |\nabla H'(\widetilde{S})|^2dx \le 0, \end{aligned}$$

therefore

$$\begin{aligned} \psi'(t) = &-d_S\int_\Omega |\nabla H'(\widetilde{S})|^2dx + \Lambda\int_\Omega H'(\widetilde{S})dx \\ &- \int_\Omega H'(\widetilde{S})\widetilde{S}(t)(\beta\widetilde{I}(t) + \mu)dx. \end{aligned}$$

Since $H'(\widetilde{S}) \le 0$,

$$\begin{aligned} \psi'(t) &\le -\int_\Omega H'(\widetilde{S})\widetilde{S}(t)(\beta\widetilde{S}(t) + \mu)dx \\ &\le C\|H'(\widetilde{S})\widetilde{S}\|_{L^1}(1 + \|\widetilde{S}\|_{L^\infty}) \\ &\le C\|H(\widetilde{S})\|_{L^1}(1 + \|\widetilde{S}\|_{L^\infty} + \|\widetilde{E}\|_{L^\infty} \\ &\qquad + \|\widetilde{I}\|_{L^\infty} + \|\widetilde{R}\|_{L^\infty}) \\ &\le C\|H(\widetilde{S})\|_{L^1}(1 + \|\widetilde{S}\|_{H^{2\eta}} + \|\widetilde{E}\|_{H^{2\eta}} \\ &\qquad + \|\widetilde{I}\|_{H^{2\eta}} + \|\widetilde{R}\|_{H^{2\eta}}) \\ &= C\psi(t)(1 + \|\widetilde{S}\|_{H^{2\eta}} + \|\widetilde{E}\|_{H^{2\eta}} \\ &\qquad + \|\widetilde{I}\|_{H^{2\eta}} + \|\widetilde{R}\|_{H^{2\eta}}). \end{aligned}$$

Therefore

$$\psi'(t) \le C\psi(t)(1 + \|\mathcal{A}^\eta\widetilde{\mathcal{U}}(t)\|).$$

Thus, by Lemma 1,

$$\psi(t) \le \psi(0)e^{C\int_0^t(1+\|\mathcal{A}^\eta\widetilde{\mathcal{U}}(\tau)\|)d\tau}.$$

Using the bound $\|\mathcal{A}^\eta\widetilde{\mathcal{U}}(\tau)\| \le C_{\mathcal{U}_0}\tau^{-\eta}$, which means that $\|\mathcal{A}^\eta\widetilde{\mathcal{U}}(\tau)\|$ is integrable in $0 \le t \le \widetilde{T}_{\mathcal{U}_0}$. Hence, $\psi(0) = 0$ implies $\psi(t) \equiv 0$, namely $\widetilde{S}(t) \ge 0$ for $0 \le t \le \widetilde{T}_{\mathcal{U}_0}$.

Now, computing the derivative of $\psi(t)$ with $u(t) = \widetilde{I}(t)$, we get

$$\begin{aligned} \psi'(t) &= \int_\Omega H'(\widetilde{I})\widetilde{I}'(t)dx \\ &= \int_\Omega H'(\widetilde{I})\Big(d_I\Delta\widetilde{I}(t) + \gamma\chi(\widetilde{E}(t)) - (\alpha+\mu)\widetilde{I}(t)\Big)dx \\ &= d_I\int_\Omega H'(\widetilde{I})\Delta\widetilde{I}(t)dx + \gamma\int_\Omega H'(\widetilde{I})\chi(\widetilde{E}(t))dx \\ &\quad - (\alpha+\mu)\int_\Omega H'(\widetilde{I})\widetilde{I}dx \\ &= -d_I\int_\Omega |\nabla H'(\widetilde{I})|^2dx + \gamma\int_\Omega H'(\widetilde{I})\chi(\widetilde{E}(t))dx \\ &\quad - 2(\alpha+\mu)\int_\Omega H(\widetilde{I})dx. \end{aligned}$$

By $H(\widetilde{I}) \ge 0$ and $H'(\widetilde{I}) < 0$ we get $\psi'(t) \le C\psi(t)$. Therefore $\psi(t) \le \psi(0)e^{Ct}$. From $\psi(0) = 0$, we get $\psi(t) \equiv 0$, this implies $\widetilde{I}(t) \ge 0$ for $0 \le t \le \widetilde{T}_{\mathcal{U}_0}$. From the same argument before, we get $\widetilde{E}(t) \ge 0$ and $\widetilde{R}(t) \ge 0$ for $0 \le t \le \widetilde{T}_{\mathcal{U}_0}$.

We now note that $\chi(\widetilde{E}(t)) = \widetilde{E}(t)$, which implies $\widetilde{\mathcal{U}}$ is a local solution of the original problem (11) too. The uniqueness of solution then implies $\widetilde{\mathcal{U}} = \mathcal{U}$. Hence $S(t) \geq 0$, $E(t) \geq 0$, $I(t) \geq 0$ and $R(t) \geq 0$ for $0 < t \leq \widetilde{T}_{\mathcal{U}_0}$. Now we have the possibilities:

- If $\widetilde{T}_{\mathcal{U}_0} \geq T_{\mathcal{U}_0}$ we have finished the proof.
- If not, we define $T_0 = \sup\{0 < T \leq T_{\mathcal{U}_0} : S(t) \geq 0, E(t) \geq 0, I(t) \geq 0, R(t) \geq 0$ for every $0 < t \leq T\}$. From
$$\int_\Omega H(S(T_0))dx = \lim_{t\to T_0^-}\int_\Omega H(S(t))dx = 0,$$
we see that $S(T_0) \geq 0$. By a similar argument, we have $E(T_0) \geq 0, I(T_0) \geq 0$ and $R(T_0) \geq 0$. So, if $T_0 = T_{\mathcal{U}_0}$, we have finished the proof.
- If $T_0 < T_{\mathcal{U}_0}$, we will consider again the problem (12) but with the initial time T_0 and the initial value $\mathcal{U}(T_0)$. Repeating the same argument as above, we conclude that there exists a $\delta > 0$ such that $S(t) \geq 0$, $E(t) \geq 0$, $I(t) \geq 0$ and $R(t) \geq 0$ for every $T_0 \leq t \leq T_0 + \delta$. This is a contradiction, hence $T_0 = T_{\mathcal{U}_0}$.

The above arguments imply the non-negativity of solutions for all $0 \leq t \leq T_{\mathcal{U}_0}$. ■

B. Boundedness of Solutions

In the next results, we show the boundedness of solutions.

Lemma 2. *Let (S, E, I, R) be a local-in-time solution to (7)-(9). Then, it holds that*
$$\|S(t)\|_{L^1} \leq e^{-\mu t}N + \frac{\Lambda|\Omega|}{\mu},$$
$$\|E(t)\|_{L^1} \leq e^{-\mu t}N + \frac{\Lambda|\Omega|}{\mu},$$
$$\|I(t)\|_{L^1} \leq e^{-\mu t}N + \frac{\Lambda|\Omega|}{\mu},$$
$$\|R(t)\|_{L^1} \leq e^{-\mu t}N + \frac{\Lambda|\Omega|}{\mu}.$$

Proof: Let
$$N(t) = \int_\Omega S(t,x) + E(t,x) + I(t,x)dx + R(t,x)dx.$$
Using the Green's first identity we infer
$$\begin{aligned}N'(t) = \int_\Omega & d_S\Delta S(t,x) + d_E\Delta E(t,x) \\ & + d_I\Delta I(t,x) + d_R\Delta R(t,x) \\ & + \Lambda - \mu S(t,x) - \mu E(t,x) \\ & - \mu I(t,x) - \mu R(t,x)dx\end{aligned}$$
$$\begin{aligned}&= d_S\int_{\partial\Omega}\frac{\partial S}{\partial\nu}ds + d_E\int_{\partial\Omega}\frac{\partial E}{\partial\nu}ds + d_I\int_{\partial\Omega}\frac{\partial I}{\partial\nu}ds \\ &\quad + d_R\int_{\partial\Omega}\frac{\partial R}{\partial\nu}ds + \int_\Omega \Lambda - \mu(S(t,x) + E(t,x) \\ &\qquad + I(t,x) + R(t,x))dx \\ &= \int_\Omega \Lambda - \mu(S(t,x) + E(t,x) + I(t,x) + R(t,x))dx \\ &\leq \Lambda|\Omega| - \mu N(t).\end{aligned}$$
By Lemma 1,
$$N(t) \leq e^{-\mu t}N + \frac{\Lambda|\Omega|}{\mu},$$
therefore
$$\|S(t) + E(t) + I(t) + R(t)\|_{L^1} \leq e^{-\mu t}N + \frac{\Lambda|\Omega|}{\mu}.$$
From $S(t) \geq 0$, $E(t) \geq 0$, $I(t) \geq 0$, and $R(t) \geq 0$ for all $0 \leq t \leq T_{\mathcal{U}_0}$ we get
$$\|S(t)\|_{L^1} \leq e^{-\mu t}N + \frac{\Lambda|\Omega|}{\mu},$$
$$\|E(t)\|_{L^1} \leq e^{-\mu t}N + \frac{\Lambda|\Omega|}{\mu},$$
$$\|I(t)\|_{L^1} \leq e^{-\mu t}N + \frac{\Lambda|\Omega|}{\mu},$$
$$\|R(t)\|_{L^1} \leq e^{-\mu t}N + \frac{\Lambda|\Omega|}{\mu}.$$
■

Now we show the existence of global solutions for the problem (7)-(9).

Theorem 5. *For any given initial data satisfying the condition (7)-(9), there exists a unique solution to the problem defined on $[0,\infty)$ and this solution remains nonnegative and bounded for all $t \geq 0$.*

Proof: Let $S, E, I, R \in \mathcal{C}([0, T_{\mathcal{U}_0}]; L^2(\Omega))$. We have that S, E, I, R are bounded in L^2 norm in $[0, T]$ with $T < T_{\mathcal{U}_0}$. For all $t \geq T$, we consider
$$\begin{aligned}\int_\Omega SS'dx &= \int_\Omega d_S S\Delta S dx + \int_\Omega \Lambda S - \mu S^2 - \beta S^2 I dx \\ &= -\int_\Omega d_S|\nabla S|^2dx + \int_\Omega \Lambda S - \mu S^2 - \beta S^2 I dx \\ &= -\int_\Omega\left(\sqrt{\frac{\mu}{2}}S - \frac{1}{2}\sqrt{\frac{2}{\mu}}\Lambda\right)^2dx - \frac{\mu}{2}\int_\Omega|S|^2dx \\ &\quad + \int_\Omega\frac{\Lambda^2}{2\mu}dx \leq -\frac{\mu}{2}\int_\Omega|S|^2dx + \frac{\Lambda^2}{2\mu}|\Omega|.\end{aligned}$$
Therefore
$$\frac{1}{2}\frac{d}{dt}\int_\Omega|S|^2dx \leq -\frac{\mu}{2}\int_\Omega|S|^2dx + \frac{\Lambda^2}{2\mu}|\Omega|.$$

By Lemma 1, we infer

$$\|S(t)\|_{L^2}^2 \leq \|S(0)\|_{L^2}^2 e^{-\mu t} + \frac{\Lambda^2}{\mu^2}|\Omega|.$$

Multiplying the third equation of the system by $I(\cdot)$, we get

$$\int_\Omega I' I dx = \int_\Omega d_I I \Delta I dx + \int_\Omega \gamma IE - (\alpha+\mu)I^2 dx,$$

and

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |I|^2 dx \\
&\qquad = -d_I \int_\Omega |\nabla I|^2 dx + \int_\Omega \gamma EI - (\alpha+\mu)I^2 dx \\
&\qquad \leq -\int_\Omega (\alpha+\mu)|I|^2 dx + \gamma \|I\|_{L^\infty}\|E\|_{L^1} \\
&\qquad \leq -(\alpha+\mu)\int_\Omega |I|^2 dx + \gamma\|I\|_{H^{2\eta}}\|E\|_{L^1} \\
&\qquad \leq -(\alpha+\mu)\int_\Omega |I|^2 dx + \gamma\|A^\eta I\|_{L^2}\|E\|_{L^1} \\
&\qquad \leq -(\alpha+\mu)\int_\Omega |I|^2 dx + \frac{C}{t^\eta} \\
&\qquad \leq -(\alpha+\mu)\int_\Omega |I|^2 dx + \frac{C}{T^\eta}.
\end{aligned}$$

By Lemma 1, we show

$$\|I(t)\|_{L^2}^2 \leq e^{-2(\alpha+\mu)t}\|I(0)\|_{L^2}^2 + \frac{C}{(\alpha+\mu)T^\eta}.$$

Now, multiplying the second equation of the system by $E(\cdot)$, we get

$$\begin{aligned}
\int_\Omega EE' dx &= \int_\Omega d_E E\Delta E dx - \int_\Omega (\mu+\gamma)|E|^2 dx \\
&\quad + \int_\Omega \beta SEI dx \\
&= -\int_\Omega d_E |\nabla E|^2 dx - \int_\Omega (\mu+\gamma)|E|^2 dx \\
&\quad + \int_\Omega \beta SEI dx.
\end{aligned}$$

By the Hölder inequality and Lemma 2, we get

$$\begin{aligned}
\int_\Omega \beta SEI dx &\leq \beta\|E\|_{L^\infty}\|SI\|_{L^1} \\
&\leq \beta\|E\|_{H^{2\eta}}\|S\|_{L^2}\|I\|_{L^2} \\
&\leq \beta\|A^\eta E\|_{L^2}\|S\|_{L^2}\|I\|_{L^2} \\
&\leq \frac{C}{t^\eta} \leq \frac{C}{T^\eta}.
\end{aligned}$$

From the previous argument, it follows that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |E|^2 dx \leq -\int_\Omega (\gamma+\mu)|E|^2 dx + \frac{C}{T^\eta}.$$

By Lemma 1, we get

$$\|E\|_{L^2}^2 \leq \|E(0)\|_{L^2}^2 e^{-2(\gamma+\mu)t} + \frac{C}{(\gamma+\mu)T^\eta}.$$

Multiplying the fourth equation of the system by $R(\cdot)$, we get

$$\begin{aligned}
\int_\Omega R' R dx &= \int_\Omega d_R R\Delta R dx + \int_\Omega \alpha IR - \mu R^2 dx \\
&= -\int_\Omega d_R |\nabla R|^2 dx + \int_\Omega \alpha IR - \mu R^2 dx \\
&\leq -\int_\Omega \left(\frac{\alpha}{\sqrt{2\mu}} I - \sqrt{\frac{\mu}{2}} R\right)^2 + \frac{\alpha^2}{2\mu} I^2 - \frac{\mu}{2} R^2 dx.
\end{aligned}$$

Therefore,

$$\frac{d}{dt}\int_\Omega |R|^2 dx + \frac{\mu}{2}\int_\Omega |R|^2 dx \leq \frac{\alpha^2}{\mu}\|I\|_{L^2} \leq C.$$

By Lemma 1, we infer

$$\|R\|_{L^2}^2 \leq \|R(0)\|_{L^2}^2 e^{-\mu t} + \frac{C}{\mu}.$$

By Proposition 2, the solution $S(\cdot)$, $E(\cdot)$, $I(\cdot)$ and $R(\cdot)$ of (7)-(9) are globally defined for all $t \in [0, \infty)$. By previous argument and Theorem 4 we obtain the unique solution of (7)-9 which is positive and defined for all $t \geq 0$. ■

IV. Stability Analysis

A. Local Stability

In this work, the basic reproductive number is given by

$$\mathcal{R}_0 = \frac{\Lambda\beta\gamma}{(\mu+\gamma)(\mu+\alpha)\mu}.$$

As we claim in the sequel, the disease-free equilibrium point is asymptotically stable for $\mathcal{R}_0 < 1$. This result is similar to the epidemic models completely described by ordinary differential equations [36].

Remark 2. *In [37] the authors show that the basic reproduction numbers for the partial differential equations epidemic models are the same for their corresponding ordinary differential equations models in several important cases.*

Let

$$\begin{aligned}
F(S,E,I,R) = (\Lambda - \beta SI - \mu S,\ \beta SI - (\mu+\gamma)E, \\
\gamma E - (\alpha+\mu)I,\ \alpha I - \mu R).
\end{aligned}$$

The system (7) has two constant equilibrium points. The disease-free equilibrium point is

$$E_0 = (S_0, 0, 0, 0) = \left(\frac{\Lambda}{\mu}, 0, 0, 0\right),$$

and the endemic equilibrium point is

$$E_1 = (S^*, E^*, I^*, R^*) = \left(\frac{(\mu+\gamma)(\mu+\alpha)}{\beta\gamma}, \frac{\mu(\mu+\alpha)}{\beta\gamma}(\mathcal{R}_0 - 1),\ \frac{\mu}{\beta}(\mathcal{R}_0 - 1),\ \frac{\alpha}{\beta}(\mathcal{R}_0 - 1)\right).$$

In the next result, we show the local asymptotically stability of disease-free equilibrium.

Theorem 6. *If $\mathcal{R}_0 < 1$, the disease-free equilibrium point E_0 is asymptotically stable.*

Proof: Let $A = DF(E_0)$ be the Jacobian matrix of $F(\cdot)$. By Theorem 1 and Theorem 2 the equilibrium point E_0 is asymptotically stable if for each nonnegative integer i the eigenvalue of the matrix $A - \mu_i D$ have negative real part where μ_i are the eigenvalues of $-\Delta$ and

$$D = \begin{pmatrix} d_S & 0 & 0 & 0 \\ 0 & d_E & 0 & 0 \\ 0 & 0 & d_I & 0 \\ 0 & 0 & 0 & d_R \end{pmatrix}.$$

Let

$$A - \mu_i D - \xi I = \begin{pmatrix} -\mu - \mu_i d_S - \xi & 0 & -\beta S_0 & 0 \\ 0 & -(\gamma+\mu) - \mu_i d_E - \xi & \beta S_0 & 0 \\ 0 & \gamma & -(\alpha+\mu) - \mu_i d_I - \xi & 0 \\ 0 & 0 & \alpha & -\mu - \mu_i d_R - \xi \end{pmatrix}.$$

Computing $\det(A - \mu_i D - \xi I) = 0$, we get

$$\det(A - \mu_i D - \xi I) = (-\mu - \mu_i d_S - \xi)(-\mu - \mu_i d_R - \xi)a,$$

where

$$\begin{aligned} a = \xi^2 &+ (\mu_i d_E + \mu_i d_I + (\alpha+\mu) + (\gamma+\mu))\xi \\ &+ \mu_i^2 d_E d_I + \mu_i(\alpha+\mu)d_E + \mu_i(\gamma+\mu)d_I \\ &+ (\alpha+\mu)(\gamma+\mu) - \beta S_0 \gamma. \end{aligned}$$

The eigenvalues are $\xi_1 = -\mu - \mu_i d_S$, $\xi_2 = -\mu - \mu_i d_R$ and the rest of the eigenvalues have negative real part if $a = 0$ has roots with a negative real part. But this is true since $\mathcal{R}_0 < 1$. This implies that the equilibrium point (S_0, E_0, I_0, R_0) is asymptotically stable. ■

Theorem 7. *If $\mathcal{R}_0 > 1$, the endemic equilibrium point E_1 is asymptotically stable.*

Proof: Let $A = DF(E_1)$, for each non-negative integer i, let

$$A - \mu_i D - \xi I = \begin{pmatrix} -\mu - \beta I^* - \mu_i d_S - \xi & 0 & -\beta S^* & 0 \\ \beta I^* & -(\gamma+\mu) - \mu_i d_E - \xi & \beta S^* & 0 \\ 0 & \gamma & -(\mu+\alpha) - \mu_i d_I - \xi & 0 \\ 0 & 0 & \alpha & -\mu - \mu_i d_R - \xi \end{pmatrix}.$$

Make $\det(A - \mu_i D - \xi I) = 0$ by

$$\det(A - \mu_i D - \xi I) = -(-\mu - \mu_i d_R - \xi)(\xi^3 + a_2\xi^2 + a_1\xi + a_0),$$

where

$$\begin{aligned} a_0 &= (\beta I^* + \mu + \mu_i d_S)\big(\mu_i(\alpha+\mu)d_E + \mu_i(\gamma+\mu)d_I \\ &\quad + \mu_i^2 d_I d_E + (\alpha+\mu)(\gamma+\mu)\big) - \beta^2 S^* I^* \gamma, \\ a_1 &= (\beta I^* + \mu + \mu_i d_S)\big(\mu_i d_E + \mu_i d_I \\ &\quad + (\alpha+\mu) + (\gamma+\mu)\big) + (\alpha+\mu)(\gamma+\mu) \\ &\quad + \mu_i(\alpha+\mu)d_E + \mu_i(\gamma+\mu)d_I + \mu_i^2 d_I d_E, \\ a_2 &= \beta I^* + \mu + \mu_i d_S + \mu_i d_E + \mu_i d_I \\ &\quad + (\alpha+\mu) + (\gamma+\mu). \end{aligned}$$

We get that $\xi_1 = -\mu - \mu_i d_R$ is an eigenvalue and the rest of the eigenvalues have negative real part if

$$\xi^3 + a_2\xi^2 + a_1\xi + a_0 = 0$$

has roots with negative real part. Using $\beta S^* \gamma = (\alpha + \mu)(\mu+\gamma)$ we have $a_2 > 0$, $a_1 > 0$ and $a_2 a_1 > a_0$. By the Routh-Hurwitz criterion we obtain $\xi^3 + a_2\xi^2 + a_1\xi + a_0 = 0$ has all roots with negative real part. From Theorem 1 and Theorem 2 we get that the equilibrium point (S^*, E^*, I^*, R^*) is asymptotically stable. ■

B. Global Stability

In this section, we apply the theory of Lyapunov functionals method to study the global stability of the SEIR model for COVID-19 with spatial diffusion. The method is based on the construction of Lyapunov functionals for partial differential equations using the knowledge of Lyapunov functions for ordinary differential systems [27].

Theorem 8. *If $\mathcal{R}_0 \leq 1$, the disease-free equilibrium point E_0 is globally asymptotically stable.*

Proof: We define the Lyapunov function on $X = L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ by

$$V(\mathcal{U}(t)) = \int_\Omega L(S,E,I,R)dx,$$

where

$$L(S,E,I,R) = \theta\left(S - S_0 - S_0 \ln \frac{S}{S_0}\right) + \frac{1}{\mu+\gamma}E + \frac{1}{\gamma}I.$$

It is easy to see that

$$\begin{aligned}\frac{dV(\mathcal{U}(t))}{dt} &= \int_\Omega L'(S,E,I,R)dx \\ &= \int_\Omega \theta\left(1 - \frac{S_0}{S}\right)S' + \frac{1}{\mu+\gamma}E' + \frac{1}{\gamma}I'dx \\ &= d_S\theta\int_\Omega \frac{(S-S_0)}{S}\Delta S dx + \frac{d_E}{\mu+\gamma}\int_\Omega \Delta E dx \\ &\quad + \frac{d_I}{\gamma}\int_\Omega \Delta I dx + \int_\Omega 2\Lambda\theta - \theta\beta SI - \theta\mu S \\ &\quad - \frac{\theta\Lambda^2}{\mu S} + \frac{\beta}{\mu+\gamma}SI + \left(\frac{\theta\beta\Lambda}{\mu} - \frac{\mu+\alpha}{\gamma}\right)Idx.\end{aligned}$$

Change $\theta = \dfrac{1}{\mu+\gamma}$,

$$\begin{aligned}\frac{dV(\mathcal{U}(t))}{dt} &= -d_S\theta S_0\int_\Omega \frac{|\nabla S|^2}{S^2}dx \\ &\quad - \int_\Omega \frac{\theta}{\mu S}(\Lambda - \mu S)^2 - \frac{\mu+\alpha}{\gamma}(1-R_0)Idx \\ &\leq -\int_\Omega \frac{\theta}{\mu S}(\Lambda-\mu S)^2 - \frac{\mu+\alpha}{\gamma}(1-R_0)Idx.\end{aligned}$$

We define

$$\varphi(t) = \int_\Omega \frac{\theta}{\mu S}(\Lambda-\mu S)^2 + \frac{(\mu+\alpha)}{\gamma}(1-R_0)Idx.$$

By integration from 1 to t, we get

$$V(\mathcal{U}(t)) - V(\mathcal{U}(1)) \leq -\int_1^t \varphi(s)ds,$$

and we have $\int_1^\infty \varphi(s)ds \leq V(\mathcal{U}(1)) < \infty$.

The positivity of $\varphi(t)$ indicates that $\varphi(t) \to 0$ as $t \to \infty$. We have the convergence to the $S(t) \to S_0$ in L^2-norm and $I(t) \to 0$ in L^1-norm.

Now we have again as in the proof of Theorem 5

$$\int_\Omega EE'dx = -\int_\Omega d_E|\nabla E|^2dx - \int_\Omega(\mu+\gamma)E^2dx + \int_\Omega \beta SEIdx.$$

By the Hölder inequality, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega |E|^2dx + (\mu+\gamma)\int_\Omega |E|^2dx \\ &\quad\leq \int_\Omega \beta SEIdx \\ &\quad\leq \int_\Omega \beta(S-S_0)EIdx + \int_\Omega \beta S_0EIdx \\ &\quad\leq \beta\|E\|_\infty\|S-S_0\|_{L^2}\|I\|_{L^2} + \beta S_0\|E\|_\infty\|I\|_{L^1} \\ &\quad\leq C\|E\|_{H^{2\eta}}(\|S-S_0\|_{L^2} + \|I\|_{L^1}) \\ &\quad\leq C\|A^\eta E\|_{L^2}(\|S-S_0\|_{L^2} + \|I\|_{L^1}) \\ &\quad\leq \frac{C}{t^\eta}(\|S-S_0\|_{L^2} + \|I\|_{L^1}).\end{aligned}$$

From Lemma 1, we have

$$\begin{aligned}\|E\|_{L^2}^2 &\leq \|E(0)\|_{L^2}^2 e^{-\frac{\mu+\gamma}{2}t} + \frac{2C}{(\mu+\gamma)} \\ &\quad\cdot \int_0^t e^{-\frac{\mu+\gamma}{2}(t-s)}s^{-\eta}(\|S(s)-S_0\|_{L^2} + \|I\|_{L^1})ds.\end{aligned}$$

By the convergence of the $S(t) \to S_0$ in L^2-norm and $I(t) \to 0$ in L^1-norm it implies that, for all $\varepsilon > 0$, there exists $T > 0$ such that $\|S-S_0\|_{L^2} + \|I\|_{L^1} < \varepsilon$ for all $t > T$, therefore

$$\begin{aligned}\|E\|_{L^2}^2 &\leq \|E(0)\|_{L^2}^2 e^{-\frac{\mu+\gamma}{2}t} + \frac{2C}{(\mu+\gamma)}e^{-\frac{\mu+\gamma}{2}t} \\ &\quad\cdot \int_0^T e^{-\frac{\mu+\gamma}{2}s}s^{-\eta}(\|S(s)-S_0\|_{L^2} + \|I\|_{L^1})ds \\ &\quad+ \frac{2C}{(\mu+\gamma)}\int_T^t e^{-\frac{\mu+\gamma}{2}(t-s)}s^{-\eta}\varepsilon ds.\end{aligned}$$

This implies that $E(t) \to 0$ as $t \to \infty$ in L^2-norm.

Using the same argument of the proof of Theorem 5 we arrive at the expression

$$\begin{aligned}&\frac{d}{dt}\int_\Omega |I|^2dx + (\alpha+\mu)\int_\Omega |I|^2dx \\ &\qquad\leq \int_\Omega \frac{\gamma^2}{2(\alpha+\mu)^2}|E|^2dx.\end{aligned}$$

By Lemma 1, we obtain

$$\begin{aligned}\|I(t)\|_{L^2}^2 &\leq e^{-(\alpha+\mu)t}\|I(0)\|_{L^2}^2 + \frac{\gamma^2}{2(\alpha+\mu)^2} \\ &\quad\cdot \int_0^t e^{-(\alpha+\mu)(t-s)}\|E(s)\|_{L^2}^2ds \\ &\leq e^{-(\alpha+\mu)t}\|I(0)\|_{L^2}^2 + \frac{\gamma^2}{2(\alpha+\mu)^2} \\ &\quad\cdot \int_0^T e^{-(\alpha+\mu)(t-s)}\|E(s)\|_{L^2}^2ds \\ &\quad+ \frac{\gamma^2}{2(\alpha+\mu)^2}\int_T^t e^{-(\alpha+\mu)(t-s)}\|E(s)\|_{L^2}^2ds.\end{aligned}$$

From the previous statement, it implies that $I(t) \to 0$ as $t \to \infty$ in L^2-norm as $t \to \infty$.

Multiplying $R(\cdot)$ in the fourth equation of the system (7), and repeating the same idea as before, we get

$$\frac{d}{dt}\int_\Omega |R|^2 dx + \frac{\mu}{2}\int_\Omega |R|^2 dx \le \frac{\alpha^2}{2\mu}\|I\|_{L^2}.$$

By Lemma 1, we get

$$\|R\|_{L^2}^2 \le \|R(0)\|_{L^2}^2 e^{-\mu t} + \frac{\alpha^2}{\mu^2}\int_0^t e^{-\mu(t-s)}\|I(s)\|_{L^2} ds.$$

From the previous statement implies that $R(t) \to 0$ as $t \to \infty$ in L^2-norm as $t \to \infty$.

Therefore, (S, E, I, R) converge to $(S_0, 0, 0, 0)$ in the $X = L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ norm. This completes the proof. ■

Theorem 9. *If $\mathcal{R}_0 > 1$ and $d_S = d_E$, the endemic equilibrium E_1 is globally asymptotically stable.*

Proof: We define a Lyapunov function

$$V(\mathcal{U}(t)) = \int_\Omega L(S, E, I, R) dx,$$

and

$$\begin{aligned} L(S, E, I, R) = \theta_1 &\left(S - S^* - S^* \ln \frac{S}{S^*}\right) \\ + \theta_2 &\left(E - E^* - E^* \ln \frac{E}{E^*}\right) \\ + \theta_3 &\left(I - I^* - I^* \ln \frac{I}{I^*}\right), \end{aligned}$$

where θ_1, θ_2, θ_3 are positive constants. Then

$$\begin{aligned} \frac{dV(\mathcal{U}(t))}{dt} = \int_\Omega &\theta_1 \left(1 - \frac{S^*}{S}\right) S' \\ &+ \theta_2 \left(1 - \frac{E^*}{E}\right) E' + \theta_3 \left(1 - \frac{I^*}{I}\right) I' dx. \end{aligned}$$

Substituting the previous equations of the system in the expressions and using the first equation in equilibrium

$$\Lambda = \beta S^* I^* + \mu S^*,$$

we get

$$\begin{aligned} \frac{dV(\mathcal{U}(t))}{dt} &= -\int_\Omega d_S \theta_1 S^* \frac{|\nabla S|^2}{S^2} + d_E E^* \theta_2 \frac{|\nabla E|^2}{E^2} \\ &+ d_I I^* \theta_3 \frac{|\nabla I|^2}{I^2} dx + \int_\Omega -\theta_1 \mu \frac{(S - S^*)^2}{S} + \theta_1 \beta S^* I^* \\ &- \theta_1 \beta S I - \theta_1 \frac{\beta S^{*2} I^*}{S} + \theta_1 \beta S^* I \\ &+ \theta_2 \beta S I - \theta_2(\mu + \gamma) E - \theta_2 \frac{\beta E^* S I}{E} + \theta_2 (\mu + \gamma) E^* \\ &+ \theta_3 \gamma E - \theta_3 (\mu + \alpha) I - \theta_3 \frac{\gamma I^* E}{I} + \theta_3 (\mu + \alpha) I^* dx. \end{aligned}$$

Making $\theta_1 = \theta_2$ and using the equation in the equilibrium,

$$\begin{aligned} \frac{\beta S^* I^*}{E^*} &= \mu + \gamma, \\ \gamma E^* &= (\mu + \alpha) I^*, \\ \beta S^* &= \frac{(\mu + \gamma)(\mu + \alpha)}{\gamma}, \end{aligned}$$

and choose θ_3 such that $\theta_3 = \theta_1 \frac{\mu + \gamma}{\gamma}$, we get

$$\begin{aligned} \theta_1 (\mu + \gamma) E^* &= \theta_1 \beta S^* I^*, \\ \theta_1 \beta S^* I - \theta_3 (\mu + \alpha) I &= 0, \\ \theta_3 \gamma E - \theta_1 (\mu + \gamma) E &= 0, \\ \theta_3 \frac{\gamma I^* E}{I} &= \frac{\beta S^* I^{*2} E}{I E^*}, \\ \theta_3 (\mu + \alpha) I^* &= \theta_1 \beta S^* I^*. \end{aligned}$$

Using the previous equalities and the relation between geometric mean and arithmetic mean, we get

$$\begin{aligned} \frac{dV(\mathcal{U}(t))}{dt} &= -\int_\Omega d_S \theta_1 S^* \frac{|\nabla S|^2}{S^2} + d_E \theta_2 E^* \frac{|\nabla E|^2}{E^2} \\ &\quad + d_I \theta_3 I^* \frac{|\nabla I|^2}{I^2} dx - \int_\Omega \theta_1 \mu \frac{(S - S^*)^2}{S} \\ &\quad - \theta_1 \beta S^* I^* \left(3 - \frac{S^*}{S} - \frac{I^* E}{I E^*} - \frac{E^* S I}{E S^* I^*}\right) dx \\ &\le -\int_\Omega \theta_1 \mu \frac{(S - S^*)^2}{S} dx. \end{aligned}$$

Defining $\psi(t) = \int_\Omega \theta_1 \mu \frac{(S - S^*)^2}{S} dx$. By integration from 1 to t, we get

$$V(\mathcal{U}(t)) - V(\mathcal{U}(1)) \le -\int_1^t \psi(s) ds,$$

where $\int_1^\infty \psi(s) ds \le V(\mathcal{U}(1)) < \infty$. The positivity of $\psi(t)$ indicates that $\psi(t) \to 0$ as $t \to \infty$.

We note that

$$\begin{aligned} \int_\Omega (S - S^*)^2 dx = \int_\Omega &\frac{(S - S^*)^2}{S} S dx \\ &\le \frac{1}{\theta_1 \mu} \|S\|_{L^\infty} \psi(t) \\ &\le \frac{1}{\theta_1 \mu} \|A^\eta S(t)\|_{L^2} \psi(t) \le \frac{C}{t^\eta} \psi(t) \to 0, \end{aligned}$$

as $t \to \infty$. This implies the convergence to $S(t) \to S^*$ in L^2-norm.

Using that $\Lambda = -d_S \Delta S^* + \beta S^* I^* + \mu S^*$, $\zeta = d_S = d_E$ and multiplying $S - S^*$ and $E - E^*$ in the first equation of (7) we get

$$\int_\Omega S_* S_*' dx = \int_\Omega \zeta S_* \Delta S_* dx - \int_\Omega \mu S_*^2 - \beta S_* S_{**} dx, \tag{13}$$

$$\int_\Omega E_* S_*' dx = \int_\Omega \zeta E_* \Delta S_* dx - \int_\Omega \mu E_* S_* - \beta E_* S_{**} dx, \tag{14}$$

where we replace for readability

$$S_* = (S - S^*),\ E_* = (E - E^*),\ S_{**} = (SI - S^* I^*).$$

By $d_E \Delta E^* + \beta S^* I^* - (\mu + \gamma) E^* = 0$ and multiplying $E - E^*$ and $S - S^*$ in second equation of (7) we get

$$\int_\Omega E_* E_*' dx = \int_\Omega \zeta E_* \Delta E_* dx - \int_\Omega (\mu + \gamma) E_*^2 dx + \int_\Omega \beta E_* S_{**} dx, \tag{15}$$

$$\int_\Omega S_* E_*' dx = \int_\Omega \zeta S_* \Delta E_* dx - \int_\Omega (\mu + \gamma) S_* E_* dx + \int_\Omega \beta S_* S_{**} dx. \tag{16}$$

Adding the previous equations (13)-(16) we infer

$$\begin{aligned}
\frac{d}{dt} \int_\Omega \frac{1}{2} (S_* + E_*)^2 dx &= -\zeta \int_\Omega (\nabla S_* + \nabla E_*)^2 dx \\
&\quad - \mu \int_\Omega (S_* + E_*)^2 dx - \gamma \int_\Omega E_*^2 dx - \gamma \int_\Omega S_* E_* dx \\
&\le -\frac{\mu}{2} \int_\Omega (S_* + E_*)^2 dx - \frac{\gamma}{2} \int_\Omega E_*^2 dx \\
&\quad - \gamma \int_\Omega S_* E_* dx - \frac{\gamma}{2} \int_\Omega S_*^2 dx + \frac{\gamma}{2} \int_\Omega S_*^2 dx \\
&\le -\frac{\mu + \gamma}{2} \int_\Omega (S_* + E_*)^2 dx + \frac{\gamma}{2} \int_\Omega S_*^2 dx.
\end{aligned}$$

This implies

$$\frac{d}{dt} \int_\Omega (S_* + E_*)^2 dx + (\mu + \gamma) \int_\Omega (S_* + E_*)^2 dx \le \gamma \int_\Omega S_*^2 dx.$$

From Lemma 1, and replacement back of S_* and E_*, we get

$$\begin{aligned}
\|S - S^* + E - E^*\|_{L^2} &\le e^{-(\mu+\gamma)t} \|S(0) - S^* + E(0) - E^*\|_{L^2} \\
&\quad + \frac{\gamma}{\mu + \gamma} \int_0^t e^{-(\mu+\gamma)(t-s)} \|S(s) - S^*\|_{L^2} ds.
\end{aligned}$$

Therefore

$$\begin{aligned}
\|E - E^*\|_{L^2} &\le \|S - S^* + E - E^*\|_{L^2} + \|S - S^*\|_{L^2} \\
&\le e^{-(\mu+\gamma)t} \|S(0) - S^* + E(0) - E^*\|_{L^2} \\
&\quad + \frac{\gamma}{\mu + \gamma} \int_0^t e^{-(\mu+\gamma)(t-s)} \|S(s) - S^*\|_{L^2} ds \\
&\quad + \|S - S^*\|_{L^2}.
\end{aligned}$$

This implies that $E(t) \to E^*$ as $t \to \infty$ in L^2-norm as $t \to \infty$.

Using that $d_I \Delta I^* + \gamma E^* - (\mu + \alpha) I^* = 0$ and multiplying $I - I^*$ in third equation of (7) we get

$$\begin{aligned}
\int_\Omega (I - I^*)'(I - I^*) dx &= \int_\Omega d_I (I - I^*) \Delta (I - I^*) dx \\
&\quad + \int_\Omega \gamma (E - E^*)(I - I^*) - (\mu + \alpha)(I - I^*)^2 dx,
\end{aligned}$$

from the same idea of Theorem 5 we have

$$\frac{1}{2} \frac{d}{dt} \int_\Omega |I - I^*|^2 dx \le -\frac{\mu + \alpha}{2} \int_\Omega (I - I^*)^2 dx + \int_\Omega \frac{\gamma^2}{2(\mu + \alpha)} (E - E^*)^2 dx.$$

From Lemma 1, we infer

$$\begin{aligned}
\|I(t) - I^*\|_{L^2}^2 &\le e^{-(\alpha+\mu)t} \|I(0) - I^*\|_{L^2}^2 \\
&\quad + \frac{\gamma^2}{(\mu + \alpha)^2} \int_0^t e^{-(\alpha+\mu)(t-s)} \|E(s) - E^*\|_{L^2}^2 ds.
\end{aligned}$$

From the previous statement, $I(t) \to I^*$ as $t \to \infty$ in L^2-norm as $t \to \infty$.

Using $d_R \Delta R^* + \alpha I^* - \mu R^* = 0$ in the fourth equation of the system (7) we get

$$(R - R^*)' = d_R \Delta (R - R^*) + \alpha (I - I^*) - \mu (R - R^*).$$

Multiplying by $R - R^*$ and using the same argument of Theorem 5 and Lemma 1 we get

$$\begin{aligned}
\|R - R^*\|_{L^2}^2 &\le \|R(0) - R^*\|_{L^2}^2 e^{-\mu t} \\
&\quad + \frac{\alpha^2}{\mu^2} \int_0^t e^{-\mu(t-s)} \|I(s) - I^*\|_{L^2} ds.
\end{aligned}$$

Thus, $R(t) \to R^*$ as $t \to \infty$ in L^2-norm.

Hence, (S, E, I, R) converges to the equilibrium point (S^*, E^*, I^*, R^*) in the $X = L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ norm, finishing the proof. ■

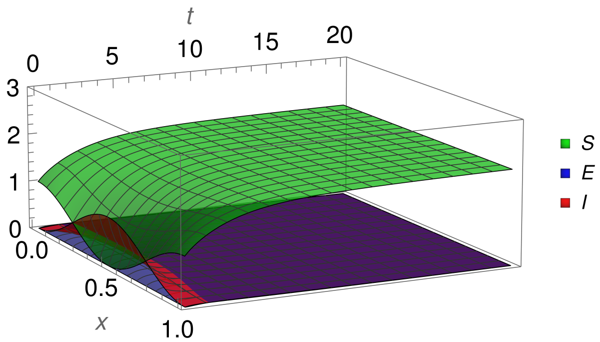
V. Numerical Examples

To illustrate the analytical results of stability more concretely, in this section we present numerical solutions in one spatial dimension on the interval $[0, 1]$ which evolve to the disease-free (E_0) and endemic (E_1) equilibria asymptotically. For that, only the three first equations of the system (7) are considered, since $R(t, x)$ has no influence on $S(t, x)$, $E(t, x)$, $I(t, x)$. Consistently with the Neumann homogeneous boundary condition, we also set the initial condition $I(0, x) = \sin^2(\pi x)$, $S(0, x) = 1 - \sin^2(\pi x)$, and $E(0, x) = 0$. As for the parameter values, these are listed in Table I.

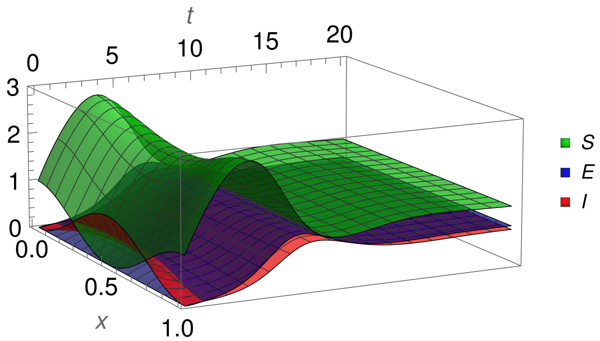
The two representative cases we address here are defined by different values of $\mathcal{R}_0$ changing the value of μ. Based on the analytical results of stability derived in the previous section, and according to the values presented in Table I, the value of $\mu = 0.5$ implies in $\mathcal{R}_0 \approx 0.88 < 1$ ensuring that $E_0 = (2, 0, 0, 0)$ is globally asymptotically stable. On the other hand, the value of $\mu = 0.1$ implies in $\mathcal{R}_0 \approx 8.26 > 1$, ensuring that $E_1 \approx (1.21, 0.80, 0.73, 7.26)$ is globally asymptotically stable. The respective numerical results for these two cases are shown in Figure 1 (a)-(b), where the constant values observed all over the domain at $t = 20$ represent their respective steady states; E_0 in Figure 1(a), and E_2 in Figure 1(b). These numerical results are in agreement with Theorem 6 and Theorem 7. Although the global stability cannot be entirely accessed by the use of numerical simulations, these results are also consistent with Theorem 8 and Theorem 9.

Table I: Parameter values used for the numerical simulations. The value of μ is used to change the stability.

Parameter	Value
Λ	1.0
β	1.0
γ	1.0
α	1.0
d_S	0.005
d_E	0.005
d_I	0.005



(a) E_0 is asymptotically stable, $\mu = 0.5$



(b) E_1 is asymptotically stable, $\mu = 0.1$

Fig. 1: Numerical results. Other parameter values are listed in Table I.

VI. Conclusions

In this paper, we studied the well-posedness and the qualitative behavior of equilibrium points to a SEIR epidemic models with spatial diffusion for the spreading of COVID-19. The well-posedness of the model was proved using the theory of abstract parabolic differential equations. The asymptotical local stability of both disease-free and endemic equilibria were established using standard linearization theory, and confirmed by illustrative numerical simulations. The asymptotical global stability of both disease-free and endemic equilibria were established using a Lyapunov function.

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